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Periodic Points of Rational Inequality in a Complex Valued Metric Space

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Abstract. In this paper, we shall prove some periodic point theorems of rational inequality in complex valued metric spaces. The first result of this type was due to Sehgal[14] and his result was generalized by Guseman[5], Khanzanchi[6], Rhoades and Ray[2] and Murthy and Pathak[10].

1. Introduction

The Banach Fixed Point Theorem is a source of inspiration for the past and present researchers of mathematics and different branches of science and technology. Even in the 21^{st} century computer scientists, physicists, applied mathematicians, etc are trying to apply Banach Contraction Principle to serve the purpose of human beings daily life.

In real analysis and functional analysis, metric space theory is a pivoting tool for the applications of many concepts. The metric space is the most general space on which one can think about applications in real life situations of this century. The concept of a topological via a metric space or the concept of a normed linear space to a topological space via a metric space is always an interesting and challenging of proves mathematics among the mathematicians. Metric fixed point theory has a wide range of applications in dynamic programming problems, variational inequalities, solutions of nonlinear differential equations, fractal dynamics, dynamical system of mathematics as well as the launching of satellites in their appropriate orbits in the space, in medicine the most appropriate diagnosis of patients, in future medical emergencies by using simulation techniques and in mock exercises on the spread of disease, etc.

The study of new space discoveries in mathematics and their basic properties are always favorite topics of interest among the mathematical research community. In this context, the concept of 2-metric spaces, introduced initially by S. Gahler[13] in his series of papers and given a new thought of new dimensions for ordinary metric spaces. Since the metric for a pair of points is non-negative real, (i.e. $[0, \infty)$) it has wide

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range of this study. The concept of probabilistic metric spaces in which the probabilistic distance between two points is considered, it has given a new height and interest for the study to know more about stars in the universe. In a similar way, the study of fuzzy metric spaces was initially done by Grabiec [9] and Micehelic[8] in which the degree of agreement and disagreement were considered.

So far the study was done around the real numbers for e.g. Metric spaces, 2-Metric spaces, Normed linear spaces, Fuzzy metric spaces, Probabilistic metric spaces, etc. Let X be a non-empty set and let $d : X \times X \to R$, $\|.\| : N \to R$, $d : X \times X \times X \to R$ and $M(x, y, t) : X \times X \times [0, 1] \to [0, 1]$, $F : X \times X \longrightarrow [0, 1]$. It was quiet natural to ask "What happens if we replace R by some other sets which are not completely ordered sets like R?" This was answered by a few of them by introducing the cone metric space, the partially ordered metric space, the modular metric space and very recently, the complex valued metric space respectively by Huang and Zhang[7], Matthew [12], Chistyakov[15], Azam, Fisher and Khan [1].

Complex Valued Metric Space: Let X be a non-empty set and let $\rho : X \times X \to \mathbb{C}$, where $\mathbb{C}$ is a set of complex numbers in which ordering is not the same as in the set of real numbers. We recall some important definitions, lemmas and theorems for our further study of common fixed points in complex valued metric spaces.

Let $\mathbb{C}$ be a set of complex numbers and $\xi_1, \xi_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:
The elements $\xi_1, \xi_2 \in \mathbb{C}$ are partially ordered denoted by

$$\xi_1 \preceq \xi_2 \Rightarrow Re(\xi_1) < Re(\xi_2), Im(\xi_1) < Im(\xi_2),$$

or

$$\xi_1 \succeq \xi_2 \Rightarrow Re(\xi_1) < Re(\xi_2), Im(\xi_1) < Im(\xi_2),$$

Two elements $\xi_1, \xi_2 \in \mathbb{C}$, and

$$\xi_1 \preceq \xi_2 \quad (\text{or} \quad \xi_1 \succeq \xi_2)$$

If one of the following conditions holds:

(i) $Re(\xi_1) = Re(\xi_2), Im(\xi_1) = Im(\xi_2)$,

(ii) $Re(\xi_1) < Re(\xi_2), Im(\xi_1) < Im(\xi_2)$,
or $Re(\xi_1) > Re(\xi_2), Im(\xi_1) > Im(\xi_2)$,

(iii) $Re(\xi_1) < Re(\xi_2), Im(\xi_1) = Im(\xi_2)$,
or $Re(\xi_1) > Re(\xi_2), Im(\xi_1) = Im(\xi_2)$,

(iv) $Re(\xi_1) = Re(\xi_2), Im(\xi_1) < Im(\xi_2)$,
or $Re(\xi_1) = Re(\xi_2), Im(\xi_1) > Im(\xi_2)$

In particular,
$\xi_1 \precsim \xi_2 \quad (\text{or} \quad \xi_1 \succsim \xi_2)$, if $\xi_1 \neq \xi_2$ and one of $(ii), (iii)$ and (iv) is satisfied.

We will also write $\xi_1 \prec \xi_2 (\text{or} \quad \xi_1 \succ \xi_2)$, if (ii) is satisfied.

Note that $0 \preceq \xi_1 \precsim \xi_2 \Longrightarrow | \xi_1 | < | \xi_2 |$,

For all $\xi_1, \quad \xi_2, \quad \xi_3 \in \mathbb{C}$

$$\xi_1 \preceq \xi_2, \xi_2 \prec \xi_3 \Longrightarrow \xi_1 \prec \xi_2.$$

Definition 1.1. *Let X be a non-empty set. Suppose that the mapping $\rho : X \times X \to \mathbb{C}$, satisfies*

(CM_1) $0 \preceq \rho(x,y)$ *for all* $x, y \in X$ *and* $\rho(x,y) = 0$ *if and only if* $x = y$;

(CM_2) $\rho(x,y) = \rho(y,x)$ *for all* $x, y \in X$;

(CM_3) $\rho(x,y) \preceq \rho(x,z) + \rho(z,y)$ *for all* $x, y, z \in X$.

Then ρ is called a complex valued metric on X and (X, ρ) is called a complex valued metric space.

Remark: For various examples on complex valued metric spaces refer [1, 4, 16].
A point x in X is called an interior point of a set $A \subseteq X$ if there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x,r) = \{y \in X : \rho(x,y) \prec r\} \subseteq A.$$

A point x of X is called a limit point of A, if there exists $B(x,r)$ centered at x with radius r which contains at least one point of A other than x. i.e. $B(x,r) \cap A_{\sim x} \neq \phi$.

A subset G of X is said to be open if each point of G is an interior point of G. A subset B of X is said to be closed if each limit point of B is in B.

The family $F = \{B(x,r) : x \in X, 0 \prec r\}$ is a sub-basis for the Hausdroff topology on X.

A sequence $\{x_n\}$ of X is said to be a convergent sequence and converges to a point $x \in X$, if for a given $\epsilon \in \mathbb{C}$ with $\epsilon \succ 0$, there exists a positive integer n_0 such that $\rho(x_n, x) \prec \epsilon$ for all $n > n_0$.

A sequence $\{x_n\}$ of X is said to be a Cauchy sequence, if for a given $\epsilon \in \mathbb{C}$ with $\epsilon \succ 0$ there exists a positive inter n_0 such that $\rho(x_n, x_m) \prec \epsilon$ for all $m, n > n_0$.

A complex valued metric space (X, ρ) is said to be complete, if every Cauchy sequence in X is a convergent sequence.

Lemma 1.2. *Let (X, ρ) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $\mid \rho(x_n, x) \mid \to 0$ as $n \to \infty$.*

Lemma 1.3. *Let (X, ρ) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $\mid \rho(x_n, x_{n+m}) \mid \to 0$ as $m, n \to \infty$.*

2. Main Results

Theorem 2.1. *Let E and F be two self mappings of a complete complex metric space (X, ρ) such that there exists positive integers $p(x)$ and $q(x)$ such that for each $x, y \in X$,*

$$\rho(E^{p(x)}x, F^{q(y)}y) \preceq \frac{\alpha(\rho(x,y))\rho(x, E^{p(x)}x)\rho(y, F^{q(y)}y)}{\rho(x,y) + \rho(x, F^{q(y)}y) + \rho(y, E^{p(x)}x)} + \beta(\rho(x,y))\rho(x,y) \tag{1}$$

where $\alpha, \beta : \mathbb{C}_+ \to [0,1)$ such that for all $x, y \in X, \alpha(x) + \beta(x) < 1$.
Then E and F have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point of X and define the sequence $\{x_n\}$

$$x_n = \begin{cases} E^{p(x_{n-1})}x_{n-1}, & \text{when } n \text{ is odd,} \\ F^{q(x_{n-1})}x_{n-1}, & \text{when } n \text{ is even} \end{cases} \tag{2}$$

for $n = 1, 2, 3 \cdots$

If $x_{2n+1} = x_{2n+2}$, then $\{x_n\}$ is a Cauchy sequence.

Now suppose that $x_{2n+1} \neq x_{2n+2}$ for each $p(x) \neq q(y)$. Then

$$\begin{aligned}
\rho(x_{2n+1}, x_{2n+2}) &= \rho(E^{p(x_{2n})}x_{2n}, F^{q(x_{2n+1})}x_{2n+1}) \\
&\leq \frac{\alpha(\rho(x_{2n}, x_{2n+1}))\rho(x_{2n}, E^{p(x_{2n})}x_{2n})\rho(x_{2n+1}, F^{q(x_{2n+1})}x_{2n+1})}{\rho(x_{2n}, x_{2n+1}) + \rho(x_{2n}, F^{q(x_{2n+1})}x_{2n+1}) + \rho(x_{2n+1}, E^{p(x_{2n})}x_{2n})} \\
&\quad +\beta(\rho(x_{2n}, x_{2n+1}))\rho(x_{2n}, x_{2n+1}) \\
&\leq \frac{\alpha(\rho(x_{2n}, x_{2n+1}))\rho(x_{2n}, x_{2n+1})\rho(x_{2n+1}, x_{2n+2})}{\rho(x_{2n}, x_{2n+1}) + \rho(x_{2n}, x_{2n+2}) + \rho(x_{2n+1}, x_{2n+1})} \\
&\quad +\beta(\rho(x_{2n}, x_{2n+1}))\rho(x_{2n}, x_{2n+1}) \\
&\leq \alpha(\rho(x_{2n}, x_{2n+1}))\rho(x_{2n}, x_{2n+1}) + \beta(\rho(x_{2n}, x_{2n+1}))\rho(x_{2n}, x_{2n+1}),
\end{aligned}$$

since $\rho(x_{2n+1}, x_{2n+2}) \leq \rho(x_{2n}, x_{2n+2}) + \rho(x_{2n}, x_{2n+1}))$.

Or equivalently

$$\rho(x_{2n+1}, x_{2n+2}) \leq \delta(\rho(x_{2n}, x_{2n+1}))\rho(x_{2n}, x_{2n+1}), \tag{3}$$

where $\delta = \alpha + \beta < 1$.

Similarly, replacing x by x_{2n+2} and y by x_{2n+3}, we have

$$\rho(x_{2n+2}, x_{2n+3}) \leq \alpha(\rho(x_{2n+1}, x_{2n+2})) + \beta(\rho(x_{2n+1}, x_{2n+2}))\rho(x_{2n+1}, x_{2n+2}),$$

or equivalently

$$\rho(x_{2n+2}, x_{2n+3}) \leq \delta(\rho(x_{2n+1}, x_{2n+2}))\rho(x_{2n+1}, x_{2n+2}) \tag{4}$$

where $\delta = \alpha + \beta < 1$.
From (3) and (4), we have

$$\rho(x_n, x_{n+1}) \leq \delta(\rho(x_{n-1}, x_n))\rho(x_{n-1}, x_n)$$

for all $n \in N$, which implies that

$$\mid \rho(x_n, x_{n+1}) \mid \ \leq \ \mid \delta(\rho(x_{n-1}, x_n)) \mid\mid \rho(x_{n-1}, x_n) \mid \ \leq \ \mid \rho(x_{n-1}, x_n) \mid . \tag{5}$$

Therefore $\{\rho(x_{n-1}, x_n)\}_{n \in N}$ is monotonically decreasing and bounded below. Hence $\mid \rho(x_{n-1}, x_n) \mid \to d$ for some $d \geq 0$.

To prove that $d = 0$, we shall assume $d > 0$. Taking the limit as $n \to \infty$ in (5), we have

$$\mid \delta(\rho(x_{n-1}, x_n)) \mid \to 1.$$

Since $\delta \in \Delta$, $\mid \rho(x_{n-1}, x_n) \mid \to 0$, is a contradiction. Therefore, we have $d = 0$.

Now, we shall show that the sequence $\{x_n\}$ is a Cauchy sequence. It is easy and enough to show that $\{x_{2n}\}$ is a Cauchy sequence.

Since X is complete, every Cauchy sequence in X is convergent and converges to a point u (say) in X.

Suppose $F(u) \neq u$. Then from (1), we have

$$\begin{aligned}\rho(x_{2n+1}, F^{q(u)}u) &= \rho(E^{p(x_{2n})}x_{2n}, F^{q(u)}u)\\ &\leq \frac{\alpha(\rho(x_{2n}, u))\rho(x_{2n}, E^{p(x_{2n})}x_{2n})\rho(u, F^{q(u)}u)}{\rho(x_{2n}, u) + \rho(x_{2n}, F^{q(u)}u) + \rho(u, E^{p(x_{2n})}x_{2n})}\\ &\quad +\beta(\rho(x_{2n}, u))\rho(x_{2n}, u).\end{aligned}$$

Letting $n \to \infty$ in the above inequality, it follows that $| \rho(u, F^{q(u)}u) | \to 0$.

Thus $F^{q(u)}u = u$. Similarly we can show that $E^{p(u)}u = u$ and so u is a periodic point of E and F.

Now we shall show that this point is unique. If possible, let v be another periodic point of E and F. i.e. $E^{p(u)}u = u$ and $F^{q(v)}v = v$. Then

$$\begin{aligned}\rho(u, v) &= \rho(E^{p(u)}u, F^{q(v)}v)\\ &\leq \frac{\alpha(\rho(u, v))\rho(u, E^{p(u)}u)\rho(v, F^{q(v)}v)}{\rho(u, v) + \rho(u, F^{q(v)}v) + \rho(u, E^{p(u)}u)} + \beta(\rho(u, v))\rho(u, v)\end{aligned}$$

and so

$$\rho(u, v) \leq \beta(\rho(u, v))\rho(u, v)$$

which implies that $u = v$, since $0 \leq \beta(\rho(u, v)) < 1$).
Hence u is a unique periodic point of E and F.

Now $Eu = EE^{p(u)}u = E^{p(u)}E(u)$ implies that $E(u)$ is a periodic point of E. From the uniqueness of $u.E(u) = u$. Similarly, $F(u) = u$. Hence, u is a common fixed point of E and F.

This completes the proof. □

As an immediate consequence of the above theorems we have the following corollaries:

Corollary 2.2. *Let E be a self mapping of a complete complex metric space (X, ρ) such that there exists positive integers $p(x)$ and $q(x)$ such that for each $x, y \in X$,*

$$\rho(E^{p(x)}x, E^{q(y)}y) \leq \frac{\alpha(\rho(x, y))\rho(x, E^{p(x)}x)\rho(y, E^{q(y)}y)}{\rho(x, y) + \rho(x, E^{q(y)}y) + \rho(y, E^{p(x)}x)} + \beta(\rho(x, y))\rho(x, y)$$

where $\alpha, \beta : \mathbb{C}_+ \to [0, 1)$ such that for all $x, y \in X, \alpha(x) + \beta(x) < 1$. Then E has a unique common fixed point in X.

Corollary 2.3. *Let E be a self mapping of a complete complex metric space (X, ρ) such that there exists positive integer p such that for each $x, y \in X$,*

$$\rho(E^p x, E^p y) \leq \frac{\alpha(\rho(x, y))\rho(x, E^p x)\rho(y, E^p y)}{\rho(x, y) + \rho(x, E^p y) + \rho(y, E^p x)} + \beta(\rho(x, y))\rho(x, y)$$

where $\alpha, \beta : \mathbb{C}_+ \to [0, 1)$ such that for all $x, y \in X, \alpha(x) + \beta(x) < 1$. Then E has a unique common fixed point in X.

Corollary 2.4. *Let E be a self mapping of a complete complex metric space (X, ρ) such that for each $x, y \in X$,*

$$\rho(Ex, Ey) \leq \frac{\alpha(\rho(x, y))\rho(x, Ex)\rho(y, Ey)}{\rho(x, y) + \rho(x, Ey) + \rho(y, Ex)} + \beta(\rho(x, y))\rho(x, y)$$

where $\alpha, \beta : \mathbb{C}_+ \to [0, 1)$ such that for all $x, y \in X, \alpha(x) + \beta(x) < 1$. Then E has a unique common fixed point in X.

Now we shall give an example to support our theorem:

3. Example

Let $X_1 = \{\mathbb{C} : Re(z) \geq 0, Im(z) = 0\}$ *and* $X_2 = \{\mathbb{C} : Re(z) = 0, Im(z) \geq 0\}$. Also we let $X = X_1 \times X_2$ *and define* $\rho : X \times X \to \mathbb{C}$ by :

$$\rho(z_1, z_2) = \begin{cases} max\{x_1, x_2\} + i max\{x_1, x_2\}: \text{ if } z_1, z_2 \in X_1 \\ max\{x_1, x_2\} + i max\{x_1, x_2\}: \text{ if } z_1, z_2 \in X_2 \\ (x_1 + y_2) + i(x_1 + y_2): \text{ if } z_1, z_2 \in X_2 \\ (x_2 + y_1) + i(x_2 + y_1): \text{ if } z_1 \in X_2 \text{ and } z_2 \in X_1 \end{cases}$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. It is very easy to see that (X, ρ) is a complete complex valued metric space.

Now we set $E^{p(x)}x = F^{q(x)}x = Tx$ and define a mapping $T : X \to X$ such that with k is any finite positive integer

$$T(z) = \begin{cases} \left(\frac{x}{k}, 0\right) & : \text{ if } z \in X_1 \\ \left(0, \frac{y}{k}\right) & : \text{ if } z \in X_2 \end{cases}$$

Now we can easily evaluate that all the conditions of the Theorem(4) satisfied here with $\alpha(t) = \frac{1}{k}$ *and* $0 < \beta(t) < \frac{k-1}{k}$ and all the conditions of the theorem satisfied and we can find $z = 0 \in X$ is unique common fixed point of T.

4. Application

Let $X = C([a, b], R^n)$, $a > 0$ and let $\rho : X \times X \to \mathbb{C}$ be defined by

$$\rho(x, y) = max_{t \in [a,b]} \parallel x(t) - y(t) \parallel_\infty \sqrt{1 + a^2} \quad e^{i \tan^{-1} a}.$$

Consider the Urysohns integral equations

$$x(t) = \int_a^b K_1(t, s, x(s))ds + g(t), \tag{6}$$

$$x(t) = \int_a^b K_2(t, s, x(s))ds + g(t), \tag{7}$$

where $t \in [a, b] \subseteq \mathrm{R}, x, g, h \in X$ and K_1, $K_2 : [a, b] \times [a, b] \times \mathrm{R}^n \to \mathrm{R}^n$.
Suppose K_1, K_1 are such that F_x, $G_x \in X$ for all $x \in X$, where

$$F_x(t) = \int_a^b K_1(t, s, x(s))ds, \tag{8}$$

$$G_x(t) = \int_a^b K_2(t, s, x(s))ds \tag{9}$$

for all $t \in [a, b]$.

If there exists two mappings α, $\beta : C_+ \to [0, 1]$ such that for all $x, y \in X$ the following holds:

(i) $\alpha(t) + \beta(t) < 1$;

(ii) the mapping $\gamma : C_+ \to [0,1]$ defined by $\gamma(x) = \frac{\alpha(x)}{1-\beta(x)}$ belongs to Γ;

(iii) $\| F_x(t) - G_y(t) + g(t) - h(t) \|_\infty \ \sqrt{1+a^2} \quad e^{i\tan^{-1}a}$
$\leq \alpha(\max_{t\in[a,b]} A(x,y)(t))A(x,y) + \beta(\max_{t\in[a,b]} A(x,y)(t))B(x,y)$,
where
$A(x,y)(t) = \| x(t) - y(t) \|_\infty \ \sqrt{1+a^2} \quad e^{i\tan^{-1}a}$
and

$$B(x,y)(t) = \frac{\| F_x(t) + g(t) - x(t) \|_\infty \quad \| G_y(t) + h(t) - y(t) \|_\infty}{\| F_x(t) + g(t) - y(t) \|_\infty \quad + \quad \| G_y(t) + h(t) - x(t) \|_\infty \quad + \quad d(x,y)}.$$

Then the system of integral equations (6) and (7) has a unique common solution.

Proof. Define $S, T : X \to X$ by $S(x) = F_x + g$ and $T(x) = G_y + h$. Then

$$\rho(Sx, Ty) = \max_{t\in[a,b]} \| F_x(t) - G_y(t) + g(t) - h(t) \|_\infty \ \sqrt{1+a^2} \quad e^{i\tan^{-1}a},$$

$$\rho(x, Sx) = \max_{t\in[a,b]} \| F_x(t) + g(t) - x(t) \|_\infty \ \sqrt{1+a^2} \quad e^{i\tan^{-1}a},$$

$$\rho(y, Ty) = \max_{t\in[a,b]} \| G_y(t) + h(t) - y(t) \|_\infty \ \sqrt{1+a^2} \quad e^{i\tan^{-1}a},$$

$$\rho(y, Sx) = \max_{t\in[a,b]} \| F_x(t) + g(t) - y(t) \|_\infty \ \sqrt{1+a^2} \quad e^{i\tan^{-1}a},$$

$$\rho(x, Ty) = \max_{t\in[a,b]} \| G_y(t) + h(t) - x(t) \|_\infty \ \sqrt{1+a^2} \quad e^{i\tan^{-1}a}.$$

Then we can easily see that for $x, y \in X$,

$$\rho(Ex, \ Fy) \leq \alpha \frac{\rho(x, \ Ex)\rho(y, \ Fy)}{\rho(x, \ y) \ + \ \rho(x, \ Fy) \ + \ \rho(y, \ Ex)} \ + \ \beta \ \ \rho(x,y).$$

By applying Theorem(2.1). we get the solution to (6) and (7) of Urysohn's Integral Equations which is unique. □

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