Filomat 31:7 (2017), 1857–1868
DOI 10.2298/FIL1707857Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

L^p Solutions of Infinite Time Interval Backward Doubly Stochastic Differential Equations

Zhaojun Zong[a], Feng Hu[a]

[a]*School of Statistics, Qufu Normal University, Qufu, 273165, China*

Abstract. In this paper, we study the existence and uniqueness theorem for L^p $(1 < p < 2)$ solutions to a class of infinite time interval backward doubly stochastic differential equations (BDSDEs). Furthermore, we obtain the comparison theorem for 1-dimensional infinite time interval BDSDEs in L^p.

1. Introduction

The theory of nonlinear backward stochastic differential equations (BSDEs for short) was developed by Pardoux and Peng [13], from which we know that there exists a unique adapted and square integrable solution to a BSDE of the type

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \quad t \in [0, T], \tag{1}$$

provided the function g (also called the generator) is Lipschitz in both variables y and z, and ξ and $(g(t, 0, 0))_{t\in[0,T]}$ are square integrable. Later, many researchers developed the theory of BSDEs and their applications in a series of papers (for example, see Briand et al. [2], Hu and Peng [8], Lepeltier and San Martin [9], Pardoux [10, 11], El Karoui et al. [7] and the references therein) under some other assumptions on the coefficients but for a fixed terminal time $T > 0$. Let us mention the contribution of Lepeltier and San Martin [9] which dealt with the quadratic of growth generator g in z and got the existence and uniqueness result in L^2. Let us mention also that when the generator g is Lipschitz continuous, a result of El Karoui et al. [7], provides of a solution when the data ξ and $\{(g(t, 0, 0)_{t\in[0,T]}\}$ are in L^p even for $p \in (1, 2)$. And in 2003, Briand et al. [2] was devoted to the generalization of this result to the case of a monotone generator for BSDEs on a fixed time interval.

Under the assumptions that terminal value $\xi = 0$ or $E[\mathrm{e}^{p\rho T}|\xi|^p] < \infty$ for some constant ρ and random terminal time T (i.e., T is a stopping time), Peng [15], Pardoux [10], Darling and Pardoux [6], Peng and Shi

---

2010 *Mathematics Subject Classification*. Primary 60H10
Keywords. Backward doubly stochastic differential equation (BDSDE), backward stochastic integral, comparison theorem
Received: 20 April 2015; Accepted: 02 February 2016
Communicated by Miljana Jovanović
Research supported by the National Natural Science Foundation of China (Nos. 11301295, 11571198 and 11501319), the Natural Science Foundation of Shandong Province (No. ZR2016JL002), the Education Department of Shandong Province the Education Department of Shandong ProvinceScience and Technology Plan Project (No. J16LI07), the Science and Technology Plan Project of Qufu Normal University (No. xkj201517), the Doctoral Foundation of Qufu Normal University, the Program for Scientific Research Innovation Team in Colleges and Universities of Shandong Province of China and the Program for Scientific Research Innovation Team in Applied Probability and Statistics of Qufu Normal University (No. 0230518).
Email address: hufengqf@163.com (Feng Hu)

[16] and other researchers investigated the problem on L^2 solutions of BSDEs. In 2000, Chen and Wang [5] obtained the existence and uniqueness theorem for L^2 solutions of infinite time interval BSDEs when $T \equiv \infty$, by the martingale representation theorem and fixed point theorem. But in the case L^p $(1 < p < 2)$, there is not the martingale representation theorem. In 2013, Zong [21] studied L^p solutions to infinite time interval BSDEs. She gave a new a priori estimate. By using this a priori estimate, she proved the existence and uniqueness of L^p solutions to infinite time interval BSDEs.

In 1994, Pardoux and Peng [14] brought forward a new kind of BSDEs, i.e., a class of backward doubly stochastic differential equations (BDSDEs for short) with two differential directions of stochastic integrals, i.e., the equations involve both a standard (forward) stochastic integral $\mathrm{d}W_t$ and a backward stochastic integral $\mathrm{d}B_t$. They have proved the existence and uniqueness of solutions to BDSDEs under the uniformly Lipschitz conditions on coefficients on a finite time interval $[0, T]$. That is, for a given fixed terminal time $T > 0$, under the uniformly Lipschitz assumptions on coefficients f, g, given $\xi \in L^2$, the following BDSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s + \int_t^T g(s, Y_s, Z_s)\mathrm{d}B_s - \int_t^T Z_s \mathrm{d}W_s, \quad t \in [0, T]. \tag{2}$$

has a unique solution $(Y_t, Z_t)_{t\in[0,T]}$. Later, many researchers applied their method in this area (for example, see Bally and Matoussi [1], Buckdahn and Ma [3, 4], Pardoux [12], Peng and Shi [17], Zhang and Zhao [19] and the references therein). Recently, inspired by [5], Zhu and Han [20] got the existence and uniqueness of L^2 solutions to infinite time interval BDSDEs by using the martingale representation theorem and fixed point theorem. In this paper, we study the existence and uniqueness theorem for L^p $(1 < p < 2)$ solutions to a class of infinite time interval BDSDEs. In order to get rid of the difficulty that there is not the martingale representation theorem in L^p $(1 < p < 2)$, we give a new a priori estimate. The proof of this a priori estimate is different from Lemma 3.1 and Proposition 3.2 in Briand et al. [2]. By using this a priori estimate, we get the existence and uniqueness of L^p $(1 < p < 2)$ solutions to infinite time interval BDSDEs. Furthermore, we obtain the comparison theorem for 1-dimensional infinite time interval BDSDEs in L^p $(1 < p < 2)$.

This paper is organized as follows. In Section 2, we introduce some notations, assumptions and lemmas. In Section 3, we prove the existence and uniqueness theorem for L^p $(1 < p < 2)$ solutions of infinite time interval BDSDEs. In Section 4, we obtain the comparison theorem for 1-dimensional infinite time interval BDSDEs in L^p $(1 < p < 2)$.

2. Preliminaries

In this section, we shall present some notations, assumptions and lemmas that are used in this paper.

Notation The Euclidean norm of a vector $x \in \mathbb{R}^d$ will be denoted by $|x|$, and for a $d \times k$ matrix A, we define $\|A\| = \sqrt{TrAA^*}$, where A^* is the transpose of A.

Let $(\Omega, \mathcal{F}, P)$ be a completed probability space, $(W_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ be two mutually independent standard Brownian motions with values in $\mathbb{R}^d$ and $\mathbb{R}^k$, respectively, defined on this space. Let $\mathcal{N}$ denote the set of all P-null subsets of $\mathcal{F}$, we define

$$\mathcal{F}_{0,t}^W := \sigma\{W_s; 0 \leq s \leq t\} \vee \mathcal{N}, \ \ \mathcal{F}_{t,\infty}^B := \sigma\{B_s - B_t; t \leq s < \infty\} \vee \mathcal{N},$$

$$\mathcal{F}_{0,\infty}^W := \bigvee_{t\geq 0} \mathcal{F}_{0,t}^W, \ \ \mathcal{F}_{\infty,\infty}^B := \bigcap_{t\geq 0} \mathcal{F}_{t,\infty}^B$$

and

$$\mathcal{F}_t := \mathcal{F}_{0,t}^W \vee \mathcal{F}_{t,\infty}^B, \ \ t \geq 0.$$

Note that $\left\{\mathcal{F}_{0,t}^W; t \geq 0\right\}$ is an increasing filtration and $\left\{\mathcal{F}_{t,\infty}^B; t \geq 0\right\}$ is a decreasing filtration, and the collection $\{\mathcal{F}_t; t \geq 0\}$ is neither increasing nor decreasing. Furthermore, we define $\mathcal{F} := \mathcal{F}_{0,\infty}^W \vee \mathcal{F}_{\infty,\infty}^B$.

We consider the following spaces:
$L^p(\Omega,\mathcal{F},P,\mathbb{R}^l) := \{\xi : \xi$ is $\mathbb{R}^l$-valued and $\mathcal{F}$-measurable random variable such that $E[|\xi|^p] < \infty, p \geq 1\}$;
$\mathcal{L}(\Omega,\mathcal{F},P,\mathbb{R}^l) := \bigcup\limits_{p>1} L^p(\Omega,\mathcal{F},P,\mathbb{R}^l)$;
$\mathcal{S}^p(\mathbb{R}^l) := \{V : V_t$ is $\mathbb{R}^l$-valued and $\mathcal{F}_t$-adapted process such that $E[\sup_{t\geq 0}|V_t|^p] < \infty, p \geq 1\}$;
$\mathcal{S}(\mathbb{R}^l) := \bigcup\limits_{p>1} \mathcal{S}^p(\mathbb{R}^l)$;
$\mathcal{L}^p(\mathbb{R}^{l\times d}) := \{V : V_t$ is $\mathbb{R}^{l\times d}$-valued and $\mathcal{F}_t$-adapted process such that $E[(\int_0^\infty \|V_s\|^2 \mathrm{d}s)^{\frac{p}{2}}] < \infty, p \geq 1\}$;
$\mathcal{L}(\mathbb{R}^{l\times d}) := \bigcup\limits_{p>1} \mathcal{L}^p(\mathbb{R}^{l\times d})$.
Assumption that $p \in (1,2)$ will be kept in the sequel.
Consider the following infinite time interval BDSDE

$$Y_t = \xi + \int_t^\infty f(s,Y_s,Z_s)\mathrm{d}s + \int_t^\infty g(s,Y_s)\mathrm{d}B_s - \int_t^\infty Z_s\mathrm{d}W_s, \quad 0 \leq t \leq \infty \tag{3}$$

We make the following assumptions:
(A.0) Let

$$f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^l \times \mathbb{R}^{l\times d} \mapsto \mathbb{R}^l$$

and

$$g : \Omega \times \mathbb{R}_+ \times \mathbb{R}^l \mapsto \mathbb{R}^{l\times k}$$

such that for any $(y,z) \in \mathbb{R}^l \times \mathbb{R}^{l\times d}$, $f(\cdot,y,z)$ and $g(\cdot,y)$ are $\mathcal{F}_t$-progressively measurable.

(A.1) $E\left[\left(\int_0^\infty \left|f(t,0,0)\right| \mathrm{d}t\right)^2\right] < \infty$, $g(\cdot,0) \in \mathcal{L}^2(\mathbb{R}^{l\times k})$;

(A.2) There exist two positive non-random functions $\alpha(t)$ and $\beta(t)$, such that for all $y_1, y_2 \in \mathbb{R}^l$, $z_1, z_2 \in \mathbb{R}^{l\times d}$,

$$\left|f(t,y_1,z_1) - f(t,y_2,z_2)\right| \leq \alpha(t)\left|y_1 - y_2\right| + \beta(t)\left\|z_1 - z_2\right\|,$$

$$\left\|g(t,y_1) - g(t,y_2)\right\| \leq \beta(t)\left|y_1 - y_2\right|,$$

where $\alpha(t)$ and $\beta(t)$ satisfy that $\int_0^\infty \alpha(t)\mathrm{d}t < \infty$, $\int_0^\infty \beta^2(t)\mathrm{d}t < \infty$;

(A.3) $E\left[\left(\int_0^\infty \left|f(t,0,0)\right| \mathrm{d}t\right)^p\right] < \infty$, $g(\cdot,0) \in \mathcal{L}^p(\mathbb{R}^{l\times k})$.

The following lemma is proven in [20].

Lemma 2.1 Let $\xi \in L^2(\Omega,\mathcal{F},P,\mathbb{R}^l)$ be given. Suppose that (A.0), (A.1) and (A.2) hold for f and g, then BDSDE (3) has a unique solution $(Y,Z) \in \mathcal{S}^2(\mathbb{R}^l) \times \mathcal{L}^2(\mathbb{R}^{l\times d})$.

Lemma 2.2 Let $\alpha \in \mathcal{S}^p(\mathbb{R}^l)$, $\beta \in \mathcal{L}^p(\mathbb{R}^l)$, $\gamma \in \mathcal{L}^p(\mathbb{R}^{l\times k})$, $\delta \in \mathcal{L}^p(\mathbb{R}^{l\times d})$ be such that:

$$\alpha_T = \alpha_\tau + \int_\tau^T \beta_s\mathrm{d}s + \int_\tau^T \gamma_s\mathrm{d}B_s + \int_\tau^T \delta_s\mathrm{d}W_s, \quad T \in [0,\infty], \quad \tau \in [0,T].$$

Then

$$\begin{aligned} |\alpha_T|^2 &= |\alpha_\tau|^2 + 2\int_\tau^T \langle \alpha_s, \beta_s\rangle \mathrm{d}s + 2\int_\tau^T \langle \alpha_s, \gamma_s\mathrm{d}B_s\rangle \\ &+ 2\int_\tau^T \langle \alpha_s, \delta_s\mathrm{d}W_s\rangle - \int_\tau^T \|\gamma_s\|^2\mathrm{d}s + \int_\tau^T \|\delta_s\|^2 \mathrm{d}s. \end{aligned}$$

The proof is very similar to that of Lemma 1.3 in [14], so we omit it.

3. Existence and Uniqueness

In this section, we prove the existence and uniqueness theorem for L^p solutions of infinite time interval BDSDEs.

Theorem 3.1 If $\xi \in L^p(\Omega, \mathcal{F}, P, \mathbb{R}^l)$ and assumptions (A.0), (A.2) and (A.3) hold, then BDSDE (3) has a unique solution $(Y, Z) \in \mathcal{S}^p(\mathbb{R}^l) \times \mathcal{L}^p(\mathbb{R}^{l\times d})$.

In order to prove Theorem 3.1, we give an a priori estimate.

Lemma 3.1 Suppose that (A.2) holds for f and g. Furthermore, each ϕ_i satisfies that $E\left[\left(\int_0^\infty |\phi_i(s)|ds\right)^p\right] < \infty$ and $\varphi_i(\cdot) \in \mathcal{L}^p(\mathbb{R}^{l\times k})$, $i = 1, 2$. For any $T \in [0, \infty]$, let $\xi_i \in L^p(\Omega, \mathcal{F}_T, P, \mathbb{R}^l)$, $(Y^i, Z^i) \in \mathcal{S}^p(\mathbb{R}^l) \times \mathcal{L}^p(\mathbb{R}^{l\times d})$ satisfy the following BDSDEs

$$Y_t^i = \xi_i + \int_t^T \left[f\left(s, Y_s^i, Z_s^i\right) + \phi_i(s)\right] \mathrm{d}s + \int_t^T \left[g\left(s, Y_s^i\right) + \varphi_i(s)\right] \mathrm{d}B_s - \int_t^T Z_s^i \mathrm{d}W_s, \quad i = 1, 2.$$

Then there exists a positive constant C_p depending only on p such that, for any $\tau \in [0, T]$,

$$\begin{aligned}
& E\left[\sup_{s\in[\tau,T]} \left|Y_s^1 - Y_s^2\right|^p + \left(\int_\tau^T \left\|Z_s^1 - Z_s^2\right\|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \\
\leq\ & C_p E\left[|\xi_1 - \xi_2|^p + \left(\int_0^\infty \left|\phi_1(s) - \phi_2(s)\right| \mathrm{d}s\right)^p + \left(\int_0^\infty \left\|\varphi_1(s) - \varphi_2(s)\right\|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \\
+\ & C_p l_{(\tau,T]} E\left[\sup_{s\in[\tau,T]} \left|Y_s^1 - Y_s^2\right|^p + \left(\int_\tau^T \left\|Z_s^1 - Z_s^2\right\|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right],
\end{aligned}$$

where $l_{(\tau,T]} = \left(\int_\tau^T \alpha(s)\mathrm{d}s + \int_\tau^T \beta^2(s)\mathrm{d}s\right)^{\frac{p}{2}} + \left(\int_\tau^T \alpha(s)\mathrm{d}s\right)^p + \left(\int_\tau^T \beta^2(s)\mathrm{d}s\right)^{\frac{p}{4}}$.

Proof. Applying Lemma 2.2 to $\left|Y_t^1 - Y_t^2\right|^2$, we have

$$\begin{aligned}
& \left|Y_\tau^1 - Y_\tau^2\right|^2 + \int_\tau^T \left\|Z_s^1 - Z_s^2\right\|^2 \mathrm{d}s \\
=\ & |\xi_1 - \xi_2|^2 + 2\int_\tau^T \left\langle Y_s^1 - Y_s^2, f\left(s, Y_s^1, Z_s^1\right) - f\left(s, Y_s^2, Z_s^2\right) + \phi_1(s) - \phi_2(s)\right\rangle \mathrm{d}s \\
-\ & 2\int_\tau^T \left\langle Y_s^1 - Y_s^2, \left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s\right\rangle + \int_\tau^T \left\|g\left(s, Y_s^1\right) - g\left(s, Y_s^2\right) + \varphi_1(s) - \varphi_2(s)\right\|^2 \mathrm{d}s \\
+\ & 2\int_\tau^T \left\langle Y_s^1 - Y_s^2, \left(g\left(s, Y_s^1\right) - g\left(s, Y_s^2\right) + \varphi_1(s) - \varphi_2(s)\right)\mathrm{d}B_s\right\rangle.
\end{aligned}$$

From the Lipshcitz assumption (A.2) on f and g, we have

$$\begin{aligned}
& 2\left\langle Y_s^1 - Y_s^2, \left(f\left(s, Y_s^1, Z_s^1\right) - f\left(s, Y_s^2, Z_s^2\right)\right)\right\rangle \\
\leq\ & 2\alpha(s)\left|Y_s^1 - Y_s^2\right|^2 + 2\beta(s)\left|Y_s^1 - Y_s^2\right|\left\|Z_s^1 - Z_s^2\right\| \\
\leq\ & 2\alpha(s)\left|Y_s^1 - Y_s^2\right|^2 + 2\beta^2(s)\left|Y_s^1 - Y_s^2\right|^2 + \tfrac{1}{2}\left\|Z_s^1 - Z_s^2\right\|^2 \\
\leq\ & 2\left(\alpha(s) + \beta^2(s)\right)\sup_{s\in[\tau,T]}\left|Y_s^1 - Y_s^2\right|^2 + \tfrac{1}{2}\left\|Z_s^1 - Z_s^2\right\|^2
\end{aligned}$$

and

$$\begin{aligned}
\left\|g\left(s, Y_s^1\right) - g\left(s, Y_s^2\right) + \varphi_1(s) - \varphi_2(s)\right\|^2 \ &\leq\ 2\beta^2(s)\left|Y_s^1 - Y_s^2\right|^2 + 2\left\|\varphi_1(s) - \varphi_2(s)\right\|^2 \\
&\leq\ 2\beta^2(s)\sup_{s\in[\tau,T]}\left|Y_s^1 - Y_s^2\right|^2 + 2\left\|\varphi_1(s) - \varphi_2(s)\right\|^2.
\end{aligned}$$

It follows that

$$
\begin{aligned}
& \tfrac{1}{2}\int_\tau^T \left\|Z_s^1 - Z_s^2\right\|^2 \mathrm{d}s \\
\leq\ & |\xi_1 - \xi_2|^2 + 2\int_0^\infty \left\|\varphi_1(s) - \varphi_2(s)\right\|^2 \mathrm{d}s \\
+\ & 4\left(\int_\tau^T \alpha(s)\mathrm{d}s + \int_\tau^T \beta^2(s)\mathrm{d}s\right) \sup_{s\in[\tau,T]} \left|Y_s^1 - Y_s^2\right|^2 + 2\int_\tau^T \left|Y_s^1 - Y_s^2\right| \left|\phi_1(s) - \phi_2(s)\right| \mathrm{d}s \\
+\ & 2\left|\int_\tau^T \left\langle Y_s^1 - Y_s^2, \left(g\left(s, Y_s^1\right) - g\left(s, Y_s^2\right) + \varphi_1(s) - \varphi_2(s)\right) \mathrm{d}B_s\right\rangle\right| \\
+\ & 2\left|\int_\tau^T \left\langle Y_s^1 - Y_s^2, \left(Z_s^1 - Z_s^2\right) \mathrm{d}W_s\right\rangle\right|.
\end{aligned}
$$

Since $2\int_\tau^T \left|Y_s^1 - Y_s^2\right| \left|\phi_1(s) - \phi_2(s)\right| \mathrm{d}s \leq \sup_{s\in[\tau,T]} \left|Y_s^1 - Y_s^2\right|^2 + \left(\int_0^\infty \left|\phi_1(s) - \phi_2(s)\right| \mathrm{d}s\right)^2$, we have

$$
\begin{aligned}
& \int_\tau^T \left\|Z_s^1 - Z_s^2\right\|^2 \mathrm{d}s \\
\leq\ & 8\left(|\xi_1 - \xi_2|^2 + \left(\int_0^\infty \left|\phi_1(s) - \phi_2(s)\right| \mathrm{d}s\right)^2 + \int_0^\infty \left\|\varphi_1(s) - \varphi_2(s)\right\|^2 \mathrm{d}s\right) \\
+\ & 8\left(1 + \int_\tau^T \alpha(s)\mathrm{d}s + \int_\tau^T \beta^2(s)\mathrm{d}s\right) \sup_{s\in[\tau,T]} \left|Y_s^1 - Y_s^2\right|^2 \\
+\ & 8\left|\int_\tau^T \left\langle Y_s^1 - Y_s^2, \left(g\left(s, Y_s^1\right) - g\left(s, Y_s^2\right) + \varphi_1(s) - \varphi_2(s)\right) \mathrm{d}B_s\right\rangle\right| \\
+\ & 8\left|\int_\tau^T \left\langle Y_s^1 - Y_s^2, \left(Z_s^1 - Z_s^2\right) \mathrm{d}W_s\right\rangle\right|.
\end{aligned}
$$

Using the following fact: if $b, a_i \geq 0$ and $b \leq \sum_{i=1}^n a_i$, then $b^p \leq \sum_{i=1}^n a_i^p$ for any $p \in (0,1)$, we have

$$
\begin{aligned}
& \left(\int_\tau^T \left\|Z_s^1 - Z_s^2\right\|^2 \mathrm{d}s\right)^{\frac{p}{2}} \\
\leq\ & c_p\left(|\xi_1 - \xi_2|^p + \left(\int_0^\infty \left|\phi_1(s) - \phi_2(s)\right| \mathrm{d}s\right)^p + \left(\int_0^\infty \left\|\varphi_1(s) - \varphi_2(s)\right\|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right) \\
+\ & c_p\left(\int_\tau^T \alpha(s)\mathrm{d}s + \int_\tau^T \beta^2(s)\mathrm{d}s\right)^{\frac{p}{2}} \sup_{s\in[\tau,T]} \left|Y_s^1 - Y_s^2\right|^p + c_p \sup_{s\in[\tau,T]} \left|Y_s^1 - Y_s^2\right|^p \\
+\ & c_p\left(\left|\int_\tau^T \left\langle Y_s^1 - Y_s^2, \left(g\left(s, Y_s^1\right) - g\left(s, Y_s^2\right) + \varphi_1(s) - \varphi_2(s)\right) \mathrm{d}B_s\right\rangle\right|^{\frac{p}{2}}\right) \\
+\ & c_p\left(\left|\int_\tau^T \left\langle Y_s^1 - Y_s^2, \left(Z_s^1 - Z_s^2\right) \mathrm{d}W_s\right\rangle\right|^{\frac{p}{2}}\right),
\end{aligned} \tag{4}
$$

where c_p is a positive constant depending only on p. By the Burkholder-Davis-Gundy inequality, we get

$$
\begin{aligned}
c_pE\left[\left|\int_\tau^T \left\langle Y_s^1 - Y_s^2, \left(Z_s^1 - Z_s^2\right) \mathrm{d}W_s\right\rangle\right|^{\frac{p}{2}}\right] &\leq d_pE\left[\left(\int_\tau^T \left|Y_s^1 - Y_s^2\right|^2 \left\|Z_s^1 - Z_s^2\right\|^2 \mathrm{d}s\right)^{\frac{p}{4}}\right] \\
&\leq d_pE\left[\sup_{s\in[\tau,T]} \left|Y_s^1 - Y_s^2\right|^{\frac{p}{2}} \left(\int_\tau^T \left\|Z_s^1 - Z_s^2\right\|^2 \mathrm{d}s\right)^{\frac{p}{4}}\right]
\end{aligned}
$$

and thus

$$
c_pE\left[\left|\int_\tau^T \left\langle Y_s^1 - Y_s^2, \left(Z_s^1 - Z_s^2\right) \mathrm{d}W_s\right\rangle\right|^{\frac{p}{2}}\right] \leq \frac{1}{2}E\left[\left(\int_\tau^T \left\|Z_s^1 - Z_s^2\right\|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] + \frac{d_p^2}{2}E\left[\sup_{s\in[\tau,T]} \left|Y_s^1 - Y_s^2\right|^p\right], \tag{5}
$$

where d_p is a positive constant depending only on p. Applying the Burkholder-Davis-Gundy inequality again, we can obtain

$$\begin{aligned}
& c_pE\left[\left|\int_\tau^T \left\langle Y_s^1 - Y_s^2, \left(g\left(s,Y_s^1\right) - g\left(s,Y_s^2\right) + \varphi_1(s) - \varphi_2(s)\right)\mathrm{d}B_s\right\rangle\right|^{\frac{p}{2}}\right] \\
\leq\ & d_pE\left[\left(\int_\tau^T \left|Y_s^1 - Y_s^2\right|^2 \left\|g\left(s,Y_s^1\right) - g\left(s,Y_s^2\right) + \varphi_1(s) - \varphi_2(s)\right\|^2 \mathrm{d}s\right)^{\frac{p}{4}}\right] \\
\leq\ & k_p\left(\int_\tau^T \beta^2(s)\mathrm{d}s\right)^{\frac{p}{4}} E\left[\sup_{s\in[\tau,T]}\left|Y_s^1 - Y_s^2\right|^p\right] + k_pE\left[\sup_{s\in[\tau,T]}\left|Y_s^1 - Y_s^2\right|^p\right] \\
+\ & k_pE\left[\left(\int_0^\infty \left\|\varphi_1(s) - \varphi_2(s)\right\|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right],
\end{aligned} \tag{6}$$

where k_p is a positive constant depending only on p. From (4), (5) and (6), we have

$$\begin{aligned}
& E\left[\left(\int_\tau^T \left\|Z_s^1 - Z_s^2\right\|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \\
\leq\ & C\left(E[|\xi_1 - \xi_2|^p] + E\left[\left(\int_0^\infty \left|\phi_1(s) - \phi_2(s)\right| \mathrm{d}s\right)^p\right] + E\left[\left(\int_0^\infty \left\|\varphi_1(s) - \varphi_2(s)\right\|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right]\right) \\
+\ & C\left(1 + l_{(\tau,T]}\right)E\left[\sup_{s\in[\tau,T]}\left|Y_s^1 - Y_s^2\right|^p\right],
\end{aligned} \tag{7}$$

where C is a positive constant depending only on p.

On the other hand, we define the filtration $\{\zeta_t; \tau \leq t \leq T\}$ by

$$\zeta_t := \mathcal{F}_{0,t}^W \vee \mathcal{F}_{0,\infty}^B.$$

Obviously, $\left\{\int_\tau^t \left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s; \tau \leq t \leq T\right\}$ is a ζ_t-martingale. Indeed, from the definition of martingale, we only need to prove that $E\left[\left|\int_\tau^t \left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s\right|\right] < \infty$. Applying Hölder's inequality and the Burkholder-Davis-Gundy inequality, we have

$$E\left[\left|\int_\tau^t \left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s\right|\right] \leq \left(E\left[\left|\int_\tau^t \left(Z_s^1 - Z_s^2\right)\mathrm{d}W_s\right|^p\right]\right)^{\frac{1}{p}} \leq \left(E\left[\left(\int_0^\infty \left\|Z_s^1 - Z_s^2\right\|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}} < \infty.$$

Thus, it follows that

$$\begin{aligned}
& Y_t^1 - Y_t^2 \\
=\ & E\left[(\xi_1 - \xi_2) + \int_t^T \left(f\left(s,Y_s^1,Z_s^1\right) - f\left(s,Y_s^2,Z_s^2\right) + \phi_1(s) - \phi_2(s)\right)\mathrm{d}s|\zeta_t\right] \\
+\ & E\left[\int_t^T \left(g\left(s,Y_s^1\right) - g\left(s,Y_s^2\right) + \varphi_1(s) - \varphi_2(s)\right)\mathrm{d}B_s|\zeta_t\right].
\end{aligned}$$

Applying Doob's inequality, we have

$$\begin{aligned}
& E\left[\sup_{t\in[\tau,T]}\left(E\left[|\xi_1 - \xi_2| + \int_\tau^T \left|f\left(s,Y_s^1,Z_s^1\right) - f\left(s,Y_s^2,Z_s^2\right) + \phi_1(s) - \phi_2(s)\right|\mathrm{d}s|\zeta_t\right]\right)^p\right] \\
\leq\ & \left(\tfrac{p}{p-1}\right)^p E\left[\left(|\xi_1 - \xi_2| + \int_\tau^T \left|f\left(s,Y_s^1,Z_s^1\right) - f\left(s,Y_s^2,Z_s^2\right) + \phi_1(s) - \phi_2(s)\right|\mathrm{d}s\right)^p\right] \\
\leq\ & D_p\left(E[|\xi_1 - \xi_2|^p] + E\left[\left(\int_0^\infty \left|\phi_1(s) - \phi_2(s)\right|\mathrm{d}s\right)^p\right] + E\left[\left(\int_\tau^T \left|f\left(s,Y_s^1,Z_s^1\right) - f\left(s,Y_s^2,Z_s^2\right)\right|\mathrm{d}s\right)^p\right]\right),
\end{aligned}$$

where D_p is a positive constant depending only on p. By the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}
& E\left[\sup_{t\in[\tau,T]}\left(E\left[\left|\int_t^T\left(g\left(s,Y_s^1\right)-g\left(s,Y_s^2\right)+\phi_1(s)-\phi_2(s)\right)\mathrm{d}B_s\right||\zeta_t\right]\right)^p\right] \\
\leq\ & 2^{p-1}E\left[\sup_{t\in[\tau,T]}\left(E\left[\left|\int_\tau^T\left(g\left(s,Y_s^1\right)-g\left(s,Y_s^2\right)+\phi_1(s)-\phi_2(s)\right)\mathrm{d}B_s\right||\zeta_t\right]\right)^p\right] \\
+\ & 2^{p-1}E\left[\sup_{t\in[\tau,T]}\left|\int_\tau^t\left(g\left(s,Y_s^1\right)-g\left(s,Y_s^2\right)+\phi_1(s)-\phi_2(s)\right)\mathrm{d}B_s\right|^p\right] \\
\leq\ & K_pE\left[\left(\int_\tau^T\left\|g\left(s,Y_s^1\right)-g\left(s,Y_s^2\right)+\phi_1(s)-\phi_2(s)\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right] \\
\leq\ & 2^{\frac{p}{2}}K_p\left(E\left[\left(\int_\tau^T\left\|g\left(s,Y_s^1\right)-g\left(s,Y_s^2\right)\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]+E\left[\left(\int_0^\infty\left\|\phi_1(s)-\phi_2(s)\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\right),
\end{aligned}$$

where K_p is a positive constant depending only on p. Thus, we can deduce that

$$\begin{aligned}
& E\left[\sup_{t\in[\tau,T]}\left|Y_t^1-Y_t^2\right|^p\right] \\
\leq\ & 2^{p-1}E\left[\sup_{t\in[\tau,T]}\left|E\left[(\xi_1-\xi_2)+\int_t^T\left(f\left(s,Y_s^1,Z_s^1\right)-f\left(s,Y_s^2,Z_s^2\right)+\phi_1(s)-\phi_2(s)\right)\mathrm{d}s|\zeta_t\right]\right|^p\right] \\
+\ & 2^{p-1}E\left[\sup_{t\in[\tau,T]}\left|E\left[\int_t^T\left(g\left(s,Y_s^1\right)-g\left(s,Y_s^2\right)+\varphi_1(s)-\varphi_2(s)\right)\mathrm{d}B_s|\zeta_t\right]\right|^p\right] \\
\leq\ & 2^{p-1}E\left[\sup_{t\in[\tau,T]}\left(E\left[|\xi_1-\xi_2|+\int_\tau^T\left|f\left(s,Y_s^1,Z_s^1\right)-f\left(s,Y_s^2,Z_s^2\right)+\phi_1(s)-\phi_2(s)\right|\mathrm{d}s|\zeta_t\right]\right)^p\right] \\
+\ & 2^{p-1}E\left[\sup_{t\in[\tau,T]}\left(E\left[\left|\int_t^T\left(g\left(s,Y_s^1\right)-g\left(s,Y_s^2\right)+\phi_1(s)-\phi_2(s)\right)\mathrm{d}B_s\right||\zeta_t\right]\right)^p\right] \\
\leq\ & L_pE\left[|\xi_1-\xi_2|^p+\left(\int_0^\infty\left|\phi_1(s)-\phi_2(s)\right|\mathrm{d}s\right)^p+\left(\int_0^\infty\left\|\varphi_1(s)-\varphi_2(s)\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right] \\
+\ & L_pE\left[\left(\int_\tau^T\left|f\left(s,Y_s^1,Z_s^1\right)-f\left(s,Y_s^2,Z_s^2\right)\right|\mathrm{d}s\right)^p\right] \\
+\ & L_pE\left[\left(\int_\tau^T\left\|g\left(s,Y_s^1\right)-g\left(s,Y_s^2\right)\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right],
\end{aligned} \tag{8}$$

where L_p is a positive constant depending only on p. From the Lipshcitz assumption (A.2) on f and g, we have

$$\begin{aligned}
& E\left[\left(\int_\tau^T\left|f\left(s,Y_s^1,Z_s^1\right)-f\left(s,Y_s^2,Z_s^2\right)\right|\mathrm{d}s\right)^p\right] \\
\leq\ & E\left[\left(\int_\tau^T\left(\alpha(s)\left|Y_s^1-Y_s^2\right|+\beta(s)\left\|Z_s^1-Z_s^1\right\|\right)\mathrm{d}s\right)^p\right] \\
\leq\ & M_p\left(\int_\tau^T\alpha(s)\mathrm{d}s\right)^pE\left[\sup_{s\in[\tau,T]}\left|Y_s^1-Y_s^2\right|^p\right] \\
+\ & M_p\left(\int_\tau^T\beta^2(s)\mathrm{d}s\right)^{\frac{p}{2}}E\left[\left(\int_\tau^T\left\|Z_s^1-Z_s^2\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]
\end{aligned} \tag{9}$$

and

$$\begin{aligned} & E\left[\left(\int_\tau^T \left\|g\left(s,Y_s^1\right)-g\left(s,Y_s^2\right)\right\|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \\ \leq\ & E\left[\left(\int_\tau^T \beta^2(s)\left|Y_s^1-Y_s^2\right|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \\ \leq\ & \left(\int_\tau^T \beta^2(s)\mathrm{d}s\right)^{\frac{p}{2}} E\left[\sup_{s\in[\tau,T]}\left|Y_s^1-Y_s^2\right|^p\right], \end{aligned} \tag{10}$$

where M_p is a positive constant depending only on p. From (8), (9) and (10), we have

$$\begin{aligned} & E\left[\sup_{s\in[\tau,T]}\left|Y_s^1-Y_s^2\right|^p\right] \\ \leq\ & C'E\left[|\xi_1-\xi_2|^p+\left(\int_0^\infty\left|\phi_1(s)-\phi_2(s)\right|\mathrm{d}s\right)^p+\left(\int_0^\infty\left\|\varphi_1(s)-\varphi_2(s)\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right] \\ +\ & C'l_{(\tau,T]}E\left[\sup_{s\in[\tau,T]}\left|Y_s^1-Y_s^2\right|^p+\left(\int_\tau^T\left\|Z_s^1-Z_s^2\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right], \end{aligned} \tag{11}$$

where C' is a positive constant depending only on p.

Combining (7) with (11), we get

$$\begin{aligned} & E\left[\sup_{s\in[\tau,T]}\left|Y_s^1-Y_s^2\right|^p+\left(\int_\tau^T\left\|Z_s^1-Z_s^2\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right] \\ \leq\ & C_pE\left[|\xi_1-\xi_2|^p+\left(\int_0^\infty\left|\phi_1(s)-\phi_2(s)\right|\mathrm{d}s\right)^p+\left(\int_0^\infty\left\|\varphi_1(s)-\varphi_2(s)\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right] \\ +\ & C_pl_{(\tau,T]}E\left[\sup_{s\in[\tau,T]}\left|Y_s^1-Y_s^2\right|^p+\left(\int_\tau^T\left\|Z_s^1-Z_s^2\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right], \end{aligned}$$

where C_p is a positive constant depending only on p. The proof of Lemma 3.1 is complete.

Proof of Theorem 3.1. Let $\xi^n := (\xi\wedge n)\vee(-n)$ and $f_n(t,y,z) := f(t,y,z)-f(t,0,0)+h_n\left(f(t,0,0)\right)$, $g_n(t,y) := g(t,y)-g(t,0)+h_n(g(t,0))$ where $h_n(f(t,0,0)) := \frac{f(t,0,0)ne^{-t}}{|f(t,0,0)|\vee(ne^{-t})}$, $h_n(g(t,0)) := \frac{g(t,0)ne^{-t}}{\|g(t,0)\|\vee(ne^{-t})}$. It is easy to check that for each n, the functions f_n and g_n satisfy (A.0), (A.1) and (A.2). Then by Lemma 2.1, BDSDE

$$Y_t^n = \xi^n + \int_t^\infty f_n\left(s,Y_s^n,Z_s^n\right)\mathrm{d}s + \int_t^\infty g_n(s,Y_s^n)\mathrm{d}B_s - \int_t^\infty Z_s^n\mathrm{d}W_s$$

has a unique solution $(Y^n,Z^n)\in\mathcal{S}^2(\mathbb{R}^l)\times\mathcal{L}^2(\mathbb{R}^{l\times d})$.

Since

$$\left(\int_0^\infty\alpha(s)\mathrm{d}s+\int_0^\infty\beta^2(s)\mathrm{d}s\right)^{\frac{p}{2}}+\left(\int_0^\infty\alpha(s)\mathrm{d}s\right)^p+\left(\int_0^\infty\beta^2(s)\mathrm{d}s\right)^{\frac{p}{4}}<\infty,$$

we can choose a strictly increasing sequence $0=t_0<t_1<\cdots<t_N<t_{N+1}=\infty$, such that

$$l_{(t_i,t_{i+1}]}\leq\frac{1}{2C_p},\ \ i=0,1,2,\cdots,N,$$

where $\alpha(\cdot)$, $\beta(\cdot)$ are the same functions defined in assumption (A.2) and

$$1_{(t_i,t_{i+1}]}=\left(\int_{t_i}^{t_{i+1}}\alpha(s)\mathrm{d}s+\int_{t_i}^{t_{i+1}}\beta^2(s)\mathrm{d}s\right)^{\frac{p}{2}}+\left(\int_{t_i}^{t_{i+1}}\alpha(s)\mathrm{d}s\right)^p+\left(\int_{t_i}^{t_{i+1}}\beta^2(s)\mathrm{d}s\right)^{\frac{p}{4}}.$$

Applying Lemma 3.1, we have

$$
\begin{aligned}
& E\left[\sup_{s\in[t_i,t_{i+1}]}\left|Y_s^{m+n}-Y_s^n\right|^p+\left(\int_{t_i}^{t_{i+1}}\left\|Z_s^{m+n}-Z_s^n\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
\leq\ & C_pE\left[\left|Y_{t_{i+1}}^{m+n}-Y_{t_{i+1}}^n\right|^p\right]\\
+\ & C_pE\left[\left(\int_0^\infty\left|h_{n+m}(f(s,0,0))-h_n(f(s,0,0))\right|\mathrm{d}s\right)^p\right]\\
+\ & C_pE\left[\left(\int_0^\infty\left\|h_{n+m}(g(s,0))-h_n(g(s,0))\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
+\ & \tfrac{1}{2}E\left[\sup_{s\in[t_i,t_{i+1}]}\left|Y_s^{m+n}-Y_s^n\right|^p+\left(\int_{t_i}^{t_{i+1}}\left\|Z_s^{m+n}-Z_s^n\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right].
\end{aligned}
$$

Thus

$$
\begin{aligned}
& E\left[\sup_{s\in[t_i,t_{i+1}]}\left|Y_s^{m+n}-Y_s^n\right|^p+\left(\int_{t_i}^{t_{i+1}}\left\|Z_s^{m+n}-Z_s^n\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
\leq\ & 2C_pE\left[\left|Y_{t_{i+1}}^{m+n}-Y_{t_{i+1}}^n\right|^p\right]\\
+\ & 2C_pE\left[\left(\int_0^\infty\left|h_{n+m}(f(s,0,0))-h_n(f(s,0,0))\right|\mathrm{d}s\right)^p\right]\\
+\ & 2C_pE\left[\left(\int_0^\infty\left\|h_{n+m}(g(s,0))-h_n(g(s,0))\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
\leq\ & 2C_pE\left[\sup_{s\in[t_{i+1},t_{i+2}]}\left|Y_s^{m+n}-Y_s^n\right|^p+\left(\int_{t_{i+1}}^{t_{i+2}}\left\|Z_s^{m+n}-Z_s^n\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
+\ & 2C_pE\left[\left(\int_0^\infty\left|h_{n+m}(f(s,0,0))-h_n(f(s,0,0))\right|\mathrm{d}s\right)^p\right]\\
+\ & 2C_pE\left[\left(\int_0^\infty\left\|h_{n+m}(g(s,0))-h_n(g(s,0))\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right],\quad i=0,1,2,\cdots,N-1.
\end{aligned}
\tag{12}
$$

In particulary, we have

$$
\begin{aligned}
& E\left[\sup_{s\geq t_N}\left|Y_s^{m+n}-Y_s^n\right|^p+\left(\int_{t_N}^{\infty}\left\|Z_s^{m+n}-Z_s^n\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
\leq\ & 2C_pE\left[|\xi^{m+n}-\xi^n|^p\right]\\
+\ & 2C_pE\left[\left(\int_0^\infty\left|h_{n+m}(f(s,0,0))-h_n(f(s,0,0))\right|\mathrm{d}s\right)^p\right]\\
+\ & 2C_pE\left[\left(\int_0^\infty\left\|h_{n+m}(g(s,0))-h_n(g(s,0))\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right].
\end{aligned}
\tag{13}
$$

From (12) and (13), it follows that

$$
\begin{aligned}
& E\left[\sup_{s\geq 0}\left|Y_s^{n+m}-Y_s^n\right|^p+\left(\int_0^{\infty}\left\|Z_s^{n+m}-Z_s^n\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
\leq\ & \sum_{i=0}^N E\left[\sup_{s\in[t_i,t_{i+1}]}\left|Y_s^{m+n}-Y_s^n\right|^p+\left(\int_{t_i}^{t_{i+1}}\left\|Z_s^{m+n}-Z_s^n\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
\leq\ & (2C_p+(2C_p)^2+\cdots+(2C_p)^{N+1})E\left[|\xi^{m+n}-\xi^n|^p\right]\\
+\ & (N+1)(2C_p+(2C_p)^2+\cdots+(2C_p)^{N+1})E\left[\left(\int_0^\infty\left|h_{n+m}(f(s,0,0))-h_n(f(s,0,0))\right|\mathrm{d}s\right)^p\right]\\
+\ & (N+1)(2C_p+(2C_p)^2+\cdots+(2C_p)^{N+1})E\left[\left(\int_0^\infty\left\|h_{n+m}(g(s,0))-h_n(g(s,0))\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
\leq\ & \overline{C}E\left[|\xi^{m+n}-\xi^n|^p\right]\\
+\ & \overline{C}E\left[\left(\int_0^\infty\left|h_{n+m}(f(s,0,0))-h_n(f(s,0,0))\right|\mathrm{d}s\right)^p\right]\\
+\ & \overline{C}E\left[\left(\int_0^\infty\left\|h_{n+m}(g(s,0))-h_n(g(s,0))\right\|^2\mathrm{d}s\right)^{\frac{p}{2}}\right],
\end{aligned}
\tag{14}
$$

where $\overline{C} = (N+1)(2C_p + (2C_p)^2 + \cdots + (2C_p)^{N+1})$. The right-hand side of Inequality (14) clearly tends to 0, as $n \to \infty$, uniformly in m, so we have a Cauchy sequence and the limit is a solution to BDSDE (3). Let us consider (Y, Z) and (Y', Z') to be two solutions of BDSDE (3). In a similar manner of the proof of Inequality (14), we can obtain

$$E\left[\sup_{s\geq 0}\left|Y_s - Y_s'\right|^p + \left(\int_0^\infty \left\|Z_s - Z_s'\right\|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] \leq 0.$$

Thus, $(Y, Z) = (Y', Z')$. The proof of Theorem 3.1 is complete.

Remark 3.1 If $f(t, 0, 0) \equiv 0$ and $g(t, 0) \equiv 0$, then by Theorem 3.1, we have: Under the assumptions (A.0) and (A.2), for each given $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P, \mathbb{R}^l)$, BDSDE (3) has a unique solution $(Y, Z) \in \mathcal{S}(\mathbb{R}^l) \times \mathcal{L}(\mathbb{R}^{l\times d})$.

4. Comparison Theorem

In this section, we obtain the comparison theorem for 1-dimensional infinite time interval BDSDEs in L^p.

Let $\xi_1, \xi_2 \in L^p(\Omega, \mathcal{F}, P, \mathbb{R})$, $(Y^1, Z^1), (Y^2, Z^2) \in \mathcal{S}^p(\mathbb{R}) \times \mathcal{L}^p(\mathbb{R}^d)$ satisfy the following BDSDEs

$$Y_t^1 = \xi_1 + \int_t^\infty f^1\left(s, Y_s^1, Z_s^1\right)\mathrm{d}s + \int_t^\infty g\left(s, Y_s^1\right)\mathrm{d}B_s - \int_t^\infty Z_s^1 \mathrm{d}W_s \tag{15}$$

and

$$Y_t^2 = \xi_2 + \int_t^\infty f^2\left(s, Y_s^2, Z_s^2\right)\mathrm{d}s + \int_t^\infty g\left(s, Y_s^2\right)\mathrm{d}B_s - \int_t^\infty Z_s^2 \mathrm{d}W_s, \tag{16}$$

respectively. Furthermore, we assume that

(A.4) $\xi_1 \leq \xi_2$, a.s., $f^1(t, 0, 0) \leq f^2(t, 0, 0)$, a.s., $f^1(t, y, z) - f^1(t, 0, 0) \leq f^2(t, y, z) - f^2(t, 0, 0)$, a.s., $\forall (t, y, z) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d$;

(A.5) There exists some constant $M > 0$, such that $\alpha(t) \leq M$ and $\beta(t) \leq M$, $\forall t \in \mathbb{R}_+$.

Then we have the following comparison theorem.

Theorem 4.1 (Comparison Theorem) Suppose that BDSDEs (15) and (16) satisfy the conditions of Theorem 3.1. Let (Y^1, Z^1) and (Y^2, Z^2) be the solutions of BDSDEs (15) and (16), respectively. If (A.4)and (A.5) hold, then $Y_t^1 \leq Y_t^2$, a.s., $\forall t \in \mathbb{R}_+$.

Proof. The main idea comes from Theorem 3.1 in Shi et al. [18]. Let $\xi_1^n := (\xi_1 \wedge n) \vee (-n)$, $\xi_2^n := (\xi_2 \wedge n) \vee (-n)$ and $f_n^1(t, y, z) := f^1(t, y, z) - f^1(t, 0, 0) + h_n(f^1(t, 0, 0))$, $f_n^2(t, y, z) := f^2(t, y, z) - f^2(t, 0, 0) + h_n(f^2(t, 0, 0))$, $g_n(t, y) := g(t, y) - g(t, 0) + h_n(g(t, 0))$ where $h_n(f^1(t, 0, 0)) := \frac{f^1(t,0,0)ne^{-t}}{|f^1(t,0,0)|\vee(ne^{-t})}$, $h_n(f^2(t, 0, 0)) := \frac{f^2(t,0,0)ne^{-t}}{|f^2(t,0,0)|\vee(ne^{-t})}$, $h_n(g(t, 0)) := \frac{g(t,0)ne^{-t}}{|g(t,0)|\vee(ne^{-t})}$. By Lemma 2.1, we know that: For each n, BDSDEs

$$Y_t^{1,n} = \xi_1^n + \int_t^\infty f_n^1\left(s, Y_s^{1,n}, Z_s^{1,n}\right)\mathrm{d}s + \int_t^\infty g_n\left(s, Y_s^{1,n}\right)\mathrm{d}B_s - \int_t^\infty Z_s^{1,n} \mathrm{d}W_s$$

and

$$Y_t^{2,n} = \xi_2^n + \int_t^\infty f_n^2\left(s, Y_s^{2,n}, Z_s^{2,n}\right)\mathrm{d}s + \int_t^\infty g_n\left(s, Y_s^{2,n}\right)\mathrm{d}B_s - \int_t^\infty Z_s^{2,n} \mathrm{d}W_s$$

have the unique solutions in $\mathcal{S}^2(\mathbb{R}) \times \mathcal{L}^2(\mathbb{R}^d)$, denoted by $(Y^{1,n}, Z^{1,n})$ and $(Y^{2,n}, Z^{2,n})$, respectively.

From (A.4), we have $\xi_1^n \le \xi_2^n$, a.s., $f_n^1(t,y,z) \le f_n^2(t,y,z)$, a.s., $\forall (t,y,z) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d$. Now we prove that $Y_t^{1,n} \le Y_t^{2,n}$, a.s., $\forall t \in \mathbb{R}_+$. Obviously, $\left(Y^{1,n} - Y^{2,n}, Z^{1,n} - Z^{2,n}\right)$ satisfies the following BDSDE

$$\begin{aligned} Y_t^{1,n} - Y_t^{2,n} &= \xi_1^n - \xi_2^n + \int_t^\infty \left[f_n^1\left(s, Y_s^{1,n}, Z_s^{1,n}\right) - f_n^2\left(s, Y_s^{2,n}, Z_s^{2,n}\right)\right] \mathrm{d}s \\ &+ \int_t^\infty \left[g_n\left(s, Y_s^{1,n}\right) - g_n\left(s, Y_s^{2,n}\right)\right] \mathrm{d}B_s - \int_t^\infty \left(Z_s^{1,n} - Z_s^{2,n}\right) \mathrm{d}W_s, \quad 0 \le t \le \infty. \end{aligned}$$

Applying Lemma 2.2 to $\left|\left(Y_t^{1,n} - Y_t^{2,n}\right)^+\right|^2$, we get

$$\begin{aligned} &\left|\left(Y_t^{1,n} - Y_t^{2,n}\right)^+\right|^2 + \int_t^\infty 1_{\{Y_s^{1,n} \ge Y_s^{2,n}\}} \left|Z_s^{1,n} - Z_s^{2,n}\right|^2 \mathrm{d}s \\ = &\left|\left(\xi_1^n - \xi_2^n\right)^+\right|^2 + 2\int_t^\infty \left(Y_s^{1,n} - Y_s^{2,n}\right)^+ \left[f_n^1\left(s, Y_s^{1,n}, Z_s^{1,n}\right) - f_n^2\left(s, Y_s^{2,n}, Z_s^{2,n}\right)\right] \mathrm{d}s \\ + &\int_t^\infty 1_{\{Y_s^{1,n} \ge Y_s^{2,n}\}} \left|g_n\left(s, Y_s^{1,n}\right) - g_n\left(s, Y_s^{2,n}\right)\right|^2 \mathrm{d}s \\ - &\ 2\int_t^\infty \left(Y_s^{1,n} - Y_s^{2,n}\right)^+ \left(Z_s^{1,n} - Z_s^{2,n}\right) \mathrm{d}W_s \\ + &\ 2\int_t^\infty \left(Y_s^{1,n} - Y_s^{2,n}\right)^+ \left(g_n\left(s, Y_s^{1,n}\right) - g_n\left(s, Y_s^{2,n}\right)\right) \mathrm{d}B_s. \end{aligned} \tag{17}$$

From (A.4), we have $\xi_2^2 - \xi_1^n \ge 0$, a.s., so

$$E\left[\left|\left(\xi_1^n - \xi_2^n\right)^+\right|^2\right] = 0. \tag{18}$$

Since $(Y^{1,n}, Z^{1,n})$ and $(Y^{2,n}, Z^{2,n})$ are in $\mathcal{S}^2(\mathbb{R}) \times \mathcal{L}^2(\mathbb{R}^d)$, it easily follows that

$$E\left[\int_t^\infty \left(Y_s^{1,n} - Y_s^{2,n}\right)^+ \left(Z_s^{1,n} - Z_s^{2,n}\right) \mathrm{d}W_s\right] = 0, \tag{19}$$

$$E\left[\int_t^\infty \left(Y_s^{1,n} - Y_s^{2,n}\right)^+ \left(g_n\left(s, Y_s^{1,n}\right) - g_n\left(s, Y_s^{2,n}\right)\right) \mathrm{d}B_s\right] = 0. \tag{20}$$

Let

$$\begin{aligned} \triangle &= 2\int_t^\infty \left(Y_s^{1,n} - Y_s^{2,n}\right)^+ \left[f_n^1\left(s, Y_s^{1,n}, Z_s^{1,n}\right) - f_n^2\left(s, Y_s^{2,n}, Z_s^{2,n}\right)\right] \mathrm{d}s \\ &= 2\int_t^\infty \left(Y_s^{1,n} - Y_s^{2,n}\right)^+ \left[f_n^1\left(s, Y_s^{1,n}, Z_s^{1,n}\right) - f_n^2\left(s, Y_s^{1,n}, Z_s^{1,n}\right)\right] \mathrm{d}s \\ &\quad +2\int_t^\infty \left(Y_s^{1,n} - Y_s^{2,n}\right)^+ \left[f_n^2\left(s, Y_s^{1,n}, Z_s^{1,n}\right) - f_n^2\left(s, Y_s^{2,n}, Z_s^{2,n}\right)\right] \mathrm{d}s \\ &= \triangle_1 + \triangle_2, \end{aligned}$$

where

$$\triangle_1 = 2\int_t^\infty \left(Y_s^{1,n} - Y_s^{2,n}\right)^+ \left[f_n^1\left(s, Y_s^{1,n}, Z_s^{1,n}\right) - f_n^2\left(s, Y_s^{1,n}, Z_s^{1,n}\right)\right] \mathrm{d}s \le 0,$$

$$\triangle_2 = 2\int_t^\infty \left(Y_s^{1,n} - Y_s^{2,n}\right)^+ \left[f_n^2\left(s, Y_s^{1,n}, Z_s^{1,n}\right) - f_n^2\left(s, Y_s^{2,n}, Z_s^{2,n}\right)\right] \mathrm{d}s.$$

From (A.2) and (A.5), it follows that

$$\begin{aligned} \triangle &\le \triangle_2 \le 2M\int_t^\infty \left(Y_s^{1,n} - Y_s^{2,n}\right)^+ \left(\left|Y_s^{1,n} - Y_s^{2,n}\right| + \left|Z_s^{1,n} - Z_s^{2,n}\right|\right) \mathrm{d}s \\ &\le 2M(1+M)\int_t^\infty \left|\left(Y_s^{1,n} - Y_s^{2,n}\right)^+\right|^2 \mathrm{d}s + \frac{1}{2}\int_t^\infty 1_{\{Y_s^{1,n} \ge Y_s^{2,n}\}} \left|Z_s^{1,n} - Z_s^{2,n}\right|^2 \mathrm{d}s. \end{aligned} \tag{21}$$

Using (A.2) and (A.5) again, we deduce

$$\int_t^\infty 1_{\{Y_s^{1,n} \ge Y_s^{2,n}\}} \left|g_n\left(s, Y_s^{1,n}\right) - g_n\left(s, Y_s^{2,n}\right)\right|^2 \mathrm{d}s \le M^2 \int_t^\infty \left|\left(Y_s^{1,n} - Y_s^{2,n}\right)^+\right|^2 \mathrm{d}s. \tag{22}$$

Taking expectation on both side of Equation (17) and noting Equations (18)-(22), we get

$$E\left[\left|\left(Y_t^{1,n}-Y_t^{2,n}\right)^+\right|^2\right] \leq \left(3M^2+2M\right)\int_t^\infty E\left[\left|\left(Y_s^{1,n}-Y_s^{2,n}\right)^+\right|^2\right]ds.$$

By Gronwall's inequality, it follows that

$$E\left[\left|\left(Y_t^{1,n}-Y_t^{2,n}\right)^+\right|^2\right] = 0, \quad \forall t \in \mathbb{R}_+.$$

That is, $Y_t^{1,n} \leq Y_t^{2,n}$, a.s., $\forall t \in \mathbb{R}_+$.

From the proof of Theorem 3.1, we know that

$$Y_t^{1,n} \to Y_t^1 \quad \text{in} \quad \mathcal{L}^p(\mathbb{R}), \quad \text{as} \quad n \to \infty$$

and

$$Y_t^{2,n} \to Y_t^2 \quad \text{in} \quad \mathcal{L}^p(\mathbb{R}), \quad \text{as} \quad n \to \infty.$$

Thus, $Y_t^1 \leq Y_t^2$, a.s., $\forall t \in \mathbb{R}_+$. The proof of Theorem 4.1 is complete.

Acknowledgements

The authors would like to thank the section editor and the anonymous referee for their constructive suggestions and valuable comments that greatly improved this paper.

References

[1] V. Bally, A. Matoussi, Weak solutions for SPDEs and backward doubly stochastic differential equations, J. Theor. Probab. 14 (2001) 125-164.
[2] P. Briand, B. Delyon, Y. Hu, E. Pardoux, L. Stoica, L^p solutions of backward stochastic differential equations, Stochastic Process. Appl. 108 (2003) 109-129.
[3] R. Buckdahn, J. Ma, Stochastic viscosity solutions for nonlinear stochastic partial differential equations I, Stochastic Process. Appl. 93 (2001) 181-204.
[4] R. Buckdahn, J. Ma, Stochastic viscosity solutions for nonlinear stochastic partial differential equations II, Stochastic Process. Appl. 93 (2001) 205-228.
[5] Z. Chen, B. Wang, Infinite time interval BSDEs and the convergence of g-martingales, J. Aust. Math. Soc. (Series A) 69 (2000) 187-211.
[6] R.W.R. Darling, E. Pardoux, Backward stochastic differential equations with random terminal time and applications to semilinear elliptic PDE, Ann. Probab. 25 (1997) 1135-1159.
[7] N. El Karoui, S. Peng, M.C. Quenez, Backward stochastic differential equations in finance, Math. Finance 7 (1997) 1-71 .
[8] Y. Hu, S. Peng, Solutions of backward-forward SDE, Probab. Theory Related Fields 103 (1995) 273-283.
[9] J.P. Lepeltier, J. San Martin, BSDEs with continuous coefficients, Statist. Probab. Lett. 32 (1997) 425-430.
[10] E. Pardoux, Generalized discontinuous BSDEs, In: N. El Karoui, L. Mazliak (Eds.), Backward Stochastic Differential Equations, In: Pitman Research Notes in Mathematics Series, Vol. 364, Longman, Harlow, pp. 207-219, 1997.
[11] E. Pardoux, BSDEs, weak convergence and homogenization of semilinear PDEs, Nonlinear Analysis, Differential Equations and Control (Montreal, QC, 1998), Kluwer Academic Publishers, Dordrecht, pp. 503-549.
[12] E. Pardoux, Stochastic partial differential equations, Fudan Lecture Notes, 2007.
[13] E. Pardoux, S. Peng, Adapted solution of a backward stochastic differential equation, Systems Control Lett. 14 (1990) 55-61.
[14] E. Pardoux, S. Peng, Backward doubly stochastic differential equations and systems of quasilinear SPDEs, Probab. Theory Related Fields 98 (1994) 209-227.
[15] S. Peng, A general stochastic maximum principle for optimal problems, SIAM J. Control Optim. 28 (1990) 966-979.
[16] S. Peng, Y. Shi, Infinite horizen forward-backward stochastic differential equations, Stochastic Process. Appl. 85 (2000) 75-92.
[17] S. Peng, Y. Shi, A type of time-symnetric foeward-backward stochastic differential equations, C.R. Math. Acad. Sci. Paris 336 (2003) 773-778.
[18] Y. Shi, Y. Gu, K. Liu, Comparison theorems of backward doubly stochastic differential equations and applications, Stoch. Ana. Appl. 23 (2005) 97-110.
[19] Q. Zhang, H. Zhao, Stationary solutions of SPDEs and infinite horizon BDSDEs, J. Funct. Anal. 252 (2007) 171-219.
[20] B. Zhu, B. Han, Backward doubly stochastic differential equations with infinite time horizon, Appl. Math. 57 (2012) 641-653.
[21] Z. Zong, L^p solutions of infinite time interval BSDEs and the corresponding g-expectations and g-martingales, Turk. J. Math. 37 (2013) 704-718.