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Some Mappings Related to Levinson's Inequality for Hilbert Space Operators

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Abstract. We observe properties of some mappings related to the Davis-Choi-Jensen inequality for Hilbert space operators. Using these results, we observe properties of some mappings related to Levinson's operator inequality. Consequently, we obtain several refinements for each of these inequalities.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote a C^*-algebra of all bounded linear operators on a (complex) Hilbert space $\mathcal{H}$ and $1_{\mathcal{H}}$ denote the identity operator. We denote by $\mathcal{B}_h(\mathcal{H})$ the real subspace of all self-adjoint operators on $\mathcal{H}$ and by $\mathcal{B}^+(\mathcal{H})$ the set of all positive operators in $\mathcal{B}_h(\mathcal{H})$.

A continuous real valued function f defined on an interval I is said to be operator convex if $f(\lambda X + (1-\lambda)Y) \le \lambda f(X) + (1-\lambda)f(Y)$ for all self-adjoint operators X, Y with spectra contained in I and all $\lambda \in [0,1]$. Let $\dot{\mathcal{F}}(I)$ denote the set of all operator convex functions on interval I.

A linear mapping $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is said to be positive if it preserves the operator order $\ge$, i.e. if $A \in \mathcal{B}^+(\mathcal{H})$ implies $\Phi(A) \in \mathcal{B}^+(\mathcal{K})$ and is called normalized if it preserves the identity operator, i.e. if $\Phi(1_{\mathcal{H}}) = 1_{\mathcal{K}}$.

We recall the Davis-Choi-Jensen inequality:

Theorem A [3, Theorem 1.20]. *If $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a normalized positive linear mapping and $f \in \dot{\mathcal{F}}(I)$ is an operator convex function on an interval I, then*

$$\Phi(f(A)) \ge f(\Phi(A)) \tag{1}$$

for every selfadjoint operator A on H whose spectrum is contained in I.

Many other results can be found in [2, 3].

Next, we recall Levisons's operator inequality. First we give the definition of classes of functions for which we observe Levisons's operator inequality.

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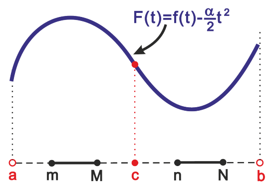
Definition 1.1. *Let $f \in \mathcal{C}(I)$ be a real valued function on an arbitrary interval I in $\mathbb{R}$ and $c \in I^\circ$, where I° is the interior of I.*
We say that $f \in \dot{\mathcal{K}}_1^c(I)$ (resp. $f \in \dot{\mathcal{K}}_2^c(I)$) if there exists a constant α such that the function $F(t) = f(t) - \frac{\alpha}{2}t^2$ is operator concave (resp. operator convex) on $I \cap (-\infty, c]$ and operator convex (resp. operator concave) on $I \cap [c, \infty)$.

Now we give Levisons's inequality for two operators and $f \in \dot{\mathcal{K}}_1^c(I)$. Many other results can be found in [4, 5].

Theorem B [4, Theorem 1].
Let $X, Y \in \mathcal{B}_h(H)$ be self-adjoint operators with spectra contained in $[m, M]$ and $[n, N]$, respectively, such that $a < m \le M \le c \le n \le N < b$. Let Φ, Ψ be normalized positive linear mappings $\Phi, \Psi : \mathcal{B}(H) \to \mathcal{B}(K)$.
If $f \in \dot{\mathcal{K}}_1^c((a, b))$ and



$$C_1 := \frac{\alpha}{2}\left[\Phi\left(X^2\right) - \Phi(X)^2\right] \le C_2 := \frac{\alpha}{2}\left[\Psi\left(Y^2\right) - \Psi(Y)^2\right], \tag{2}$$

then

$$\Phi\big(f(X)\big) - f\big(\Phi(X)\big) \le C_1 \le C_2 \le \Psi\big(f(Y)\big) - f\big(\Psi(Y)\big). \tag{3}$$

S.S. Dragomir in [1] observe Hermite-Hadamard's type inequalities for operator as follows. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any self-adjoint operators A and B with spectra in I we have (see [1, Theorem 1]):*

$$\left(f\left(\frac{A+B}{2}\right) \le\right) \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right)\right] \le \int_0^1 f\big((1-t)A + tB\big)\,dt \le \frac{1}{2}\left[f\left(\frac{A+B}{2}\right) + \frac{f(A)+f(B)}{2}\right] \left(\le \frac{f(A)+f(B)}{2}\right)$$

and (see [1, Corollary 1]):

$$0 \le \int_0^1 f\big((1-t)A + tB\big)\,dt - f\left(\frac{A+B}{2}\right) \le \frac{f(A)+f(B)}{2} - \int_0^1 f\big((1-t)A + tB\big)\,dt.$$

In the proof of the first result he introduce real-valued function $\varphi_{x,A,B} : [0, 1] \to \mathbb{R}$ defined by $\varphi_{x,A,B}(t) = \langle f((1-t)x + ty)x, x\rangle$.

In [1, Section 3] Dragomir gives two definitions
- the closed operator segment $[A, B] = \{ (1-t)A + tB,\ t \in [0, 1] \}$, for two distinct self-adjoint operators A, B;
-the operator-valued functional

$$\Delta_f(A, B; t) = (1-t) f(A) + t f(B) - f((1-t)A + tB) \ge 0,$$

for an operator convex function $f : I \to \mathbb{R}$ defined on the interval I and operators A, B with spectra in I; and proves an operator quasi-linearity property for the functional $\Delta_f(\cdot, \cdot; t)$:

Theorem C [4, Theorem 1]. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for each two distinct self-adjoint operators A, B with spectra contained in I and $C \in [A, B]$ we have*

$$(0 \le)\ \Delta_f(A, C; t) + \Delta_f(C, B; t) \le \Delta_f(A, B; t) \tag{4}$$

for each $t \in [0, 1]$, i.e. the functional $\Delta_f(\cdot, \cdot; t)$ is operator superadditive as a function of interval.

If $[C, D] \subset [A, B]$, *then*

$$(0 \leq)\ \Delta_f(C, D; t) \leq \Delta_f(A, B; t) \tag{5}$$

for each $t \in [0, 1]$, *i.e. the the functional* $\Delta_f(\cdot, \cdot; t)$ *is operator monotone as a function of interval.*

Inspired by Dragomir's results, we observe some mappings related to Levinson's operator inequality. To obtain these results, we give appropriate mappings related to the Davis-Choi-Jensen inequality. As application, we obtained some refinements of inequalities (1) and (3).

2. Jensen's Mapping and its Properties

We define Jensen's mapping $\mathfrak{J}_\Phi : \overset{\bullet}{\mathcal{F}}([m, M]) \times \mathcal{B}_h(\mathcal{H}) \times \mathcal{B}_h(\mathcal{H}) \times [0, 1] \to \mathcal{B}_h(\mathcal{H})$ as

$$\mathfrak{J}_\Phi(f, A, B, t) = \Phi\Big(f\,(tA + (1 - t)B)\Big), \tag{6}$$

where $m1_\mathcal{H} \leq A, B \leq M1_\mathcal{H}$ and $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a normalized positive linear mapping.

The following properties of the mapping (6) are obvious. We omit the proof.

Lemma 2.1. *Mapping* $\mathfrak{J}_\Phi$ *defined by* (6) *satisfied:*

(i) $\mathfrak{J}_\Phi(f, A, B, \cdot)$ *is convex on* $[0, 1]$, *that is*

$$\mathfrak{J}_\Phi(f, A, B, \alpha\, t_1 + \beta\, t_2) \leq \alpha\, \mathfrak{J}_\Phi(f, A, B, t_1) + \beta\, \mathfrak{J}_\Phi(f, A, B, t_2) \tag{7}$$

for every $\alpha, \beta \in [0, 1]$ *such that* $\alpha + \beta = 1$ *and for every* $t_1, t_2 \in [0, 1]$.

(ii) $\mathfrak{J}_\Phi(f, \cdot, \cdot, t)$ *is operator convex on* $\mathcal{B}_h(\mathcal{H}) \times \mathcal{B}_h(\mathcal{H})$, *that is*

$$\mathfrak{J}_\Phi(f, \alpha A_1 + \beta A_2, \alpha B_1 + \beta B_2, t) \leq \alpha\, \mathfrak{J}_\Phi(f, A_1, B_1, t) + \beta\, \mathfrak{J}_\Phi(f, A_2, B_2, t)$$

for every $\alpha, \beta \in [0, 1]$ *such that* $\alpha + \beta = 1$ *and for every* $A_1, A_2, B_1, B_2 \in \mathcal{B}_h(\mathcal{H})$.

The following three theorems contain another properties of the mapping (6) for some special case of operator B.

Theorem 2.2. *Let* $\mathfrak{J}_\Phi(f, A, B, t)$ *be a mapping defined by* (6) *and* $B = \bar{A} := \Phi(A)$.
If $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ *preserve the operator* $\bar{A}$, *then* $\mathfrak{J}_\Phi(f, A, \bar{A}, \cdot)$ *is monotone increasing on* $[0, 1]$, *that is*

$$\mathfrak{J}_\Phi(f, A, \bar{A}, t_1) \leq \mathfrak{J}_\Phi(f, A, \bar{A}, t_2) \qquad \text{for every} \quad 0 \leq t_1 \leq t_2 \leq 1.$$

So,

$$\inf_{t \in [0,1]} \mathfrak{J}_\Phi(f, A, \bar{A}, t) = f(\Phi(A)), \qquad \sup_{t \in [0,1]} \mathfrak{J}_\Phi(f, A, \bar{A}, t) = \Phi(f(A)). \tag{8}$$

Proof. Using the Davis-Choi-Jensen inequality (1) and condition $\Phi(\bar{A}) = \bar{A}$, we obtain that

$$\mathfrak{J}_\Phi(f, A, \bar{A}, t) = \Phi\Big(f\,(tA + (1 - t)\bar{A})\Big) \geq f\Big(\Phi\Big(tA + (1 - t)\bar{A}\Big)\Big) = f(\bar{A}) = \mathfrak{J}_\Phi(f, A, \bar{A}, 0)$$

holds for every $t \in [0, 1]$. Since $\mathfrak{J}_\Phi(f, A, \bar{A}, \cdot)$ is a convex on $[0, 1]$, setting $\alpha = \frac{t_2 - t_1}{t_2}$ and $\beta = \frac{t_1}{t_2}$ (for $0 < t_1 < t_2 \leq 1$) in (7) we obtain

$$\mathfrak{J}_\Phi\left(f, A, \bar{A}, \frac{t_2 - t_1}{t_2}\, s_1 + \frac{t_1}{t_2}\, s_2\right) \leq \frac{t_2 - t_1}{t_2}\, \mathfrak{J}_\Phi(f, A, \bar{A}, s_1) + \frac{t_1}{t_2}\, \mathfrak{J}_\Phi(f, A, \bar{A}, s_2).$$

Now, setting $s_1 = 0$ and $s_2 = t_2$ in the above inequality we obtain

$$0 \leq \frac{\mathfrak{J}_\Phi(f, A, \bar{A}, t_1) - \mathfrak{J}_\Phi(f, A, \bar{A}, 0)}{t_1 - 0} \leq \frac{\mathfrak{J}_\Phi(f, A, \bar{A}, t_2) - \mathfrak{J}_\Phi(f, A, \bar{A}, t_1)}{t_2 - t_1}$$

for every $0 < t_1 < t_2 \leq 1$. It follows that $\mathfrak{J}_\Phi(f, A, B, \cdot)$ is monotone increasing on $[0, 1]$.

Finally, applying this monotone increasing mapping, we obtain the minimum and maximum values in (13). □

Theorem 2.3. *Let $\mathfrak{I}_\Phi(f,A,B,t)$ be a mapping defined by* (6)*, where $\Phi:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{K})$, $B=\langle\Phi(A)u,u\rangle 1_{\mathcal{H}}$ and $u\in\mathcal{K}$ with $\|u\|=1$.*
Then $\mathfrak{I}_\Phi(f,A,\langle\Phi(A)u,u\rangle 1_{\mathcal{H}},\cdot)$ is monotone increasing on $[0,1]$ *and*

$$\inf_{t\in[0,1]}\mathfrak{I}_\Phi(f,A,\langle\Phi(A)u,u\rangle 1_{\mathcal{H}},t)=f(\langle\Phi(A)u,u\rangle)1_{\mathcal{H}},\qquad \sup_{t\in[0,1]}\mathfrak{I}_\Phi(f,A,\langle\Phi(A)u,u\rangle 1_{\mathcal{H}},t)=\Phi(f(A)).$$

The proof is the same as one of Theorem 2.2 and we omit it.

Remark 2.4. *Under the assumptions of Theorem 2.2 the following refinement of the Davis-Choi-Jensen inequality* (1)

$$f\Big(\Phi(A)\Big)\le\Phi\Big(f\left(t_1A+(1-t_1)\bar{A}\right)\Big)\le\Phi\Big(f\left(t_2A+(1-t_2)\bar{A}\right)\Big)\le\Phi\Big(f(A)\Big)\tag{9}$$

holds for every $0\le t_1\le t_2\le 1$.

Since f is continuous on $[m,M]$, the operator valued integral $\int_0^1 f\Big(tA+(1-t)B\Big)dt$ exists for any self-adjoint operators A and B with spectra in $[m,M]$. Since the mapping Φ is linear and continuous, then

$$\int_0^1\Phi\Big(f\left(tA+(1-t)B\right)\Big)dt=\Phi\Big(\int_0^1 f\left(tA+(1-t)B\right)dt\Big).$$

Finally, integrating the inequality (9) *over $t\in[0,1]$, we obtain another refinement of the Davis-Choi-Jensen inequality:*

$$f\Big(\Phi(A)\Big)\le\Phi\Big(\int_0^1 f\left(tA+(1-t)\bar{A}\right)dt\Big)\le\Phi\Big(f(A)\Big).\tag{10}$$

But, under the assumptions of Theorem 2.3 we obtain the following inequality:

$$f\Big(\langle\Phi(A)u,u\rangle\Big)1_{\mathcal{K}}\le\Phi\Big(\int_0^1 f\left(tA+(1-t)\langle\Phi(A)u,u\rangle 1_{\mathcal{H}}\right)dt\Big)\le\Phi\Big(f(A)\Big).$$

Also, it is obvious that we obtain the inequality with the scalar product by using (10)*:*

$$f\Big(\Big\langle\Phi(A)u,u\Big\rangle\Big)\le\Big\langle\Phi\Big(\int_0^1 f(tA+(1-t)\langle\Phi(A)u,u\rangle 1_{\mathcal{H}})dt\Big)u,u\Big\rangle\le\langle\Phi\left(f(A)\right)u,u\rangle,$$

where $u\in\mathcal{K}$, $\|u\|=1$.

Example 2.5. *We give examples with the power function and the trace mapping.*
(1) Let the assumptions of Theorem 2.2 be hold. If A is a positive self-adjoint operator with spectra contained in $[m,M]$ for some $0<m<M$, then

$$\mathfrak{I}_\Phi(t^p,A,\bar{A},t)=\Phi\Big((tA+(1-t)\bar{A})^p\Big),\quad p\in[-1,0]\cup[1,2],$$

is an example of (6) *and refinements of the Davis-Choi-Jensen inequality as in* (9) *and* (10) *hold. But, if $p\in[0,1]$, then the reverse inequalities are valid in* (9) *and* (10)*.*

(2) Let $A\in\mathcal{M}_n$ be a hermite matrix with spectra contained in $[m,M]$, f be an operator convex function on $[m,M]$ and $\Phi(A)=\frac{1}{n}Tr(A)I_n$. Then Φ is the normalized positive linear mapping and preserve the operator $\Phi(A)$. So,

$$\mathfrak{I}_{Tr}(f,A,\Phi(A),t)=\frac{1}{n}Tr\left(f\Big(tA+(1-t)\frac{Tr(A)}{n}I_n\Big)\right)I_n$$

is another example of (6)*.*

Theorem 2.6. *Let $\mathfrak{J}_\Phi(f, A, B, t)$ be a mapping defined by* (6), *where A is a positive self-adjoint operator, $B = M1_\mathcal{H}$ where M is the upper bound of A, and $f : [0, \infty) \to [0, \infty)$, $f(0) \le 0$, is an operator convex function. Then $\mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, \cdot)$ is monotone decreasing on* $[0, 1]$ *and*

$$\inf_{t\in[0,1]} \mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, t) = \Phi(f(A)), \qquad \sup_{t\in[0,1]} \mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, t) = f(M)1_\mathcal{K}. \tag{11}$$

Proof. Let $t \in (0, 1)$. Operator convexity of f and positive linearity of Φ give

$$\begin{aligned} &\mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, t) = \Phi\Big(f\,(tA + (1-t)M1_\mathcal{H})\Big) \le t\Phi\Big(f(A)\Big) + (1-t)\Phi\Big(f(M)1_\mathcal{H}\Big) \\ \le\;& tf(M)1_\mathcal{K} + (1-t)f(M)1_\mathcal{K} = f(M)1_\mathcal{K} = \mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, 0). \end{aligned}$$

Next, [6, Corollary 2.9] provides that $f(X+Y) \ge f(X) + f(Y)$ holds if $X \le N1_\mathcal{H} \le X+Y$ and $Y \le N1_\mathcal{H} \le X+Y$ for some scalar N, and $f : [0, \infty) \to [0, \infty)$, $f(0) \le 0$, is convex. Moreover, using [6, Corollary 2.5] we have $f(\alpha X) \le \alpha f(X)$ for $0 \le \alpha \le 1$. Setting $X := \frac{1}{2}A$, $Y := \frac{1}{2}M1_\mathcal{H}$ and $N := \frac{1}{2}M$, we have

$$f\left(\frac{1}{2}A + \frac{1}{2}M1_\mathcal{H}\right) \ge f\left(\frac{1}{2}A\right) + f\left(\frac{1}{2}M\right)1_\mathcal{H}.$$

It follows

$$\begin{aligned} \mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, 1/2) &= \Phi\Big(f\Big(\frac{1}{2}A + \frac{1}{2}M1_\mathcal{H}\Big)\Big) \ge \Phi\Big(f\Big(\frac{1}{2}A\Big)\Big) + f\Big(\frac{1}{2}M\Big)1_\mathcal{K} \\ &\ge 2\Phi\Big(f\Big(\frac{1}{2}A\Big)\Big) \ge \Phi\Big(\Big(f(A)\Big)\Big) = \mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, 1). \end{aligned}$$

Now, since, $\mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, \cdot)$ is convex function on $[0, 1]$, then

$$\begin{aligned} 0 \;&\ge\; \frac{\mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, 1) - \mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, 1/2)}{1/2} \\ &\ge\; \frac{\mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, 1/2) - \mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, t_2)}{1/2 - t_2} \ge \frac{\mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, t_2) - \mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, t_1)}{t_2 - t_1} \\ \Rightarrow\; &\quad \mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, t_2) \le \mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, t_1) \quad \text{for } 1/2 \ge t_2 > t_1 \ge 0 \end{aligned}$$

that is, $\mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, \cdot)$ is a monotone decreasing on $[0, 1/2]$. Similarly, we obtain that $\mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, t) \ge \mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, 1)$, which gives that $\mathfrak{J}_\Phi(f, A, M1_\mathcal{H}, \cdot)$ is a monotone decreasing on $[1/2, 1]$. □

Remark 2.7. *Applying Theorem 2.6 we have the following extension of the Davis-Choi-Jensen inequality* (1)

$$\begin{aligned} &f(\Phi(A)) \le \Phi\Big(f(A)\Big) \le \Phi\Big(f\,(t_2A + (1-t_2)M1_\mathcal{H})\Big) \\ \le\;& \Phi\Big(f\,(t_1A + (1-t_1)M1_\mathcal{H})\Big) \le f(M)1_\mathcal{K}, \quad \textit{for every } 0 \le t_1 \le t_2 \le 1 \end{aligned}$$

and

$$f(\Phi(A)) \le \Phi\Big(f(A)\Big) \le \Phi\Big(\int_0^1 f\,(tA + (1-t)M1_\mathcal{H})\Big)\mathrm{d}t \le f(M)1_\mathcal{K}.$$

Now, if we define trivial Jensen's mapping $\bar{\mathfrak{J}} : \dot{\mathcal{F}}([m, M]) \times \mathcal{B}_h(\mathcal{H}) \times \mathcal{B}_h(\mathcal{K}) \times [0, 1] \to \mathcal{B}_h(\mathcal{K})$ as

$$\bar{\mathfrak{J}}_\Phi(f, A, B, t) = t\Phi\Big(f(A)\Big) + (1-t)f(B), \tag{12}$$

where $m1_\mathcal{H} \le A, B \le M1_\mathcal{H}$ and $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a normalized positive linear mapping, then the following results hold. We omit the proof.

Lemma 2.8. *Mapping $\tilde{\mathfrak{J}}_\Phi$ defined by* (12) *has the following properties:*
(i) $\tilde{\mathfrak{J}}_\Phi(f, A, B, \cdot)$ *is linear on* $[0,1]$, *that is*

$$\tilde{\mathfrak{J}}_\Phi(f, A, B, \alpha t_1 + \beta t_2) = \alpha \tilde{\mathfrak{J}}_\Phi(f, A, B, t_1) + \beta \tilde{\mathfrak{J}}_\Phi(f, A, B, t_2)$$

for every $\alpha, \beta \in [0,1]$ *such that* $\alpha + \beta = 1$ *and for every* $t_1, t_2 \in [0,1]$.
(ii) $\tilde{\mathfrak{J}}_\Phi(f, \cdot, \cdot, t)$ *is operator convex on* $\mathcal{B}_h(\mathcal{H}) \times \mathcal{B}_h(\mathcal{K})$.

We can observe special cases of operator B in the mapping (12) similar to Theorems 2.2, 2.3 and 2.6. We give one of these results.

Theorem 2.9. *Let* $\tilde{\mathfrak{J}}_\Phi(f, A, B, t)$ *be a mapping defined by* (12) *and* $B = \bar{A} := \Phi(A)$.
Then $\tilde{\mathfrak{J}}_\Phi(f, A, \bar{A}, \cdot)$ *is monotone increasing on* $[0,1]$ *and*

$$\inf_{t\in[0,1]} \tilde{\mathfrak{J}}_\Phi(f, A, \bar{A}, t) = f(\Phi(A)), \qquad \sup_{t\in[0,1]} \tilde{\mathfrak{J}}_\Phi(f, A, \bar{A}, t) = \Phi(f(A)). \tag{13}$$

Proof. By using (1) we have

$$\tilde{\mathfrak{J}}_\Phi(f, A, \Phi(A), t) = t\Phi\Big(f(A)\Big) + (1-t)f(\bar{A}) \geq tf(\bar{A}) + (1-t)f(\bar{A}) = f(\bar{A}) = \tilde{\mathfrak{J}}_\Phi(f, A, \Phi(A), 0).$$

Next, similar to the proof of Theorem 2.2, we obtain that $\tilde{\mathfrak{J}}_\Phi(f, A, \Phi(A), \cdot)$ is monotone increasing on $[0,1]$. □

Remark 2.10. *1) Under the assumptions of Theorem 2.9 we can give refinement of the Davis-Choi-Jensen inequality* (1)*, similarly as in Remark 2.4.*

2) Under the assumptions of Theorems 2.2 and 2.9, we obtain another refinements of (1)

$$f\Big(\Phi(A)\Big) \leq \Phi\Big(f\left(tA + (1-t)\bar{A}\right)\Big) \leq t\Phi\Big(f\left(A\right)\Big) + (1-t)f\left(\bar{A}\right) \leq \Phi\Big(f(A)\Big)$$

for every $t \in [0,1]$ *and*

$$f\Big(\Phi(A)\Big) \leq \Phi\Big(\int_0^1 f\left(tA + (1-t)\bar{A}\right)dt\Big) \leq \frac{1}{2}\,\Phi\Big(f\left(A\right)\Big) + \frac{1}{2}\,f\left(\bar{A}\right) \leq \Phi\Big(f(A)\Big).$$

3) It is easy to prove that the above results hold for series of operators. For example, similarly to (6) *we define Jensen's mapping* $\mathfrak{J}_\Phi : \dot{\mathcal{F}}([m, M]) \times \mathcal{B}_h(\mathcal{H}) \times \mathcal{B}_h(\mathcal{H}) \times [0,1] \times \mathbb{N} \to \mathcal{B}_h(\mathcal{H})$ *as*

$$\mathfrak{J}_{\mathbf{\Phi}}(f, \mathbf{A}, \mathbf{B}, t, k) = \sum_{i=1}^{k} \Phi_i\Big(f\left(tA_i + (1-t)B_i\right)\Big),$$

where $\mathbf{A} = (A_1, \ldots, A_k)$, $\mathbf{B} = (A_1, \ldots, B_k)$ *are two* $k-$*tuples of self-adjoint operators* $A_i, B_i \in \mathcal{B}_h(\mathcal{H})$ *with spectra contained in* $[m, M]$ *for some* $m < M$ *and* $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_k)$ *be a unital* $k-$*tuple of positive linear mappings* $\Phi_i : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ *(i.e.* $\sum_{i=1}^n \Phi_i(1_\mathcal{H}) = 1_\mathcal{H}$*).*

As a special case, we set $f(t) = t^r$, $r \in [-1,0] \cup [1,2]$, *and* $\Phi_i(X) = p_iX$, *for some positive scalars* p_i *such that* $\sum_{i=1}^k p_i = 1$. *It is obvious that* $\mathbf{\Phi}$ *preserve the operator* $\bar{A} = \sum_{i=1}^{k} p_iA_i$. *So, inequalities in 2) become:*

$$\bar{A}^r \leq \sum_{i=1}^{k} p_i\left(tA_i + (1-t)\bar{A}\right)^r \leq t\sum_{i=1}^{k} p_iA_i^r + (1-t)\bar{A}^r \leq \sum_{i=1}^{k} p_iA_i^r,\ t \in [0,1]$$

and

$$\bar{A}^r \leq \sum_{i=1}^{k} p_i \int_0^1 \left(tA_i + (1-t)\bar{A}\right)^r dt \leq \frac{1}{2}\,\sum_{i=1}^{k} p_iA_i^r + \frac{1}{2}\,\bar{A}^r \leq \sum_{i=1}^{k} p_iA_i^r.$$

3. Levinson's Mapping and its Properties

We define Levinson's mapping $\mathfrak{L}_{\Phi,\Psi}$ as a difference between the corresponding Jensen's mappings (6), i.e.

$$\mathfrak{L}_{\Phi,\Psi}(f,X,Y,t) = \mathfrak{J}_{\Psi}(f,Y,\Psi(Y),t) - \mathfrak{J}_{\Phi}(f,X,\Phi(X),t), \tag{14}$$

where $\Phi,\Psi : \mathcal{B}(H) \to \mathcal{B}(H)$ are normalized positive linear mappings, $f \in \dot{\mathcal{K}}_1^c((a,b))$, $X,Y \in \mathcal{B}_h(H)$ are self-adjoint operators with spectra contained in $[m,M]$ and $[n,N]$, such that $a < m \le M \le c \le n \le N < b$ and $t \in [0,1]$. Therefore,

$$\mathfrak{L}_{\Phi,\Psi}(f,X,Y,t) = \Psi\Big(f\left(tY + (1-t)\Psi(Y)\right)\Big) - \Phi\Big(f\left(tX + (1-t)\Phi(X)\right)\Big). \tag{15}$$

Now, we prove our first result that the properties of convexity and monotonicity of the mapping (14) hold.

Theorem 3.1. *Let $\mathfrak{L}_{\Phi,\Psi}$ be a mapping defined by* (14)*, let Φ preserve the operator $\bar{X} := \Phi(X)$ and the product of operators X and $\bar{X}$, and let Ψ, analogous, preserve $\bar{Y} := \Psi(Y)$ and the product of Y and $\bar{Y}$.*
If

$$C_1 := \frac{\alpha}{2}\left[\Phi\Big(X^2\Big) - \Phi(X)^2\right] \le C_2 := \frac{\alpha}{2}\left[\Psi\Big(Y^2\Big) - \Psi(Y)^2\right],$$

holds (see (2)*), then $\mathfrak{L}_{\Phi,\Psi}(f,X,Y,\cdot)$ is convex and monotone increasing on* $[0,1]$*. So,*

$$\inf_{t\in[0,1]} \mathfrak{L}_{\Phi,\Psi}(f,X,Y,t) = f(\Psi(Y)) - f(\Phi(X)), \qquad \sup_{t\in[0,1]} \mathfrak{L}_{\Phi,\Psi}(f,X,Y,t) = \Psi(f(Y)) - \Phi(f(X)). \tag{16}$$

Proof. Let the constant α be as in Definition 1.1, such that $F(s) = f(s) - \frac{\alpha}{2}s^2$ is an operator concave function on $[a,c]$ and operator convex on $[c,b]$. Then we can apply Lemma 2.1 and Theorem 2.2 to the mappings

$$\mathfrak{J}_{\Psi}(F,Y,\bar{Y},t) = \Psi\Big(F\left(tY + (1-t)\bar{Y}\right)\Big) \quad \text{and} \quad -\mathfrak{J}_{\Phi}(F,X,\bar{X},t) = \Psi\Big(-F\left(tX + (1-t)\bar{X}\right)\Big)$$

and obtain that $\mathfrak{J}_{\Psi}(F,Y,\bar{Y},\cdot)$ and $-\mathfrak{J}_{\Phi}(F,X,\bar{X},\cdot)$ are convex and monotone increasing on $[0,1]$. So the mapping

$$\mathfrak{L}_{\Phi,\Psi}(F,X,Y,\cdot) = \mathfrak{J}_{\Psi}(F,Y,\bar{Y},\cdot) - \mathfrak{J}_{\Phi}(F,X,\bar{X},\cdot)$$

has the same properties. Next,

$$\begin{aligned} \mathfrak{J}_{\Psi}(F,Y,\bar{Y},t) &= \Psi\Big(f\left(tY + (1-t)\bar{Y}\right)\Big) - \tfrac{\alpha}{2}\Psi\Big(\left(tY + (1-t)\bar{Y}\right)^2\Big) \\ &= \Psi\Big(f\left(Y + (1-t)\bar{Y}\right)\Big) - \tfrac{\alpha}{2}\Big(t^2\Psi(Y^2) + t(1-t)\left(\Psi(Y\bar{Y}) + \Psi(\bar{Y}Y)\right) + (1-t)^2\Psi\left(\bar{Y}^2\right)\Big) \end{aligned}$$

Since $\Psi(\bar{Y}) = \bar{Y}$ and Ψ preserve the product of operators Y and $\bar{Y}$, it follows that

$$\begin{aligned} \mathfrak{J}_{\Psi}(F,Y,\bar{Y},t) &= \Psi\Big(f\left(Y + (1-t)\bar{Y}\right)\Big) - \tfrac{\alpha}{2}\Big(t^2\Psi(Y^2) + 2t(1-t)\bar{Y}^2 + (1-t)^2\bar{Y}^2\Big) \\ &= \Psi\Big(f\left(Y + (1-t)\bar{Y}\right)\Big) - t^2C_2 - \tfrac{\alpha}{2}\bar{Y}^2, \end{aligned} \tag{17}$$

where $C_2 = \frac{\alpha}{2}\left[\Psi\Big(Y^2\Big) - \bar{Y}^2\right]$. Similarly, we have

$$-\mathfrak{J}_{\Phi}(F,X,\bar{X},t) = -\Phi\Big(f\left(X + (1-t)\bar{X}\right)\Big) + t^2C_1 + \frac{\alpha}{2}\bar{X}^2, \tag{18}$$

where $C_1 = \frac{\alpha}{2}\left[\Psi\Big(X^2\Big) - \bar{X}^2\right]$.

Using (17) and (18) we obtain

$$\begin{aligned} & \mathfrak{L}_{\Phi,\Psi}(f,X,Y,t) = \Psi\Big(f\left(Y+(1-t)\bar{Y}\right)\Big) - \Phi\Big(f\left(X+(1-t)\bar{X}\right)\Big) \\ = \ & \mathfrak{J}_{\Psi}(F,Y,\bar{Y},t) + t^2 C_2 + \frac{\alpha}{2}\bar{Y}^2 - \mathfrak{J}_{\Phi}(F,X,\bar{X},t) - t^2 C_1 - \frac{\alpha}{2}\bar{X}^2 \\ = \ & \mathfrak{L}_{\Phi,\Psi}(F,X,Y,t) + t^2\left(C_2 - C_1\right) + \frac{\alpha}{2}\left(\bar{Y}^2 - \bar{X}^2\right). \end{aligned} \tag{19}$$

We define mappings: $\mathfrak{L}_1(C_1,C_2,t) = t^2\left(C_2 - C_1\right)$ and $\mathfrak{L}_2(X,Y,\alpha) = \frac{\alpha}{2}\left(\bar{Y}^2 - \bar{X}^2\right)$. Since $C_1 \le C_2$ then the mapping $\mathfrak{L}_1(C_1,C_2,\cdot)$ is convex and monotone increasing on $[0,1]$. Also, the constant mapping $\mathfrak{L}_2(X,Y,\alpha)$ has the same properties.

Taking into account properties of mappings $\mathfrak{L}_{\Phi,\Psi}, \mathfrak{L}_1, \mathfrak{L}_2$, it follows from (19) that $\mathfrak{L}_{\Phi,\Psi}(f,X,Y,\cdot)$ is convex and monotone increasing on $[0,1]$. □

Now, we define trivial Levinson's mapping $\bar{\mathfrak{L}}_{\Phi,\Psi}$ as a difference between the corresponding Jensen's mappings (12), i.e.

$$\bar{\mathfrak{L}}_{\Phi,\Psi}(f,X,Y,t) = \bar{\mathfrak{J}}_{\Psi}(f,Y,\Psi(Y),t) - \bar{\mathfrak{J}}_{\Phi}(f,X,\Phi(X),t), \tag{20}$$

where $\Phi,\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ are normalized positive linear mappings, $f \in \dot{\mathcal{K}}_1^c((a,b))$, $X,Y \in \mathcal{B}_h(H)$ are self-adjoint operators with spectra contained in $[m,M]$ and $[n,N]$, such that $a < m \le M \le c \le n \le N < b$ and $t \in [0,1]$. Therefore,

$$\bar{\mathfrak{L}}_{\Phi,\Psi}(f,X,Y,t) = t\Psi\Big(f(Y)\Big) + (1-t)f(\Psi(Y)) - \Big[t\Phi\Big(f(X)\Big) + (1-t)f(\Phi(X))\Big].$$

Next, we show that the mapping (20) has the properties convexity and monotonicity.

Theorem 3.2. *Let $\bar{\mathfrak{L}}_{\Phi,\Psi}$ be a mapping defined by* (20).
If $C_1 \le C_2$ holds (see (2)*), then $\bar{\mathfrak{L}}_{\Phi,\Psi}(f,X,Y,\cdot)$ is convex and monotone increasing on* $[0,1]$. *So,*

$$\inf_{t\in[0,1]} \bar{\mathfrak{L}}_{\Phi,\Psi}(f,X,Y,t) = f(\Psi(Y)) - f(\Phi(X)), \qquad \sup_{t\in[0,1]} \bar{\mathfrak{L}}_{\Phi,\Psi}(f,X,Y,t) = \Psi(f(Y)) - \Phi(f(X)).$$

Proof. The proof is similar to the one for Theorem 3.1. We give a short version. Using Lemma 2.8 and Theorem 2.9 we obtain that the mapping

$$\bar{\mathfrak{L}}_{\Phi,\Psi}(F,X,Y,\cdot) = \bar{\mathfrak{J}}_{\Psi}(F,Y,\bar{Y},\cdot) - \bar{\mathfrak{J}}_{\Phi}(F,X,\bar{X},\cdot)$$

is convex and monotone increasing on $[0,1]$. We have

$$\bar{\mathfrak{L}}_{\Phi,\Psi}(f,X,Y,t) = \bar{\mathfrak{L}}_{\Phi,\Psi}(F,X,Y,\cdot) + t\,\left(C_2 - C_1\right) + \frac{\alpha}{2}\left(\bar{Y}^2 - \bar{X}^2\right).$$

We define mapping: $\mathfrak{L}_3(C_1,C_2,t) = t\,\left(C_2 - C_1\right)$. Since $C_1 \le C_2$ then the mapping $\mathfrak{L}_3(C_1,C_2,\cdot)$ is convex and monotone increasing on $[0,1]$. Taking into account properties of mappings $\bar{\mathfrak{L}}_{\Phi,\Psi}, \mathfrak{L}_3, \mathfrak{L}_2$ (define in the proof of Theorem 3.1), it follows that $\bar{\mathfrak{L}}_{\Phi,\Psi}(f,X,Y,\cdot)$ is convex and monotone increasing on $[0,1]$. □

Remark 3.3.
1) Using the Davis-Choi-Jensen inequality (1), *we have that*

$$\bar{\mathfrak{L}}_{\Phi,\Psi}(F,X,Y,t) \ge \mathfrak{L}_{\Phi,\Psi}(F,X,Y,t).$$

Next, using the proofs of Theorem 3.1 and 3.2 we have that

$$\bar{\mathfrak{L}}_{\Phi,\Psi}(f,X,Y,t) - \mathfrak{L}_{\Phi,\Psi}(f,X,Y,t) \ge t(1-t)(C_2 - C_1) \ge 0 \tag{21}$$

if $C_2 \geq C_1$.

We remark that Levinson's inequality (3) *can we read as follows*

$$f\Big(\Psi(Y)\Big) - f\Big(\Phi(X)\Big) \leq \Psi\Big(f(Y)\Big) - \Phi\Big(f(X)\Big). \tag{22}$$

If all assumptions of Theorem 3.1 and 3.2 valid, then we have the following refinement of (22) *by using* (21)*:*

$$\begin{aligned} & f\Big(\Psi(Y)\Big) - f\Big(\Phi(X)\Big) \\ \leq \ & \Psi\Big(f\left(tY + (1-t)\Psi(Y)\right)\Big) - \Phi\Big(f\left(tX + (1-t)\Phi(X)\right)\Big) \\ \leq \ & t\Psi\Big(f(Y)\Big) + (1-t)f(\Psi(Y)) - t\Phi\Big(f(X)\Big) - (1-t)f(\Phi(X)) \\ \leq \ & \Psi\Big(f(Y)\Big) - \Phi\Big(f(X)\Big) \end{aligned}$$

for every $t \in [0,1]$ *and*

$$\begin{aligned} & f\Big(\Psi(Y)\Big) - f\Big(\Phi(X)\Big) \\ \leq \ & \Psi\Big(\int_0^1 f\left(tY + (1-t)\Psi(Y)\right) dt\Big) - \Phi\Big(\int_0^1 f\left(tX + (1-t)\Phi(X)\right) dt\Big) \\ \leq \ & \tfrac{1}{2}\left[\Psi\Big(f(Y)\Big) + f(\Psi(Y)) - \Phi\Big(f(X)\Big) - f(\Phi(X))\right] \\ \leq \ & \Psi\Big(f(Y)\Big) - \Phi\Big(f(X)\Big) \end{aligned}$$

2) Similar to Remark 2.10, we have versions of (14) *and* (20) *for series of operators.*

Let $\mathbf{X} = (X_1, \ldots, X_{k_1})$ *be an* k_1*−tuple and* $\mathbf{Y} = (Y_1, \ldots, Y_{k_2})$ *be a* k_2*−tuple of self-adjoint operators* $X_i, Y_j \in \mathcal{B}_h(\mathcal{H})$ *with spectra contained in* $[m, M]$ *and* $[n, N]$*, respectively, such that* $a < m \leq M \leq c \leq n \leq N < b$ *for some* $a, b, c \in \mathbb{R}$*. Let* $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_{k_1})$ *be a unital* k_1*−tuple which preserve the operator* $\bar{X} = \sum_{i=1}^{k_1} \Phi_i(X_i)$ *and the product of operators* X_i *and* $\bar{X}$*. Let* $\mathbf{\Psi} = (\Psi_1, \ldots, \Psi_{k_2})$ *be a unital* k_2*−tuple which preserve the operator* $\bar{Y} = \sum_{i=1}^{k_2} \Psi_i(Y_i)$ *and the product of operators* Y_i *and* $\bar{Y}$*.*

We define

$$\begin{aligned} \mathfrak{L}_{\mathbf{\Phi},\mathbf{\Psi}}(f, \mathbf{X}, \mathbf{Y}, t) = \ & \textstyle\sum_{i=1}^{k_2} \Psi_i\Big(f\left(tY_i + (1-t)\bar{Y}\right)\Big) - \sum_{i=1}^{k_1} \Phi\Big(f\left(tX_i + (1-t)\bar{X}\right)\Big), \\ \bar{\mathfrak{L}}_{\mathbf{\Phi},\mathbf{\Psi}}(f, \mathbf{X}, \mathbf{Y}, t) = \ & t\textstyle\sum_{i=1}^{k_2} \Psi\Big(f(Y_i)\Big) + (1-t)f(\bar{Y}) - t\sum_{i=1}^{k_1} \Phi\Big(f(X)\Big) - (1-t)f(\bar{X}). \end{aligned}$$

where $\bar{X} = \sum_{i=1}^{k_1} \Phi_i(X_i)$ *and* $\bar{Y} = \sum_{i=1}^{k_2} \Psi_i(Y_i)$*.*

If $f \in \dot{\mathcal{K}}_1^c((a,b))$ *and*

$$\alpha\left[\textstyle\sum_{i=1}^{k_1} \Phi_i\Big(X_i^2\Big) - \bar{X}^2\right] \leq \alpha\left[\sum_{j=1}^{k_2} \Psi_j\Big(Y_j^2\Big) - \bar{Y}^2\right],$$

then $\mathfrak{L}_{\mathbf{\Phi},\mathbf{\Psi}}(f, \mathbf{X}, \mathbf{Y}, \cdot)$ *and* $\bar{\mathfrak{L}}_{\mathbf{\Phi},\mathbf{\Psi}}(f, \mathbf{X}, \mathbf{Y}, \cdot)$ *are convex and monotone increasing on* $[0,1]$*. So,*

$$\inf_{t\in[0,1]} \mathfrak{L}_{\mathbf{\Phi},\mathbf{\Psi}}(f, \mathbf{X}, \mathbf{Y}, t) = \inf_{t\in[0,1]} \bar{\mathfrak{L}}_{\mathbf{\Phi},\mathbf{\Psi}}(f, \mathbf{X}, \mathbf{Y}, t) = f\left(\textstyle\sum_{i=1}^{k_2} \Psi_i(Y_i)\right) - f\left(\sum_{i=1}^{k_1} \Phi_i(X)\right),$$

$$\sup_{t\in[0,1]} \mathfrak{L}_{\mathbf{\Phi},\mathbf{\Psi}}(f, \mathbf{X}, \mathbf{Y}, t) = \sup_{t\in[0,1]} \bar{\mathfrak{L}}_{\mathbf{\Phi},\mathbf{\Psi}}(f, \mathbf{X}, \mathbf{Y}, t) = \textstyle\sum_{i=1}^{k_2} \Psi_i\left(f(Y_i)\right) - \sum_{i=1}^{k_1} \Phi_i\left(f(X)\right).$$

The interested reader can construct other Levinson's mappings using the remaining Jensen's mappings given in Section 2.

Finally, inspired by Dragomir's research given in Theorem B, we define the operator valued-functional related to Levinson's inequality as a difference between respective mappings $\bar{\mathfrak{L}}_{\Phi,\Psi}$ and $\mathfrak{L}_{\Phi,\Psi}$, i.e.

$$\begin{aligned} & \Delta_{\Phi,\Psi}(f, A, B; C, D, t) \\ = \ & (1-t)\Psi\Big(f(C)\Big) + t\Psi\Big(f(D)\Big) - \Psi\Big(f\left((1-t)C + tD\right)\Big) \\ - \ & \Big[(1-t)\Phi\Big(f(A)\Big) + t\Phi\Big(f(B)\Big) - \Phi\Big(f\left((1-t)A + tB\right)\Big)\Big] \end{aligned}$$

where $\Phi, \Psi : \mathcal{B}(H) \to \mathcal{B}(H)$ are normalized positive linear mappings, $A, B, C, D \in \mathcal{B}_h(H)$ are self-adjoint operators with spectra of A, C contained in $[m, M]$ and spectra of B, D contained in $[n, N]$, such that $a < m \leq M \leq c \leq n \leq N < b$, $f \in \overset{\bullet}{\mathcal{K}}{}_1^c((a,b))$ and $t \in [0,1]$.

For the sake of convenience let us define operator functions:

$$\delta_\Phi(A, B) = \Phi\Big((A - B)^2\Big) \tag{23}$$

and

$$\Delta_\Phi(f, A, B, t) = (1-t)\Phi\Big(f(A)\Big) + t\Phi\Big(f(B)\Big) - \Phi\Big(f\left((1-t)A + tB\right)\Big).$$

So, we can read

$$\Delta_{\Phi,\Psi}(f, A, B; C, D, t) = \Delta_\Psi(f, C, D, t) - \Delta_\Phi(f, A, B, t). \tag{24}$$

Now, we show an operator quasi-linearity property for the mapping (24), as operator superadditive and operator monotone as a function of intervals.

Theorem 3.4. *Let $\Delta_{\Phi,\Psi}$ be a mapping defined by* (24).

If $\alpha\, \delta_\Psi(C, D) \geq \alpha\, \delta_\Phi(A, B)$, then for every $A_1 = (1-s)A + sB \in [A, B]$ and $C_1 = (1-s)C + sD \in [C, D]$, we have

$$0 \leq \Delta_{\Phi,\Psi}(f, A, A_1; C, C_1, t) + \Delta_{\Phi,\Psi}(f, A_1, B; C_1, D, t) \leq \Delta_{\Phi,\Psi}(f, A, B; C, D, t). \tag{25}$$

Moreover, if $B_1 = (1-r)A + rB$ and $D_1 = (1-r)C + rD$ so that $[A_1, B_1] \subset [A, B]$ and $[C_1, D_1] \subset [C, D]$, then

$$0 \leq \Delta_{\Phi,\Psi}(f, A_1, B_1; C_1, D_1, t) \leq \Delta_{\Phi,\Psi}(f, A, B; C, D, t). \tag{26}$$

Proof. (i) Let the constant α be as in Definition 1.1, such that $F(s) = f(s) - \frac{\alpha}{2}s^2$ is an operator concave function on $[a, c]$ and operator convex on $[c, b]$. Then $F\left((1-t)C + tD\right) \leq (1-t)F(C) + tF(D)$. Since Ψ is positive linear mapping it follows

$$\begin{aligned} 0 \leq &\quad \Delta_\Psi(F, C, D, t) \\ = &\quad \Delta_\Psi(f, C, D, t) - \frac{\alpha}{2}\Psi\left((1-t)C^2 + tD^2 - ((1-t)C + tD)^2\right) \\ = &\quad \Delta_\Psi(f, C, D, t) - \frac{\alpha}{2}t(1-t)\Psi\Big((C - D)^2\Big). \end{aligned}$$

Also,

$$0 \leq -\Delta_\Phi(F, A, B, t) = -\Delta_\Phi(f, A, B, t) + \frac{\alpha}{2}t(1-t)\Phi\Big((A - B)^2\Big).$$

Using that

$$\Delta_{\Phi,\Psi}(F, A, B; C, D, t) = \Delta_\Psi(F, C, D, t) - \Delta_\Phi(F, A, B, t) \geq 0$$

and $\alpha\, \delta_\Psi(C, D) \geq \alpha\, \delta_\Phi(A, B)$, we obtain positive sign of $\Delta_{\Phi,\Psi}(f, A, B; C, D, t)$ for every $t \in [0,1]$, since

$$\Delta_{\Phi,\Psi}(f, A, B; C, D, t) = \Delta_{\Phi,\Psi}(F, A, B; C, D, t) + t(1-t)\frac{\alpha}{2}\left(\delta_\Psi(C, D) - \delta_\Phi(A, B)\right) \geq 0. \tag{27}$$

(ii) Applying positive linear mapping Ψ to (4), we obtain

$$\Delta_\Psi(F, C, C_1, t) + \Delta_\Psi(F, C_1, D, t) \leq \Delta_\Psi(F, C, D, t) \tag{28}$$

for every $C_1 \in [C, D]$. Since $C_1 = (1-s)C + sD$ for some $s \in (0, 1)$, then

$$\begin{aligned}\delta_\Psi(C, C_1) &= \Psi\Big((C-(1-s)C-sD)^2\Big) = s^2\delta_\Psi(C, D),\\ \delta_\Psi(C_1, D) &= \Psi\Big(((1-s)C+sD-D)^2\Big) = (1-s)^2\delta_\Psi(C, D).\end{aligned}$$

So (28) give

$$\begin{aligned}\Delta_\Psi(f, C, C_1, t) - \tfrac{\alpha}{2}t(1-t)s^2\delta_\Psi(C, D) + \Delta_\Psi(f, C_1, D, t) - \tfrac{\alpha}{2}t(1-t)(1-s)^2\delta_\Psi(C, D)\\ \leq \Delta_\Psi(f, C, D, t) - \tfrac{\alpha}{2}t(1-t)\delta_\Psi(C, D).\end{aligned}$$

It follows

$$\Delta_\Psi(f, C, C_1, t) + \Delta_\Psi(f, C_1, D, t) + s(1-s)t(1-t)\cdot\alpha\delta_\Psi(C, D) \leq \Delta_\Psi(f, C, D, t). \tag{29}$$

Similarly, we obtain

$$-\Delta_\Phi(f, A, A_1, t) - \Delta_\Phi(f, A_1, B, t) - s(1-s)t(1-t)\cdot\alpha\delta_\Phi(A, B) \leq -\Delta_\Phi(f, A, B, t). \tag{30}$$

Summing (29) and (30), applying (27) and using that

$$s(1-s)t(1-t)\ \alpha\left(\delta_\Psi(C, D) - \delta_\Phi(A, B)\right) \geq 0,$$

we obtain

$$\begin{aligned}0 \leq\ & \Delta_{\Phi,\Psi}(f, A, A_1; C, C_1, t) + \Delta_{\Phi,\Psi}(f, A_1, B; C_1, D, t)\\ \leq\ & \Delta_{\Phi,\Psi}(f, A, A_1; C, C_1, t) + \Delta_{\Phi,\Psi}(f, A_1, B; C_1, D, t) + s(1-s)t(1-t)\ \alpha\left(\delta_\Psi(C, D) - \delta_\Phi(A, B)\right)\\ \leq\ & \Delta_{\Phi,\Psi}(f, A, B; C, D, t),\end{aligned}$$

which give the desired inequality (25).
(iii) Let $A_1 = (1-s)C+sD$, $B_1 = (1-r)A+rB$, $C_1 = (1-s)C+sD$ and $D_1 = (1-r)C+rD$ so that $[A_1, B_1] \subset [A, B]$ and $[C_1, D_1] \subset [C, D]$. Applying positive linear mapping Ψ to (5), we obtain

$$\begin{aligned}\Delta_\Psi(f, C_1, D_1, t) - \tfrac{\alpha}{2}t(1-t)\delta_\Psi(C_1, D_1) &\leq \Delta_\Psi(f, C, D, t) - \tfrac{\alpha}{2}t(1-t)\delta_\Psi(C, D)\\ -\Delta_\Phi(f, A_1, B_1, t) + \tfrac{\alpha}{2}t(1-t)\delta_\Phi(A_1, B_1) &\leq -\Delta_\Phi(f, A, B, t) + \tfrac{\alpha}{2}t(1-t)\delta_\Phi(A, B)\end{aligned}$$

Summing the above inequalities and applying (27), we obtain

$$\begin{aligned}0 \leq\ & \Delta_{\Phi,\Psi}(f, A_1, B_1; C_1, D_1, t) - \tfrac{\alpha}{2}t(1-t)\left(\delta_\Psi(C_1, D_1) - \delta_\Phi(A_1, B_1)\right)\\ \leq\ & \Delta_{\Phi,\Psi}(f, A, B; C, D, t) - \tfrac{\alpha}{2}t(1-t)\left(\delta_\Psi(C, D) - \delta_\Phi(A, B)\right),\end{aligned}$$

Since

$$\delta_\Psi(C_1, D_1) - \delta_\Phi(A_1, B_1) = (r-s)^2\left(\delta_\Psi(C, D) - \delta_\Phi(A, B)\right)$$

it follows that for every $r, s \in [0, 1]$

$$\begin{aligned}0 \leq\ & \Delta_{\Phi,\Psi}(f, A_1, B_1; C_1, D_1, t)\\ \leq\ & \Delta_{\Phi,\Psi}(f, A, B; C, D, t) - \Big(1-(r-s)^2\Big)t(1-t)\tfrac{\alpha}{2}\left(\delta_\Psi(C, D) - \delta_\Phi(A, B)\right)\\ \leq\ & \Delta_{\Phi,\Psi}(f, A, B; C, D, t)\end{aligned}$$

hods. Then the desired inequality (26) holds. □

Remark 3.5. *From the proof of Theorem 3.4 it obviously follows that the inequality* (25) *holds if the condition:* $A_1 = (1-s)A+sB \in [A,B]$ *and* $C_1 = (1-s)C+sD \in [C,D]$ *is replaced by the weaker condition:* $A_1 = (1-s_1)A+s_1B \in [A,B]$ *and* $C_1 = (1-s_2)C+s_2D \in [C,D]$ *such that* $s_2(1-s_2) \geq s_1(1-s_1)$.
Similarly, the inequality (26) *holds if the condition:* $B_1 = (1-r)A+rB$ *and* $D_1 = (1-r)C+rD$ *is replaced by the weaker condition:* $B_1 = (1-r_1)A+r_1B \in [A,B]$ *and* $D_1 = (1-r_2)C+r_2D \in [C,D]$ *such that* $r_2(1-r_2) \geq r_1(1-r_1)$.

Applying Theorem 3.4 we are able to state the following bounds.

Corollary 3.6. *Let* $\Phi, \Psi : \mathcal{B}(H) \to \mathcal{B}(H)$ *are normalized positive linear mappings,* $A, C \in \mathcal{B}_h(H)$ *are self-adjoint operators with spectra of* A *and* C *contained in* $[m,M]$ *and* $[n,N]$*, respectively, such that* $a < m \leq M \leq c \leq n \leq N < b$, $\bar{A} = \Phi(A)$ *and* $\bar{C} = \Psi(C)$.

If $\alpha\, \delta_\Psi(C,\bar{C}) \geq \alpha\, \delta_\Phi(A,\bar{A})$, $B = (1-s)A + s\bar{A}$ *and* $D = (1-s)C + s\bar{C}$*, then*

$$\begin{aligned}
&\inf_{\substack{B\in[A,\bar{A}]\\ D\in[C,\bar{C}]}} \Big\{\Psi\big(f\left((1-t)C+tD\right)\big) + \Psi\big(f\left((1-t)D+t\bar{C}\right)\big) - \Psi\big(f\left(D\right)\big) \\
&\qquad -\Phi\big(f\left((1-t)A+tB\right)\big) - \Phi\big(f\left((1-t)B+t\bar{A}\right)\big) + \Phi\big(f\left(B\right)\big)\Big\} \qquad (31)\\
=\ & \Psi\big(f\left((1-t)C+t\bar{C}\right)\big) - \Phi\big(f\left((1-t)A+t\bar{A}\right)\big)
\end{aligned}$$

holds for every $f \in \overset{\bullet}{\mathcal{K}}{}_1^c((a,b))$ *and* $t \in [0,1]$.
Moreover, if $B_1 = (1-s)A + s\bar{A}$, $B_2 = (1-r)A + rB$, $D_1 = (1-s)C + s\bar{C}$ *and* $D_2 = (1-r)C + rD$ *for* $r,s \in [0,1]$*, then*

$$\begin{aligned}
&\sup_{\substack{B_1,B_2\in[A,\bar{A}]\\ D_1,D_2\in[C,\bar{C}]}} \Big\{(1-t)\Psi\big(f\left(D_1\right)\big) + t\Psi\big(f\left(D_2\right)\big) - \Psi\big(f\left((1-t)D_1+tD_2\right)\big) \\
&\qquad -(1-t)\Phi\big(f\left(B_1\right)\big) - t\Phi\big(f\left(B_2\right)\big) + \Phi\big(f\left((1-t)B_1+tB_2\right)\big)\Big\} \qquad (32)\\
=\ & (1-t)\Psi\big(f\left(C\right)\big) + t\Psi\big(f(\bar{C})\big) - \Psi\big(f\left((1-t)C+t\bar{C}\right)\big)\\
-\ & (1-t)\Phi\big(f\left(A\right)\big) - t\Phi\big(f(\bar{A})\big) + \Phi\big(f\left((1-t)A+t\bar{A}\right)\big).
\end{aligned}$$

If Φ *and* Ψ *preserve the operator* $f(\bar{A})$ *and* $f(\bar{C})$*, respectively, then supremum in* (32) *is equal to* $\bar{\mathfrak{L}}_{\Phi,\Psi}(f,A,C,t) - \mathfrak{L}_{\Phi,\Psi}(f,A,C,t)$*, where* $\bar{\mathfrak{L}}_{\Phi,\Psi}$ *and* $\mathfrak{L}_{\Phi,\Psi}$ *are Levinson's mapping defined by* (14) *and* (20).

Proof. Replacing B by $\bar{A}$, A_1 by B and D by $\bar{C}$, C_1 by D in (25) and using (24) we obtain

$$\Delta_\Psi(f,C,D,t) - \Delta_\Phi(f,A,B,t) + \Delta_\Psi(f,D,\bar{C},t) - \Delta_\Phi(f,B,\bar{A},t) \leq \Delta_\Psi(f,C,\bar{C},t) - \Delta_\Phi(f,A,\bar{A},t). \qquad (33)$$

We have

$$\begin{aligned}
& \Delta_\Psi(f,C,D,t) + \Delta_\Psi(f,D,\bar{C},t) - \Delta_\Psi(f,C,\bar{C},t)\\
=\ & (1-t)\Psi\big(f(C)\big) + t\Psi\big(f(D)\big) - \Psi\big(f\left((1-t)C+tD\right)\big) + (1-t)\Psi\big(f(D)\big) + t\Psi\big(f(\bar{C})\big)\\
-\ & \Psi\big(f\left((1-t)D+t\bar{C}\right)\big) - (1-t)\Psi\big(f(C)\big) - t\Psi\big(f(\bar{C})\big) + \Psi\big(f\left((1-t)C+t\bar{C}\right)\big)\\
=\ & -\Psi\big(f\left((1-t)C+tD\right)\big) - \Psi\big(f\left((1-t)D+t\bar{C}\right)\big) + \Psi\big(f(D)\big) + \Psi\big(f\left((1-t)C+t\bar{C}\right)\big)
\end{aligned}$$

and, similarly,

$$\begin{aligned}
& -\Delta_\Phi(f,A,B,t) - \Delta_\Phi(f,B,\bar{A},t) + \Delta_\Phi(f,A,\bar{A},t)\\
=\ & \Phi\big(f\left((1-t)A+tB\right)\big) + \Phi\big(f\left((1-t)B+t\bar{A}\right)\big) - \Phi\big(f(B)\big) - \Phi\big(f\left((1-t)A+t\bar{A}\right)\big).
\end{aligned}$$

So the inequality (33) becomes

$$\begin{aligned} & \Psi\Big(f((1-t)C+tD)\Big)+\Psi\Big(f((1-t)D+t\bar{C})\Big)-\Psi\Big(f(D)\Big) \\ - & \Phi\Big(f((1-t)A+tB)\Big)-\Phi\Big(f((1-t)B+t\bar{A})\Big)+\Phi\Big(f(B)\Big) \\ \geq & \Psi\Big(f((1-t)C+t\bar{C})\Big)-\Phi\Big(f((1-t)A+t\bar{A})\Big). \end{aligned} \tag{34}$$

Since the equality case in (34) is realized for either s=0 ($B=A$ and $D=C$) or $s=1$ ($B=\bar{A}$ and $D=\bar{C}$), we get the desired bound (31).

The bound (32) is obvious by the monotonicity property of the functional $\Delta_{\Phi,\Psi}(f,\cdot,\cdot;\cdot,\cdot,t)$ as a function of intervals $[A,\bar{A}]$ and $[C,\bar{C}]$, respectively. □

We can also consider the following functional

$$\begin{aligned} \Theta_{\Phi,\Psi}(f,A,B;C,D) &= \Psi\Big(f(C)\Big)+\Psi\Big(f(D)\Big)-\Psi\Big(\int_0^1 f((1-t)C+tD)\,dt\Big) \\ &- \Phi\Big(f(A)\Big)-\Phi\Big(f(B)\Big)+\Phi\Big(\int_0^1 f((1-t)A+tB)\,dt\Big) \end{aligned}$$

where $\Phi,\Psi:\mathcal{B}(H)\to\mathcal{B}(H)$ are normalized positive linear mappings, $A,B,C,D\in\mathcal{B}_h(H)$ are self-adjoint operators with spectra of A,C contained in $[m,M]$ and spectra of B,D contained in $[n,N]$, such that $a<m\leq M\leq c\leq n\leq N<b$ and $f\in\dot{\mathcal{K}}_1^c((a,b))$.

We observe that

$$\begin{aligned} & \Theta_{\Phi,\Psi}(f,A,B;C,D) \\ = & \int_0^1 \Delta_{\Phi,\Psi}(f,A,B;C,D,t)dt=\int_0^1 \Delta_{\Phi,\Psi}(f,A,B;C,D,1-t)dt \ \geq \ 0 \end{aligned}$$

holds. Utilising this representation, we obtain the following result.

Corollary 3.7. *Let $\Phi,\Psi:\mathcal{B}(H)\to\mathcal{B}(H)$ are normalized positive linear mappings, $A,C\in\mathcal{B}_h(H)$ are self-adjoint operators with spectra of A and C contained in $[m,M]$ and $[n,N]$, respectively, such that $a<m\leq M\leq c\leq n\leq N<b$, $\bar{A}:=\Phi(A)$ and $\bar{C}:=\Psi(C)$.*

If $\alpha\,\delta_\Psi(C,\bar{C})\geq\alpha\,\delta_\Phi(A,\bar{A})$, $B=(1-s)A+s\bar{A}$ and $D=(1-s)C+s\bar{C}$, then

$$\begin{aligned} & \inf_{\substack{B\in[A,\bar{A}]\\ D\in[C,\bar{C}]}} \Big\{\Psi\Big(\int_0^1 f((1-t)C+tD)\,dt\Big)+\Psi\Big(\int_0^1 f((1-t)D+t\bar{C})\,dt\Big)-\Psi\Big(f(D)\Big) \\ & \qquad -\Phi\Big(\int_0^1 f((1-t)A+tB)\,dt\Big)-\Phi\Big(\int_0^1 f((1-t)B+t\bar{A})\,dt\Big)+\Phi\Big(f(B)\Big)\Big\} \\ = \ & \Psi\Big(\int_0^1 f((1-t)C+t\bar{C})\,dt\Big)-\Phi\Big(\int_0^1 f((1-t)A+t\bar{A})\,dt\Big) \end{aligned}$$

holds for every $f\in\dot{\mathcal{K}}_1^c((a,b))$.

Moreover, if $B_1=(1-s)A+s\bar{A}$, $B_2=(1-r)A+rB$, $D_1=(1-s)C+s\bar{C}$ and $D_2=(1-r)C+rD$ for $r,s\in[0,1]$, then

$$\begin{aligned} & \sup_{\substack{B_1,B_2\in[A,\bar{A}]\\ D_1,D_2\in[C,\bar{C}]}} \Big\{\Psi\Big(f(D_1)\Big)+\Psi\Big(f(D_2)\Big)-\Psi\Big(\int_0^1 f((1-t)D_1+tD_2)\,dt\Big) \\ & \qquad -\Phi\Big(f(B_1)\Big)-\Phi\Big(f(B_2)\Big)+\Phi\Big(\int_0^1 f((1-t)B_1+tB_2)\,dt\Big)\Big\} \\ = \ & \Psi\Big(f(C)\Big)+\Psi\Big(f(\bar{C})\Big)-\Psi\Big(\int_0^1 f((1-t)C+t\bar{C})\,dt\Big) \\ - \ & \Phi\Big(f(A)\Big)-\Phi\Big(f(\bar{A})\Big)+\Phi\Big(\int_0^1 f((1-t)A+t\bar{A})\,dt\Big). \end{aligned}$$

Example 3.8. *Putting the power function $f(t) = t^r$ for $t \in (0, c]$, $f(t) = d \cdot t^s$ for $t \in [c, \infty)$, where $c > 0$ and $k = c^{r-s}$ in the above, we can get perhaps interesting results.*

(1) If $r \in [0, 1]$, $s \in [-1, 0] \cup [1, 2]$, we have mappings

$$\begin{aligned}\Delta_{\Phi,\Psi}(r,s,A,B;C,D,t) &= d\left\{(1-t)\Psi\Big(C^s\Big)+t\Psi\Big(D^s\Big)-\Psi\Big(((1-t)C+tD)^s\Big)\right\}\\ &- (1-t)\Phi\Big(A^r\Big)-t\Phi\Big(B^r\Big)+\Phi\Big(((1-t)A+tB)^r\Big),\\ \Theta_{\Phi,\Psi}(r,s,A,B;C,D) &= d\left\{\Psi\Big(C^s\Big)+\Psi\Big(D^s\Big)-\Psi\Big(\int_0^1((1-t)C+tD)^s\,dt\Big)\right\}\\ &- \Phi\Big(A^r\Big)-\Phi\Big(B^r\Big)+\Phi\Big(\int_0^1((1-t)A+tB)^r\,dt\Big).\end{aligned}$$

If $\alpha\,\delta_\Psi(C,\bar{C}) \geq \alpha\,\delta_\Phi(A,\bar{A})$, then it follows from Theorem (3.4) that these mappings are operator superadditive and operator monotone as functions of intervals. Also, we can state the proper bounds for the power function by applying Corollaries 3.6-3.7.

(2) It is easy to prove that the above results hold for series of operators. For example, let $\mathbf{A} = \mathrm{diag}(A_1, \dots, A_{k_1})$, $\mathbf{B} = \mathrm{diag}(B_1, \dots, B_{k_1})$, $\mathbf{C} = \mathrm{diag}(C_1, \dots, C_{k_2})$, $\mathbf{D} = \mathrm{diag}(D_1, \dots, D_{k_2})$ such that $A_i, B_i, C_j, D_J \in \mathcal{B}_h(\mathcal{H})$ with spectra of A_i, C_j and B_i, D_j contained in $[m, M]$ and $[n, N]$, respectively, and $a < m \leq M \leq c \leq n \leq N < b$. Then $\mathbf{A}, \mathbf{B} \in \mathcal{B}_h(\underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{k_1})$ and $\mathbf{C}, \mathbf{D} \in \mathcal{B}_h(\underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{k_2})$, with spectra of $\mathbf{A}, \mathbf{C}$ and $\mathbf{B}, \mathbf{D}$ contained in $[m, M]$ and $[n, N]$, respectively. Also, let be $\mathbf{\Phi}(\mathrm{diag}(X_1, \dots, X_{k_1})) = \sum_{i=1}^{k_1} p_i X_i$ and $\mathbf{\Psi}(\mathrm{diag}(Y_1, \dots, Y_{k_2})) = \sum_{i=1}^{k_2} q_i Y_i$, where $\mathbf{p} = (p_1, \dots, p_{k_1})$ and $\mathbf{q} = (q_1, \dots, q_{k_w})$ are k_1 and k_2-tuples of positive scalars such that $\sum_{i=1}^{n} p_i = 1$ and $\sum_{j=1}^{k} q_j = 1$. Then we obtain that $\mathbf{\Phi} : \mathcal{B}(\underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{k_1}) \to \mathcal{B}(\mathcal{H})$ and $\mathbf{\Psi} : \mathcal{B}(\underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{k_2}) \to \mathcal{B}(\mathcal{H})$ are normalized positive linear mappings.

Then, putting these mappings and f in 1) we obtain that the mapping

$$\begin{aligned}\Delta_{\mathbf{p},\mathbf{q}}(r,s,\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D},t) &= d\left\{(1-t)\sum_{i=1}^{k_2} q_i C_i^s + t\sum_{i=1}^{k_2} q_i D_i^s - \sum_{i=1}^{k_2} q_i\left((1-t)C_i+tD_i\right)^s\right\}\\ &- (1-t)\sum_{i=1}^{k_1} p_i A_i^r + t\sum_{i=1}^{k_1} p_i B_i^r - \sum_{i=1}^{k_1} p_i\left((1-t)A_i+tB_i\right)^r\end{aligned}$$

is operator superadditive and operator monotone as functions of intervals if $r \in [0, 1]$, $s \in [-1, 0] \cup [1, 2]$, $d = c^{r-s}$ and

$$\alpha\Big[\sum_{i=1}^{k_2} q_i C_i^2 - \bar{C}^2\Big] \geq \alpha\Big[\sum_{i=1}^{k_1} p_i A_i^2 - \bar{A}^2\Big], \textit{ where } \bar{C} := \sum_{i=1}^{k_2} q_i C_i \textit{ and } \bar{A} := \sum_{i=1}^{k_1} p_i A_i.$$

Moreover, we define $\bar{\mathbf{A}} = \mathrm{diag}(\underbrace{\bar{A}, \dots, \bar{A}}_{k_1})$ and

$$[\mathbf{A}, \bar{\mathbf{A}}] = \{\mathbf{B} = (B_1, \dots, B_{k_1}) \,|\, B_i = (1-t)A_i + t\bar{A},\ t \in [0,1], i = 1, \dots, k_1\}$$

and by analogy $[\mathbf{C}, \bar{\mathbf{C}}]$. We have the following bounds

$$\begin{aligned}&\inf_{\substack{\mathbf{B}\in[\mathbf{A},\bar{\mathbf{A}}]\\ \mathbf{D}\in[\mathbf{C},\bar{\mathbf{C}}]}} \left\{d\sum_{i=1}^{k_2} q_i\left((1-t)C_i+tD_i\right)^s + d\sum_{i=1}^{k_2} q_i\left((1-t)D_i+t\bar{C}\right)^s - d\sum_{i=1}^{k_2} q_i D_i^s\right.\\ &\qquad\left. - \sum_{i=1}^{k_1} p_i\left((1-t)A_i+tB_i\right)^r - \sum_{i=1}^{k_1} p_i\left((1-t)B_i+t\bar{A}\right)^r + \sum_{i=1}^{k_1} p_i B_i^r\right\}\\ &\qquad = d\sum_{i=1}^{k_2} q_i\left((1-t)C_i+t\bar{C}\right)^s - \sum_{i=1}^{k_1} p_i\left((1-t)A_i+t\bar{A}\right)^r\end{aligned}$$

and similarly, we can give results for supremum. Or, we can observe the mapping

$$\Theta_{\mathbf{p},\mathbf{q}}(r,s,\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}) = \int_0^1 \Delta_{\mathbf{p},\mathbf{q}}(r,s,\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D},t)\,dt$$

with similar properties.

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