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Iterative Hermitian R-Conjugate Solutions to General Coupled Sylvester Matrix Equations

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Abstract. For a given symmetric orthogonal matrix R, i.e., $R^T = R$, $R^2 = I$, a matrix $A \in \mathbb{C}^{n\times n}$ is termed Hermitian R-conjugate matrix if $A = A^H$, $RAR = \overline{A}$. In this paper, an iterative method is constructed for finding the Hermitian R-conjugate solutions of general coupled Sylvester matrix equations. Convergence analysis shows that when the considered matrix equations have a unique solution group then the proposed method is always convergent for any initial Hermitian R-conjugate matrix group under a loose restriction on the convergent factor. Furthermore, the optimal convergent factor is derived. Finally, two numerical examples are given to demonstrate the theoretical results and effectiveness.

1. Introduction

In this paper, the following notations and definitions are used. Let $\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$ be the set of all $m \times n$ real matrices and complex matrices, respectively. For a given matrix A, the notations $\mathrm{tr}(A)$, $\overline{A}$, A^T, A^H, $\lambda_{max}(A)$, $\lambda_{min}(A)$, $cond(A)$, $\rho(A)$ and $\|A\| = \sqrt{\mathrm{tr}(A^H A)}$ denote its trace, conjugate, transpose, conjugate transpose, maximal eigenvalue, minimal eigenvalue, condition number, spectral radius and Frobenius norm, respectively. For two matrices $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times n}$, $A \otimes B$ is their Kronecker product. The symbol $vec(\cdot)$ is a vector formed by the columns of given matrix $A = (a_1, a_2, \ldots, a_n)$, i.e., $vec(A) = (a_1^T, a_2^T, \ldots, a_n^T)^T$.

Definition 1.1. *For a given symmetric orthogonal matrix $R \in \mathbb{R}^{n\times n}$, i.e., $R^T = R$, $R^2 = I_n$, a matrix $A \in \mathbb{C}^{n\times n}$ is termed Hermitian R-conjugate matrix if $A = A^H$, $RAR = \overline{A}$. The set of all $n \times n$ Hermitian R-conjugate matrices is denoted by $\mathbb{HRC}^{n\times n}$.*

Centro-Hermitian matrix and related matrices, such as Hermitian Toeplitz matrix, generalized centro-Hermitian matrix, and so on, have been widely investigated, which naturally appear in digital signal processing and other areas [1–4]. As an extension of centro-Hermitian matrix and its related matrices, (R, S)-conjugate matrix was defined by Trench [5] as: a matrix $A \in \mathbb{C}^{m\times n}$ is (R, S)-conjugate if $RAS = \overline{A}$, where R, S are two given symmetric orthogonal matrices. In [6], Chang, Wang and Song gave the expression

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of the (R, S)-conjugate solution of $AX = C$, $XB = D$ by matrix decompositions. In particular, if a (R, S)-conjugate matrix is Hermitian, we also require that $R = S$. In this sense, the Hermitian R-conjugate matrix is in fact a Hermitian generalized centro-Hermitian matrix. In [7], Dong, Wang and Zhang discussed the Hermitian R-conjugate solution of system $AX = C$, $XB = D$ by matrix decompositions. Chang, Duan and Wang [8] derived the expression of the solution to the Hermitian R-conjugate generalized procrustes problem by matrix decompositions.

Consider the following general coupled Sylvester matrix equations

$$\sum_{j=1}^{p} A_{ij}X_jB_{ij} = C_i, \quad i = 1, 2, \ldots, p, \tag{1}$$

where $A_{ij} \in \mathbb{C}^{m\times n}$, $B_{ij} \in \mathbb{C}^{n\times s}$ and $C_i \in \mathbb{C}^{m\times s}$ are given matrices, $X_j \in \mathbb{C}^{n\times n}$ are the unknown matrices to be determined. The coupled matrix equations have wide applications in many areas. For example, in stability analysis of linear jump systems with Markovian transitions [9, 10], the coupled Lyapunov matrix equations are required to be solved. For stability analysis of control system and robust control [11], we need to solve the coupled Sylvester matrix equations $AX + YB = C$ and $DX + YE = F$ where A, B, C, D, E and F are known. In addition, one naturally encounters the coupled Sylvester matrix equations when dealing with the problems of reordering eigenvalues of regular matrix pairs [12], or computing an additive decomposition of a generalized transform matrix equations [13]. Owing to their important applications, many iterative methods have been proposed to solve the coupled matrix equations.

By extending the idea of the CGNE method, some finite iterative algorithms have been proposed to solve the different kinds of coupled matrix equations over reflexive, generalized bisymmetric, generalized centro-symmetric, (P, Q)-reflexive and common matrices, for more details, see [14–24] and the references therein. The gradient-based iterative (GI) algorithm is another kind of effective algorithm for solving the coupled matrix equations, which was first proposed by Ding and Chen [25–27] with using the hierarchical identification principle. In [28, 29], the optimal parameter of the GI method was derived for computing the solutions and the weighted least squares solutions of the general coupled matrix equations. In order to improve the convergent rate of the GI method, two variants of the GI method were proposed to solve the Sylvester equations in [30, 31]. Meanwhile, the GI method was extended to solve the common solutions, the generalized centro-symmetric solutions, generalized bisymmetric solutions, reflexive and anti-reflexive solutions of some coupled matrix equations, see [32–36] for further details on this topic. However, the optimal convergent factors of these extended GI methods were not given in computing such constraint solutions.

In addition, some other iterative methods were proposed to solve the coupled matrix equations. In [37, 38], some Krylov subspace methods were presented to solve the general coupled matrix equations. Li and Huang [39] presented a matrix LSQR method for computing the constrained solutions of the generalized coupled Sylvester matrix equations. By developing the Richardson iterative method, Salkuyeh and Beik [40] obtained the solutions of the general coupled matrix equations. In [41], Beik used a one-dimensional projection technique to improve the convergent rate of the GI method for solving the general coupled Sylvester matrix equations over reflexive matrices. More recently, Hajarian [42–45] solved some coupled Sylvester matrix equations by using the matrix forms of the CGS method, Bi-CGSTAB method, GPBiCG algorithms, BiCOR method and CORS method, respectively. However, by the previous iterative methods, we cannot obtain the Hermitian R-conjugate solutions of the matrix equations (1). Therefore, we are interested in constructing a new iterative method to solve the matrix equations (1) over Hermitian R-conjugate matrices.

The remainder of this paper is organized as follows. In Section 2, an iterative method is proposed to solve the matrix equations (1) over Hermitian R-conjugate matrices. The convergence of the proposed method is proved and the optimal convergent factor is derived. In Section 3, two numerical examples are offered to illustrate the efficiency of the proposed method. Finally, we end the paper with a brief conclusion in Section 4.

2. Main Results

In this section, we first derive the solvability conditions of the matrix equations (1) over Hermitian R-conjugate matrices.

Lemma 2.1. *A necessary and sufficient condition of the consistency of the matrix equations (1) over Hermitian R-conjugate matrices is that the following matrix equations*

$$\begin{cases} \sum_{j=1}^{p} A_{ij}X_jB_{ij} = C_i, \\ \sum_{j=1}^{p} B_{ij}^H X_j A_{ij}^H = C_i^H, \\ \sum_{j=1}^{p} \overline{A}_{ij}RX_jR\overline{B}_{ij} = \overline{C}_i, \\ \sum_{j=1}^{p} \overline{B}_{ij}^H RX_jR\overline{A}_{ij}^H = \overline{C}_i^H, \quad i = 1, 2, \ldots, p, \end{cases} \tag{2}$$

are consistent.

Proof. If the matrix equations (1) have solutions $X_j \in \mathbb{HRC}^{n\times n}$, $j = 1, 2, \ldots, p$, i.e., $X_j = X_j^H, RX_jR = \overline{X}_j$, it is easy to get that X_j are also the solutions of the matrix equations (2). Conversely, if the matrix equations (2) have solutions $X_j \in \mathbb{C}^{n\times n}$, let $X_j^* = \frac{X_j + X_j^H + R\overline{(X_j + X_j^H)}R}{4}$, then $X_j^* \in \mathbb{HRC}^{n\times n}$, and

$$\begin{aligned} \sum_{j=1}^{p} A_{ij}X_j^*B_{ij} &= \frac{1}{4}\sum_{j=1}^{p} A_{ij}X_jB_{ij} + \frac{1}{4}\sum_{j=1}^{p} A_{ij}X_j^HB_{ij} + \frac{1}{4}\sum_{j=1}^{p} A_{ij}R\overline{X}_jRB_{ij} + \frac{1}{4}\sum_{j=1}^{p} A_{ij}R\overline{X}_j^HRB_{ij} \\ &= \frac{1}{4}(C_i + C_i + C_i + C_i) = C_i. \end{aligned}$$

Therefore, X_j^* are the Hermitian R-conjugate solutions of the matrix equations (1). So the solvability of the matrix equations (1) is equivalent to that of the matrix equations (2). □

For further details on the consistency of the matrix equations (2), we refer to see Dmytryshyn et al. [46]. By using Kronecker product, the matrix equations (2) can be rewritten as $Mx = b$ with

$$M = \begin{pmatrix} B_{11}^H \otimes A_{11} & B_{12}^H \otimes A_{12} & \cdots & B_{1p}^H \otimes A_{1p} \\ A_{11} \otimes B_{11}^H & A_{12} \otimes B_{12}^H & \cdots & A_{1p} \otimes B_{1p}^H \\ \overline{B}_{11}^H R \otimes \overline{A}_{11}R & \overline{B}_{12}^H R \otimes \overline{A}_{12}R & \cdots & \overline{B}_{1p}^H R \otimes \overline{A}_{1p}R \\ \overline{A}_{11}R \otimes \overline{B}_{11}^H R & \overline{A}_{12}R \otimes \overline{B}_{12}^H R & \cdots & \overline{A}_{1p}R \otimes \overline{B}_{1p}^H R \\ \vdots & \vdots & \vdots & \vdots \\ B_{p1}^H \otimes A_{p1} & B_{p2}^H \otimes A_{p2} & \cdots & B_{pp}^H \otimes A_{pp} \\ A_{p1} \otimes B_{p1}^H & A_{p2} \otimes B_{p2}^H & \cdots & A_{pp} \otimes B_{pp}^H \\ \overline{B}_{p1}^H R \otimes \overline{A}_{p1}R & \overline{B}_{p2}^H R \otimes \overline{A}_{p2}R & \cdots & \overline{B}_{pp}^H R \otimes \overline{A}_{pp}R \\ \overline{A}_{p1}R \otimes \overline{B}_{p1}^H R & \overline{A}_{p2}R \otimes \overline{B}_{p2}^H R & \cdots & \overline{A}_{pp}R \otimes \overline{B}_{pp}^H R \end{pmatrix}, x = \begin{pmatrix} vec(X_1) \\ vec(X_2) \\ \vdots \\ vec(X_p) \end{pmatrix}, \quad b = \begin{pmatrix} vec(C_1) \\ vec(C_1^H) \\ vec(\overline{C}_1) \\ vec(\overline{C}_1^H) \\ \vdots \\ vec(C_p) \\ vec(C_p^H) \\ vec(\overline{C}_p) \\ vec(\overline{C}_p^H) \end{pmatrix}. \tag{3}$$

Then, we have the following theorem.

Theorem 2.1. *The matrix equations (1) have a unique Hermitian R-conjugate solution group* $(X_1^*, X_2^*, \ldots, X_p^*)$ *if and only if* $\mathrm{rank}(M, b) = \mathrm{rank}(M)$ *and M has a full column rank; in this case, the Hermitian R-conjugate solution group* $(X_1^*, X_2^*, \ldots, X_p^*)$ *is given by*

$$X_j^* = \frac{X_j + X_j^H + R\overline{(X_j + X_j^H)}R}{4},$$

with

$$\begin{pmatrix} vec(X_1) \\ vec(X_2) \\ \vdots \\ vec(X_p) \end{pmatrix} = (M^H M)^{-1} M^H b,$$

and the corresonding homogeneous matrix equations (1) with $C_i = 0, i = 1, 2, \ldots, p$, *have the unique Hermitian R-conjugate solution group* $(X_1^*, X_2^*, \ldots, X_p^*) = 0$.

From Theorem 2.1, the Hermitian R-conjugate solutions of the matrix equations (1) can be obtained by solving the linear system $Mz = b$. In this case, we will encounter the problem of dimensionality which leads to computational difficulties. Therefore, we tend to solve the original system (1) over Hermitian R-conjugate matrices instead of the linear system $Mz = b$.

Algorithm 2.1.

Step 1: Input matrices $A_{ij} \in \mathbb{C}^{m\times n}$, $B_{ij} \in \mathbb{C}^{n\times s}$ and $C_i \in \mathbb{C}^{m\times s}$. Choose arbitrary initial matrices $X_1(1), X_2(1), \ldots, X_p(1) \in \mathbb{HRC}^{n\times n}$, symmetric orthogonal matrix $R \in \mathbb{R}^{n\times n}$, and a parameter μ as

$$0 < \mu < \frac{2}{\sum_{i=1}^{p}\sum_{j=1}^{p} \|A_{ij}\|^2 \|B_{ij}\|^2}; \tag{4}$$

Step 2: Compute

$R_i(1) = C_i - \sum_{j=1}^{p} A_{ij} X_j(1) B_{ij}, \quad i = 1, 2, \ldots, p;$

Step 3: For $k = 1, 2, \ldots$, compute

$X_j(k+1) = X_j(k) + \frac{\mu}{4}[\sum_{i=1}^{p} A_{ij}^H R_i(k) B_{ij}^H + \sum_{i=1}^{p} B_{ij} R_i^H(k) A_{ij} + \sum_{i=1}^{p} R\overline{A_{ij}^H R_i(k) B_{ij}^H} R + \sum_{i=1}^{p} R\overline{B_{ij} R_i^H(k) A_{ij}} R], \quad j = 1, 2, \ldots, p,$

$R_i(k+1) = C_i - \sum_{j=1}^{p} A_{ij} X_j(k+1) B_{ij}, \quad i = 1, 2, \ldots, p.$

Obviously, it can be seen that $X_1(k), X_2(k), \ldots, X_p(k) \in \mathbb{HRC}^{n\times n}$ for $k = 1, 2, \ldots$. Next, we review a definition and then prove the convergence of Algorithm 2.1.

Definition 2.1. [47] *In the space* $\mathbb{C}^{m\times n}$ *over the field* $\mathbb{R}$, *an inner product can be defined as*

$$\langle A, B\rangle = Re\Big[tr(A^H B)\Big]$$

for $A, B \in \mathbb{C}^{m\times n}$. *This inner product space is defined as* $(\mathbb{C}^{m\times n}, \mathbb{R}, \langle\cdot,\cdot\rangle)$.

From Definition 2.1, we can find that the inner product space $(\mathbb{C}^{m\times n}, \mathbb{R}, \langle\cdot,\cdot\rangle)$ is $2mn$-dimensional. It is known that for any matrices A and B with suitable dimensions, $Re\Big[tr(AB)\Big] = Re\Big[tr(\overline{AB})\Big] = Re\Big[tr(BA)\Big] = Re\Big[tr(A^T B^T)\Big] = Re\Big[tr(A^H B^H)\Big]$. In addition, it is easy to get that $\|A\|^2 = tr(A^H A) = Re\Big[tr(A^H A)\Big]$.

Theorem 2.2. *If the matrix equations (1) have a unique Hermitian R-conjugate solution group $(X_1^*, X_2^*, \ldots, X_p^*)$, then the iterative solution group $(X_1(k), X_2(k), \ldots, X_p(k))$ generated by Algorithm 2.1 converges to $(X_1^*, X_2^*, \ldots, X_p^*)$ for any initial Hermitian R-conjugate matrix group $(X_1(1), X_2(1), \ldots, X_p(1))$, that is,*

$$\lim_{k\to\infty} X_j(k) = X_j^*, \quad j = 1, 2, \ldots, p. \tag{5}$$

Proof. First, we define the error matrices as

$$\widetilde{X}_j(k) := X_j(k) - X_j^*, \quad j = 1, 2, \ldots, p. \tag{6}$$

It is obvious that $\widetilde{X}_j(k) \in \mathbb{HRC}^{n\times n}$, $j = 1, 2, \ldots, p$. Then we have

$$R_i(k) = C_i - \sum_{j=1}^{p} A_{ij} X_j(k) B_{ij} = -\sum_{j=1}^{p} A_{ij} \widetilde{X}_j(k) B_{ij}, \quad i = 1, 2, \ldots, p.$$

For simplicity, we use the following notations:

$$\Delta_i(k) = -R_i(k) = \sum_{j=1}^{p} A_{ij} \widetilde{X}_j(k) B_{ij}, \quad i = 1, 2, \ldots, p. \tag{7}$$

Therefore, by Algorithm 2.1, for $j = 1, 2, \ldots, p$, we can obtain

$$\begin{aligned} \widetilde{X}_j(k+1) = \widetilde{X}_j(k) - \tfrac{\mu}{4}[&\textstyle\sum_{i=1}^{p} A_{ij}^H \Delta_i(k) B_{ij}^H + \sum_{i=1}^{p} B_{ij} \Delta_i^H(k) A_{ij} \\ &+ \textstyle\sum_{i=1}^{p} R\overline{A_{ij}^H \Delta_i(k) B_{ij}^H} R + \sum_{i=1}^{p} R\overline{B_{ij} \Delta_i^H(k) A_{ij}} R]. \end{aligned} \tag{8}$$

Since $\|RAR\| = \|A\| = \|\overline{A}\|$, $\|A + B\| \leq \|A\| + \|B\|$ for any appropriately dimensioned matrices A, B, from (8) we can get

$$\begin{aligned} \|\widetilde{X}_j(k+1)\|^2 = \;& Re\Big[tr\Big(\widetilde{X}_j^H(k+1)\widetilde{X}_j(k+1)\Big)\Big] \\ =\;& \|\widetilde{X}_j(k)\|^2 - \tfrac{\mu}{2} Re\Big[tr\Big(\widetilde{X}_j^H(k)(\textstyle\sum_{i=1}^{p} A_{ij}^H \Delta_i(k) B_{ij}^H + \sum_{i=1}^{p} B_{ij} \Delta_i^H(k) A_{ij} \\ & + \textstyle\sum_{i=1}^{p} R\overline{A_{ij}^H \Delta_i(k) B_{ij}^H} R + \sum_{i=1}^{p} R\overline{B_{ij} \Delta_i^H(k) A_{ij} R})\Big)\Big] \\ & + \tfrac{\mu^2}{16} \|\textstyle\sum_{i=1}^{p} A_{ij}^H \Delta_i(k) B_{ij}^H + \sum_{i=1}^{p} B_{ij} \Delta_i^H(k) A_{ij} + \sum_{i=1}^{p} R\overline{A_{ij}^H \Delta_i(k) B_{ij}^H} R + \sum_{i=1}^{p} R\overline{B_{ij} \Delta_i^H(k) A_{ij}} R\|^2 \\ \leq\;& \|\widetilde{X}_j(k)\|^2 - \tfrac{\mu}{2} Re\Big[tr\Big(\textstyle\sum_{i=1}^{p} B_{ij} \Delta_i(k)^H A_{ij} \widetilde{X}_j(k) + \sum_{i=1}^{p} A_{ij}^H \Delta_i(k) B_{ij}^H \widetilde{X}_j(k) \\ & + \textstyle\sum_{i=1}^{p} R\overline{B}_{ij} \overline{\Delta}_i^H(k) \overline{A}_{ij} R \widetilde{X}_j(k) + \sum_{i=1}^{p} R\overline{A}_{ij}^H \overline{\Delta}_i(k) \overline{B}_{ij}^H R \widetilde{X}_j(k)\Big)\Big] \\ & + \tfrac{\mu^2}{16}\Big(\|\textstyle\sum_{i=1}^{p} A_{ij}^H \Delta_i(k) B_{ij}^H\| + \|\sum_{i=1}^{p} B_{ij} \Delta_i^H(k) A_{ij}\| + \|\sum_{i=1}^{p} R\overline{A_{ij}^H \Delta_i(k) B_{ij}^H} R\| + \|\sum_{i=1}^{p} R\overline{B_{ij} \Delta_i^H(k) A_{ij}} R\|\Big)^2 \\ \leq\;& \|\widetilde{X}_j(k)\|^2 - \tfrac{\mu}{2} Re\Big[tr\Big(\textstyle\sum_{i=1}^{p} A_{ij} \widetilde{X}_j(k) B_{ij} \Delta_i^H(k) + \sum_{i=1}^{p} B_{ij}^H \widetilde{X}_j(k) A_{ij}^H \Delta_i(k) \\ & + \textstyle\sum_{i=1}^{p} \overline{A}_{ij} R \widetilde{X}_j(k) R \overline{B}_{ij} \overline{\Delta}_i^H(k) + \sum_{i=1}^{p} \overline{B}_{ij}^H R \widetilde{X}_j(k) R \overline{A}_{ij}^H \overline{\Delta}_i(k)\Big)\Big] + \mu^2 \|\sum_{i=1}^{p} A_{ij}^H \Delta_i(k) B_{ij}^H\|^2 \\ \leq\;& \|\widetilde{X}_j(k)\|^2 - 2\mu Re\Big[tr\Big(\textstyle\sum_{i=1}^{p} A_{ij} \widetilde{X}_j(k) B_{ij} \Delta_i^H(k)\Big)\Big] + \mu^2 \sum_{i=1}^{p} (\|A_{ij}\|^2 \|B_{ij}\|^2) \sum_{i=1}^{p} \|\Delta_i(k)\|^2. \end{aligned} \tag{9}$$

Now, we define the nonnegative function $\widetilde{Z}(k)$ by

$$\widetilde{Z}(k) = \sum_{j=1}^{p} \|\widetilde{X}_j(k)\|^2. \tag{10}$$

It follows that

$$\begin{aligned}
\widetilde{Z}(k+1) &= \sum_{j=1}^{p} \|\widetilde{X}_j(k+1)\|^2 \\
&\leq \sum_{j=1}^{p} \|\widetilde{X}_j(k)\|^2 - 2\mu \sum_{j=1}^{p} Re\Big[tr\Big(\sum_{i=1}^{p} A_{ij}\widetilde{X}_j(k)B_{ij}\Delta_i^H(k)\Big)\Big] + \mu^2 \sum_{j=1}^{p}\Big[\sum_{i=1}^{p}(\|A_{ij}\|^2\|B_{ij}\|^2)\sum_{i=1}^{p}\|\Delta_i(k)\|^2\Big] \\
&= \widetilde{Z}(k) - 2\mu \sum_{i=1}^{p} Re\Big[tr\Big((\sum_{j=1}^{p} A_{ij}\widetilde{X}_j(k)B_{ij})\Delta_i^H(k)\Big)\Big] + \mu^2(\sum_{i=1}^{p}\|\Delta_i(k)\|^2)\sum_{j=1}^{p}\sum_{i=1}^{p}(\|A_{ij}\|^2\|B_{ij}\|^2) \\
&= \widetilde{Z}(k) - 2\mu \sum_{i=1}^{p} Re\Big[tr\Big(\Delta_i(k)\Delta_i^H(k)\Big)\Big] + \mu^2(\sum_{i=1}^{p}\|\Delta_i(k)\|^2)\sum_{i=1}^{p}\sum_{j=1}^{p}(\|A_{ij}\|^2\|B_{ij}\|^2) \\
&= \widetilde{Z}(k) - 2\mu \sum_{i=1}^{p}\|\Delta_i(k)\|^2 + \mu^2(\sum_{i=1}^{p}\|\Delta_i(k)\|^2)\sum_{i=1}^{p}\sum_{j=1}^{p}(\|A_{ij}\|^2\|B_{ij}\|^2) \\
&= \widetilde{Z}(k) - 2\mu\Big[1 - \frac{\mu}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}(\|A_{ij}\|^2\|B_{ij}\|^2)\Big]\sum_{i=1}^{p}\|\Delta_i(k)\|^2 \\
&\leq \widetilde{Z}(1) - 2\mu\Big[1 - \frac{\mu}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}(\|A_{ij}\|^2\|B_{ij}\|^2)\Big]\sum_{m=1}^{k}(\sum_{i=1}^{p}\|\Delta_i(m)\|^2).
\end{aligned}$$

According to

$$0 < \mu < \frac{2}{\sum_{i=1}^{p}\sum_{j=1}^{p}\|A_{ij}\|^2\|B_{ij}\|^2},$$

we have

$$\sum_{m=1}^{\infty}(\sum_{i=1}^{p}\|\Delta_i(m)\|^2) < \infty.$$

For the necessary condition of the series convergence, we have

$$\lim_{m\to\infty}\Delta_i(m) = \lim_{m\to\infty}\sum_{j=1}^{p} A_{ij}\widetilde{X}_j(m)B_{ij} = 0, \quad i = 1, 2, \ldots, p.$$

According to Theorem 2.1, we can get

$$\lim_{m\to\infty}\widetilde{X}_j(m) = 0, \quad j = 1, 2, \ldots, p.$$

The proof is completed. □

Remark 2.1. In practical operation, we can choose a relatively large μ, and even do not meet the inequality (4), which may also converge to the Hermitian R-conjugate solutions. This is because that the control inequality (4) is only a sufficient condition but not a necessary condition and we magnify the inequality too large during the proof. This will be demonstrated in Example 3.1 given later.

Next, we discuss the optimal choice of the factor μ. Submitting (7) into (8), we have

$$\begin{aligned}
\widetilde{X}_j(k+1) = \widetilde{X}_j(k) - \frac{\mu}{4}[&\underline{\sum_{i=1}^{p} A_{ij}^H(\sum_{j=1}^{p} A_{ij}\widetilde{X}_j(k)B_{ij})B_{ij}^H} + \underline{\sum_{i=1}^{p} B_{ij}(\sum_{j=1}^{p} A_{ij}\widetilde{X}_j(k)B_{ij})^H A_{ij}} \\
&+ \sum_{i=1}^{p} RA_{ij}^H(\sum_{j=1}^{p} A_{ij}\widetilde{X}_j(k)B_{ij})B_{ij}^H R + \sum_{i=1}^{p} RB_{ij}(\sum_{j=1}^{p} A_{ij}\widetilde{X}_j(k)B_{ij})^H A_{ij}R].
\end{aligned} \tag{11}$$

By employing Kronecker product and vectorization operator, it is not difficult to obtain

$$\begin{pmatrix} vec(\widetilde{X}_1(k+1)) \\ vec(\widetilde{X}_2(k+1)) \\ \vdots \\ vec(\widetilde{X}_p(k+1)) \end{pmatrix} = (I_{pn^2} - \tfrac{\mu}{4}\Phi) \begin{pmatrix} vec(\widetilde{X}_1(k)) \\ vec(\widetilde{X}_2(k)) \\ \vdots \\ vec(\widetilde{X}_p(k)) \end{pmatrix}, \tag{12}$$

with

$$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1p} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{p1} & \phi_{p2} & \cdots & \phi_{pp} \end{pmatrix}, \tag{13}$$

where

$$\phi_{st} = \sum_{i=1}^{p} (B_{is}B_{it}^H \otimes A_{is}^H A_{it} + A_{is}^H A_{it} \otimes B_{is}B_{it}^H + R\overline{B_{is}B_{it}^H}R \otimes R\overline{A_{is}^H A_{it}}R + R\overline{A_{is}^H A_{it}}R \otimes R\overline{B_{is}B_{it}^H}R). \tag{14}$$

Obviously, the matrix $\Phi = M^H M$ is a Hermitian matrix where matrix M is defined as (3). According to Theorem 2.1, if the matrix equations (1) have a unique Hermitian R-conjugate solution group, then the matrix Φ is also positive definite.

Lemma 2.2. *Suppose the matrix equations (1) have a unique Hermitian R-conjugate solution group. Then Algorithm 2.1 converges for any initial Hermitian R-conjugate matrix group if and only if the convergent factor μ satisfies the following condition*

$$0 < \mu < \frac{8}{\lambda_{max}(\Phi)}. \tag{15}$$

Proof. Since Φ is a Hermitian positive definite matrix, the iterative matrix $I_{pn^2} - \frac{\mu}{4}\Phi$ is Hermitian too. Then the spectral radius of the iterative matrix $I_{pn^2} - \frac{\mu}{4}\Phi$ is identical to $\max\{|1 - \frac{\mu\lambda_{\min}(\Phi)}{4}|, |1 - \frac{\mu\lambda_{\max}(\Phi)}{4}|\}$.

From $\rho(I_{pn^2} - \frac{\mu}{4}\Phi) < 1$, i.e.,

$$\max\{|1 - \frac{\mu\lambda_{\min}(\Phi)}{4}|, |1 - \frac{\mu\lambda_{\max}(\Phi)}{4}|\} < 1,$$

we have

$$0 < \frac{\mu\lambda_{min}(\Phi)}{4} < 2 \quad \text{and} \quad 0 < \frac{\mu\lambda_{\max}(\Phi)}{4} < 2.$$

Thus

$$0 < \mu < \frac{8}{\lambda_{max}(\Phi)},$$

and the proof is completed. □

It should be noted that when the matrix M is a column reduced-rank matrix, i.e., $M^H M$ is singular, the proposed method is also available, and the semi-convergence can be obtained by the analogous strategy applied in [40].

Lemma 2.3. [48] *Let $a, b \in R$ and $\mu > 0$, then we have*
(a) If $b > a > 0$, then $\min\limits_{0<\mu<2/b} \{\max\{|1 - \mu a|, |1 - \mu b|\}\} = \frac{b-a}{b+a}$, and the minimizer can be reached at the point $\mu = \frac{2}{a+b}$;

(b) If $b > 0 > a$, *then* $\min_{0<\mu}\{\max\{|1-\mu a|, |1-\mu b|\}\} > 1$, *and for any* $\mu > 0$ *we have* $\max\{|1-\mu a|, |1-\mu b|\} > 1$;

(c) If $a < b < 0$, *then* $\min_{0<\mu}\{\max\{|1-\mu a|, |1-\mu b|\}\} > 1$, *and for any* $\mu > 0$ *we have* $\max\{|1-\mu a|, |1-\mu b|\} > 1$.

Theorem 2.3. *Suppose the matrix equations (1) have a unique Hermitian R-conjugate solution group. When* $0 < \mu < \frac{8}{\lambda_{max}(\Phi)}$, *Algorithm 2.1 converges and the optimal convergent factor should be*

$$\mu_{opt} = \frac{8}{\lambda_{min}(\Phi) + \lambda_{max}(\Phi)}. \tag{16}$$

Moreover, if μ *is chosen as (16), then*

$$\left\| \begin{array}{c} vec(\widetilde{X}_1(k+1)) \\ vec(\widetilde{X}_2(k+1)) \\ \vdots \\ vec(\widetilde{X}_p(k+1)) \end{array} \right\|_2 \leq \left(\frac{cond(\Phi)-1}{cond(\Phi)+1}\right)^k \left\| \begin{array}{c} vec(\widetilde{X}_1(1)) \\ vec(\widetilde{X}_2(1)) \\ \vdots \\ vec(\widetilde{X}_p(1)) \end{array} \right\|_2 . \tag{17}$$

Proof. According to (12), we can see that the optimal convergent factor μ should been chosen to minimize the spectral radius $\rho(I_{pn^2} - \frac{\mu}{4}\Phi)$. As Φ is Hermitian positive definite, we have $\lambda_{min}(\Phi) > 0$ and $\lambda_{max}(\Phi) > 0$. Then by Lemma 2.3, the optimal convergent factor can be taken as

$$\mu_{opt} = \frac{8}{\lambda_{min}(\Phi) + \lambda_{max}(\Phi)}.$$

Moreover,

$$\rho(I_{pn^2} - \frac{\mu_{opt}}{4}\Phi) = \frac{\lambda_{max}(\Phi) - \lambda_{min}(\Phi)}{\lambda_{max}(\Phi) + \lambda_{min}(\Phi)} = \frac{cond(\Phi) - 1}{cond(\Phi) + 1},$$

and (17) holds. □

3. Numerical Experiments

In this section, we give two examples to illustrate the effectiveness of the proposed algorithm.

Example 3.1 Consider the Hermitian R-conjugate solutions of the following generalized coupled Sylvester matrix equations

$$\begin{cases} A_{11}X_1B_{11} + A_{12}X_2B_{12} = C_1 \\ A_{21}X_1B_{21} + A_{22}X_2B_{22} = C_2 \end{cases}, \tag{18}$$

with

$$A_{11} = \begin{pmatrix} 1+i & 1 \\ i & -1 \end{pmatrix}, \quad B_{11} = \begin{pmatrix} 1 & i \\ 2 & 1-i \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 2-i & 0 \\ 1 & i \end{pmatrix}, B_{12} = \begin{pmatrix} 1 & i \\ 2 & 1 \end{pmatrix},$$

$$A_{21} = \begin{pmatrix} i & 1 \\ i & -i \end{pmatrix}, \quad B_{21} = \begin{pmatrix} 1 & -i \\ 0 & 1+i \end{pmatrix}, \quad A_{22} = \begin{pmatrix} -i & 1+i \\ 1 & i \end{pmatrix}, \quad B_{22} = \begin{pmatrix} 1+i & -i \\ i & 1 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 10i & 4+8i \\ -2+14i & -4+10i \end{pmatrix}, \quad C_2 = \begin{pmatrix} 4i & 0 \\ -6+2i & -2+4i \end{pmatrix}.$$

Let R be as follows:

$$R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Hermitian R-conjugate solutions of the matrix equations (18) can be obtained as follows

$$X_1^* = \begin{pmatrix} 0 & 2i \\ -2i & 0 \end{pmatrix}, \qquad X_2^* = \begin{pmatrix} 0 & 2i \\ -2i & 4 \end{pmatrix}.$$

According to (13) and (14), the matrix

$$\Phi = \begin{pmatrix} 40 & -2i & -2i & 4 & -4 & -18i & -18i & 0 \\ 2i & 60 & 12 & -4i & -14i & 4 & 6 & -4i \\ 2i & 12 & 60 & -4i & -14i & 6 & 4 & -4i \\ 4 & 4i & 4i & 64 & 0 & -16i & -16i & -8 \\ -4 & 14i & 14i & 0 & 72 & 20i & 20i & 8 \\ 18i & 4 & 6 & 16i & -20i & 90 & -16 & 8i \\ 18i & 6 & 4 & 16i & -20i & -16 & 90 & 8i \\ 0 & 4i & 4i & -8 & 8 & -8i & -8i & 44 \end{pmatrix}.$$

From Theorem 2.3, we can get $\mu_{opt} = 0.0584$ and $\frac{8}{\lambda_{max}(\Phi)} = 0.0662$, which are more than $\frac{2}{\|A_{11}\|^2\|B_{11}\|^2+\|A_{12}\|^2\|B_{12}\|^2+\|A_{21}\|^2\|B_{21}\|^2+\|A_{22}\|^2\|B_{22}\|^2} = 0.0154$. Take the initial Hermitian R-conjugate matrix pair $(X_1(1), X_2(1)) = 0$ and $\mu = 0.0284, 0.0384, 0.0484, 0.0584, 0.0650$, respectively. Applying Algorithm 2.1 to compute $(X_1(k), X_2(k))$, the iterative errors $r(k) = log_{10}\sqrt{\|R_1(k)\|^2 + \|R_2(k)\|^2}$ versus k are shown in Fig. 1. According to Fig. 1, it is clear that the larger the convergent factor μ, the faster the convergent rate and when the convergent factor μ is taken to be 0.0584, the convergent rate is the fastest. However, when the convergent factor μ is greater than 0.0584 but less than 0.0662, the convergent rate becomes slow. Also, in Fig. 2, we plot the relationship of the iterative number k versus μ, which further verifies the theoretical findings.

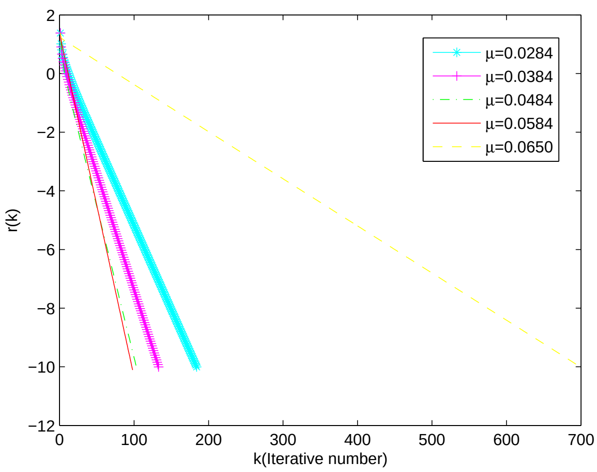

Figure 1: $r(k)$ versus k with different μ for Example 3.1.

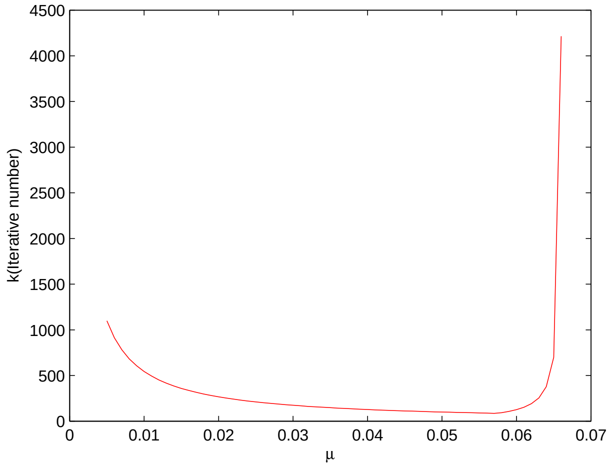

Figure 2: Iterative number k versus μ for Example 3.1.

Example 3.2 Consider the Hermitian R-conjugate solutions of the matrix equations (18) with

$$rand('state', 0),$$

$$A_{11} = tril(rand(p, p), 1) * i - diag(2 + diag(rand(p))),$$

$$B_{11} = tril(rand(p, p), 1) + diag(diag(rand(p))) * i,$$

$$A_{12} = tril(rand(p, p), 1) - diag(2 + diag(rand(p))) * i,$$

$$B_{12} = triu(rand(p, p), 1) - diag(1.5 + diag(rand(p))) * i,$$

$$A_{21} = tril(rand(p, p), 1) * i + diag(2 + diag(rand(p))),$$

$$B_{21} = tril(rand(p, p), 1) + diag(2 + diag(rand(p))) * i,$$

$$A_{22} = tril(rand(p, p), 1) - diag(1 + diag(rand(p))) * i,$$

$$B_{22} = triu(rand(p, p), 1) + diag(2 + diag(rand(p))) * i.$$

Here, the C_1, C_2 are chosen such that the Hermitian Toeplitz matrices $X_1^* = \mathrm{tridiag}(i, 2, -i), X_2^* = \mathrm{tridiag}(1+i, 2, 1-i)$ are the Hermitian R-conjugate solutions with respect to $R = \mathrm{fliplr}(eye(p))$. When $p = 10$, the sequences pair $(X_1(k), X_2(k))$ are obtained with $\mu_{opt} = 0.0027$ and the initial matrix pair $(X_1(1), X_2(1)) = 0$. We show the numerical results in Fig. 3, where

$$\delta(k) = log_{10} \sqrt{\frac{\|X_1(k) - X_1^*\|^2 + \|X_2(k) - X_2^*\|^2}{\|X_1^*\|^2 + \|X_2^*\|^2}}.$$

Obviously, both $r(k)$ and $\delta(k)$ decrease and converge to zero as k increases.

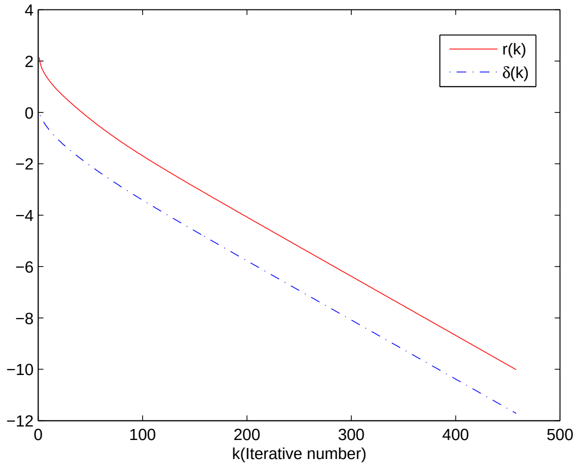

Figure 3: $r(k)$ and $\delta(k)$ versus k for Example 3.2.

4. Conclusions

In this paper, we have constructed an iterative method to solve the general coupled Sylvester matrix equations over Hermitian R-conjugate matrices. When the considered coupled matrix equations have a unique Hermitian R-conjugate solution group, some conditions have been established to guarantee the convergence of the proposed method. The optimal convergent factor has been also derived. Finally, the efficiency of the proposed method is verified by two numerical experiments.

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