Filomat 31:8 (2017), 2477–2497
DOI 10.2298/FIL1708477S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Orthogonal Polynomials Associated with an Inverse Spectral Transform. The Cubic Case

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Abstract. The purpose of this work is to give some new algebraic properties of the orthogonality of a monic polynomial sequence $\{Q_n\}_{n\geq 0}$ defined by

$$Q_n(x) = P_n(x) + s_nP_{n-1}(x) + t_nP_{n-2}(x) + r_nP_{n-3}(x), \quad n \geq 1,$$

where $r_n \neq 0, n \geq 3$, and $\{P_n\}_{n\geq 0}$ is a given sequence of monic orthogonal polynomials. Essentially, we consider some cases in which the parameters r_n, s_n, and t_n can be computed more easily. Also, as a consequence, a matrix interpretation using LU and UL factorization is done. Some applications for Laguerre, Bessel and Tchebychev orthogonal polynomials of second kind are obtained.

1. Introduction and Preliminaries

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic orthogonal polynomials with respect to a regular linear functional u. We define a new sequence of monic polynomials $\{Q_n\}_{n\geq 0}$ by the M-N type linear structure relation

$$Q_n(x) + \sum_{i=1}^{M-1} a_{i,n}Q_{n-i}(x) = P_n(x) + \sum_{i=1}^{N-1} b_{i,n}P_{n-i}(x), \; n \geq 1,$$

where M and N are fixed positive integer numbers, and $\{a_{i,n}\}_n$ and $\{b_{i,n}\}_n$ are sequences of complex numbers with $a_{M-1,n}b_{N-1,n} \neq 0$. The study of the regularity of the sequence $\{Q_n\}_{n\geq 0}$ is said to be an inverse problem. This problem has been studied in some particular cases. Indeed, the relations of types 1-2 and 2-1 have been studied in [9], the 1-3 type relation in [2], the 2-2 type relation in [4] and the 2-3 type relation in [1]. In addition, the 1-N type relation with constant coefficients has been analyzed in [3].
Recently, in [8] and for $M = 1, N = 4$, F. Marcelln and S. Varma determine necessary and sufficient conditions such that $\{Q_n\}_{n\geq 0}$ becomes also orthogonal.
This article is a continuation of [8]. It deals with some new results about the sequence $\{Q_n\}_{n\geq 0}$ defined by

$$Q_n(x) = P_n(x) + s_nP_{n-1}(x) + t_nP_{n-2}(x) + r_nP_{n-3}(x), \quad r_n \neq 0, \quad n \geq 3.$$

2010 *Mathematics Subject Classification*. Primary 33C45; Secondary 42C05
Keywords. Orthogonal polynomials, Recurrence relations, Linear functionals, Jacobi matrices.
Received: 23 February 2016; Accepted: 28 April 2016
Communicated by Dragan S. Djordjević
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Firstly, we give some new results concerning the regularity conditions of the sequence $\{Q_n\}_{n\geq 0}$. In particular, we obtain a new characterization of the orthogonality of this sequence with respect to a linear functional v, in terms of the coefficients of a cubic polynomial q such that $q(x)v = u$. Indeed, it is known [17] that up to some natural conditions the M - N type structure relation leads to a rational transformation $\Phi u = \Psi v$ where Φ and Ψ are polynomials. Secondly, since the cases 1-2 and 1-3 type structure relation have been already considered in previous works (see [2, 9]), we obtain necessary and sufficient conditions so that the above 1-4 relation can be decomposed in three 1-2 relations or two relations of types 1-2 and 1-3 and then proceed by iteration. This study is based on the factorization of $q(x)$. We will study the case when $\{P_n\}_{n\geq 0}$ is symmetric and $\{Q_n\}_{n\geq 0}$ is quasi-antisymmetric. In any situation, the matrix interpretation of this problem in terms of monic Jacobi matrices is done carefully. Finally, we give a detailed study of three examples.

Now, we are going to introduce some basic definitions and results. The field of complex numbers is denoted by $\mathbb{C}$. The vector space of polynomials with coefficients in $\mathbb{C}$ is denoted by $\mathcal{P}$ and its dual space is presented as $\mathcal{P}'$. We will simply call polynomial every element of $\mathcal{P}$ and linear functional to the elements in $\mathcal{P}'$. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle , n \geq 0$, the moments of u.
For any linear functional v and any polynomial h let hv, δ_c, and $(x-c)^{-1}v$ be the linear functionals defined by: $\langle hv, f\rangle := \langle v, hf\rangle$, $\langle \delta_c, f\rangle := f(c)$ and $\left\langle (x-c)^{-1}v, f\right\rangle := \langle v, \theta_c f\rangle$ where $\left(\theta_c f\right)(x) = \dfrac{f(x)-f(c)}{x-c}$, $c \in \mathbb{C}$, $f \in \mathcal{P}$. Then, it is straightforward to prove that for $c \in \mathbb{C}$, and $v \in \mathcal{P}'$, we have [15]

$$(x-c)^{-1}((x-c)v) = v - (v)_0\delta_c, \tag{1.1}$$

$$(x-c)((x-c)^{-1}v) = v. \tag{1.2}$$

A linear functional u is called regular if there exists a sequence of polynomials $\{P_n\}_{n\geq 0}$ ($\deg P_n \leq n$) such that $\langle u, P_nP_m\rangle = r_n\delta_{n,m}$, $r_n \neq 0$, $n \geq 0$.
Then $\deg P_n = n, n \geq 0$ and we can always suppose each P_n is monic. In such a case, the sequence $\{P_n\}_{n\geq 0}$ is unique. It is said to be the sequence of monic orthogonal polynomials with respect to u. In the sequel it will be denoted as SMOP. It is a very well known fact that the sequence $\{P_n\}_{n\geq 0}$ satisfies the recurrence relation (see, for instance, the monograph by Chihara [6])

$$\begin{aligned} &P_{n+2}(x) = (x-\beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x) , \quad n \geq 0 , \\ &P_1(x) = x - \beta_0 , \qquad P_0(x) = 1 , \end{aligned} \tag{1.3}$$

with $\left(\beta_n, \gamma_{n+1}\right) \in \mathbb{C}\times\mathbb{C} - \{0\}$, $n \geq 0$. By convention we set $\gamma_0 = (u)_0$.
The linear functional u is said to be normalized if $(u)_0 = 1$. In this paper, we suppose that any linear functional will be normalized.

2. Some Algebraic Properties

In the sequel $\{P_n\}_{n\geq 0}$ denotes a SMOP with respect to a regular linear functional u. By giving three sequences of complex numbers $\{s_n\}_{n\geq 1}$, $\{t_n\}_{n\geq 2}$, and $\{r_n\}_{n\geq 3}$, we define a new sequence of monic polynomials $\{Q_n\}_{n\geq 0}$ such that

$$\begin{aligned} Q_1(x) &= P_1(x) + s_1, \\ Q_2(x) &= P_2(x) + s_2P_1(x) + t_2, \\ Q_n(x) &= P_n(x) + s_nP_{n-1}(x) + t_nP_{n-2}(x) + r_nP_{n-3}(x), \quad n \geq 3, \quad \text{with } r_n \neq 0,\ n \geq 3. \end{aligned} \tag{2.1}$$

Let us recall the following result:

Theorem 2.1. *[8] $\{Q_n\}_{n\geq 0}$ is an SMOP if and only if as well as $\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3 \neq 0$ with*

$$\begin{aligned} s_{n-1}\tilde{\gamma}_n &= s_n\gamma_{n-1} + t_n(\beta_{n-2} - \tilde{\beta}_n) + r_n - r_{n+1},\ n \geq 2, && (2.2)\\ t_{n-1}\tilde{\gamma}_n &= t_n\gamma_{n-2} + r_n(\beta_{n-3} - \tilde{\beta}_n),\ n \geq 3, && (2.3)\\ r_{n-1}\tilde{\gamma}_n &= r_n\gamma_{n-3},\ n \geq 4, && (2.4) \end{aligned}$$

where

$$\begin{aligned} \tilde{\beta}_n &= \beta_n + s_n - s_{n+1},\ n \geq 0, && (2.5)\\ \tilde{\gamma}_n &= \gamma_n + t_n - t_{n+1} + s_n(\beta_{n-1} - \beta_n - s_n + s_{n+1}),\ n \geq 1, && (2.6) \end{aligned}$$

with $s_0 = t_0 = t_1 = r_0 = r_1 = r_2 = 0$.
Furthermore, $\{Q_n\}_{n\geq 0}$ satisfies the three-term recurrence relation

$$\begin{aligned} &Q_{n+2}(x) = (x - \tilde{\beta}_{n+1})Q_{n+1}(x) - \tilde{\gamma}_{n+1}Q_n(x),\ n \geq 0,\\ &Q_1(x) = x - \tilde{\beta}_0, \qquad Q_0(x) = 1. \end{aligned} \tag{2.7}$$

Remark 1. When $\{Q_n\}_{n\geq 0}$ is an SMOP, then (2.2)-(2.4) can be written as

$$\begin{aligned} r_3 &= t_2(\beta_1 - \beta_2 - s_2 + s_3) + s_2\gamma_1 - s_1[\gamma_2 + t_2 - t_3 + s_2(\beta_1 - \beta_2 - s_2 + s_3)],\\ r_4 &= r_3 + t_3(\beta_2 - \beta_3 - s_3 + s_4) + s_3\gamma_2 - s_2[\gamma_3 + t_3 - t_4 + s_3(\beta_2 - \beta_3 - s_3 + s_4)], \end{aligned} \tag{2.8}$$

$$s_{n+5} = s_{n+4} + \beta_{n+4} - \beta_{n+1} + \frac{t_{n+3}}{r_{n+3}}\gamma_{n+1} - \frac{t_{n+4}}{r_{n+4}}\gamma_{n+2},\ n \geq 0, \tag{2.9}$$

$$t_{n+5} = t_{n+4} + s_{n+4}(\beta_{n+3} - \beta_{n+4} - s_{n+4} + s_{n+5}) - \frac{r_{n+4}}{r_{n+3}}\gamma_{n+1} + \gamma_{n+4},\ n \geq 0, \tag{2.10}$$

$$r_{n+5} = r_{n+4}\Big(1 - \tfrac{s_{n+3}}{r_{n+3}}\gamma_{n+1}\Big) + s_{n+4}\gamma_{n+3} + t_{n+4}(\beta_{n+2} - \beta_{n+4} - s_{n+4} + s_{n+5}),\ n \geq 0, \tag{2.11}$$

where the initial conditions are

(a) $t_2,\ t_3,\ t_4,\ s_1,\ s_2,\ s_3,$ and $\begin{cases} s_4 = s_3 + \beta_3 - \beta_0 - \frac{t_3}{r_3}\gamma_1, & \text{if } t_2 = 0,\\ s_4 = \frac{t_3\gamma_1 + r_3(\beta_0 - \beta_3 - s_3) - t_2(\gamma_3 + t_3 - t_4 + s_3(\beta_2 - \beta_3 - s_3))}{t_2 s_3 - r_3}, & \text{if } t_2 \neq 0,\ t_2 s_3 - r_3 \neq 0. \end{cases}$

(b) $t_2,\ t_3,\ s_1,\ s_2,\ s_3,\ s_4,$ and $t_4 = s_3(\beta_2 - \beta_3 - s_3) + \gamma_3 + t_3 - \dfrac{t_3\gamma_1 + r_3(\beta_0 - \beta_3 - s_3)}{t_2}$, if $t_2 \neq 0,\ r_3 = t_2 s_3$.

Furthermore, $t_2,\ t_3,\ t_4,\ s_1,\ s_2,\ s_3,\ s_4$ verify $\begin{cases} \gamma_1 - t_2 + s_1(\beta_0 - \beta_1 - s_1 + s_2) \neq 0,\\ \gamma_2 + t_2 - t_3 + s_2(\beta_1 - \beta_2 - s_2 + s_3) \neq 0,\\ \gamma_3 + t_3 - t_4 + s_3(\beta_2 - \beta_3 - s_3 + s_4) \neq 0. \end{cases}$

Theorem 2.2. *The following statements are equivalent:*

(i) *$\{Q_n\}_{n\geq 0}$ is an SMOP with $(\tilde{\beta}_n)_n$ and $(\tilde{\gamma}_n)_n$ given by (2.5) and (2.6) the corresponding sequences of recurrence coefficients.*

(ii) *It holds $\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3 \neq 0$ together with the initial conditions (2.8) and*

$$t_2(\gamma_3 + t_3 - t_4 + s_3(\beta_2 - \beta_3 - s_3 + s_4)) = t_3\gamma_1 + r_3(\beta_2 - \beta_3 - s_3 + s_4). \tag{2.12}$$

and there exist three complex numbers a, b and c such that, for $n \geq 1$

$$A_n := \frac{t_{n+2}}{r_{n+2}}\gamma_n - \beta_n - \beta_{n+1} - \beta_{n+2} + s_{n+3} = a, \tag{2.13}$$

$$\begin{aligned} B_n := &\frac{1}{r_{n+2}}\gamma_n\Big[s_{n+2}\gamma_{n+1} + t_{n+2}(s_{n+3} - \beta_{n+2} - \beta_{n+1})\Big] - \gamma_{n+1} - \gamma_{n+2} - \gamma_{n+3} + t_{n+4} \\ &- (s_{n+1} - \beta_{n+2} - \beta_{n+1})(\beta_{n+1} + \beta_n) + s_{n+3}(s_{n+3} - s_{n+4} - \beta_{n+2} - \beta_{n+3}) - \beta^2_{n+1} = b, \end{aligned} \tag{2.14}$$

$$\begin{aligned} C_n := &\frac{1}{r_{n+2}}\gamma_n\Big[\gamma_{n+1}\Big(\gamma_{n+2} + s_{n+2}(s_{n+3} - \beta_{n+2})\Big) + t_{n+2}\Big(s_{n+3}(\beta_{n+3} - \beta_{n+2} + s_{n+3} - s_{n+4}) \\ &+ \beta_{n+1}(\beta_{n+2} - s_{n+3}) - \gamma_{n+2} - \gamma_{n+3} + t_{n+4}\Big)\Big] + \gamma_{n+1}(\beta_{n+2} - s_{n+3}) \\ &+ (s_{n+4} - \beta_{n+3})(\beta_{n+2}\beta_{n+1} - \gamma_{n+2}) + \beta_{n+1}(\gamma_{n+3} - t_{n+4}) + r_{n+4} = c. \end{aligned} \tag{2.15}$$

Furthermore, if u and v are the linear functionals associated with the sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, respectively, then

$$q(x)v = -ku, \tag{2.16}$$

with $q(x) = x^3 + ax^2 + bx + c$, $k \in \mathbb{C} - \{0\}$.

Proof. Notice that, by Remark 1, $\{Q_n\}_{n\geq 0}$ is an SMOP if and only if the condition $\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3 \neq 0$ and the initial conditions (2.8), (2.12) and the above Eqs (2.9)-(2.11) hold. To conclude the proof we need to show that Eqs. (2.13)-(2.15) are equivalent to (2.9)-(2.11).

• We first prove that (2.9)-(2.11) ⇒ (2.13)-(2.15). Using (2.9), we get,

$$A_{n+1} = A_{n+2}, \quad n \geq 0. \tag{2.17}$$

Hence (2.13). Now, we will deduce (2.14).
Multiplying the expression (2.11) by γ_{n+2}/r_{n+4}, we obtain

$$\begin{aligned} \frac{s_{n+3}}{r_{n+3}}\gamma_{n+1}\gamma_{n+2} + \frac{t_{n+4}}{r_{n+4}}\gamma_{n+2}(s_{n+4} - \beta_{n+2} - \beta_{n+3}) = &\frac{s_{n+4}}{r_{n+4}}\gamma_{n+3}\gamma_{n+2} + \frac{t_{n+4}}{r_{n+4}}\gamma_{n+2}(s_{n+5} - \beta_{n+3} - \beta_{n+4}) \\ &+ \Big(1 - \frac{r_{n+5}}{r_{n+4}}\Big)\gamma_{n+2}. \end{aligned}$$

Besides, from (2.10) we have, for $n \geq 0$

$$\begin{aligned} &\frac{s_{n+3}}{r_{n+3}}\gamma_{n+1}\gamma_{n+2} + \frac{t_{n+4}}{r_{n+4}}\gamma_{n+2}(s_{n+4} - \beta_{n+2} - \beta_{n+3}) = \frac{s_{n+4}}{r_{n+4}}\gamma_{n+3}\gamma_{n+2} + \frac{t_{n+4}}{r_{n+4}}\gamma_{n+2}(s_{n+5} - \beta_{n+3} - \beta_{n+4}) \\ &+ \gamma_{n+2} - \gamma_{n+5} - t_{n+5} + t_{n+6} - s_{n+5}(\beta_{n+4} - \beta_{n+5} - s_{n+5} + s_{n+6}). \end{aligned} \tag{2.18}$$

Using (2.17) in the expression of $\frac{t_{n+4}\gamma_{n+2}}{r_{n+4}}$ which appears in the right hand side of the above formula, we obtain

$$B_{n+2} = B_{n+1}, \quad n \geq 0. \tag{2.19}$$

Hence (2.14). Now, we will deduce (2.15).
Multiplying (2.10) by $\gamma_{n+2}\gamma_{n+3}/r_{n+4}$, we get

$$\begin{aligned} &\frac{\gamma_{n+1}\gamma_{n+2}\gamma_{n+3}}{r_{n+3}} + (s_{n+4} - \beta_{n+3})\frac{s_{n+4}}{r_{n+4}}\gamma_{n+2}\gamma_{n+3} = \frac{\gamma_{n+2}\gamma_{n+3}\gamma_{n+4}}{r_{n+4}} + (s_{n+5} - \beta_{n+4})\frac{s_{n+4}}{r_{n+4}}\gamma_{n+2}\gamma_{n+3} \\ &+ \Big[\frac{t_{n+4} - t_{n+5}}{r_{n+4}}\Big]\gamma_{n+2}\gamma_{n+3}. \end{aligned}$$

Using (2.9) and (2.10), we have, for $n \geq 0$

$$\frac{t_{n+5}\gamma_{n+3}}{r_{n+4}}\gamma_{n+2} = [\gamma_{n+5} + t_{n+5} - t_{n+6} + s_{n+5}(\beta_{n+4} - \beta_{n+5} - s_{n+5} + s_{n+6})]$$
$$[\frac{t_{n+4}}{r_{n+4}}\gamma_{n+2} - \beta_{n+2} + \beta_{n+5} - s_{n+5} + s_{n+6}],$$

and, therefore, for $n \geq 0$

$$\begin{aligned}&\frac{\gamma_{n+1}\gamma_{n+2}\gamma_{n+3}}{r_{n+3}} + (s_{n+4} - \beta_{n+3})\frac{s_{n+4}}{r_{n+4}}\gamma_{n+2}\gamma_{n+3} = \frac{\gamma_{n+2}\gamma_{n+3}\gamma_{n+4}}{r_{n+4}} + (s_{n+5} - \beta_{n+4})\frac{s_{n+4}}{r_{n+4}}\gamma_{n+2}\gamma_{n+3}\\&+ \frac{t_{n+4}}{r_{n+4}}\gamma_{n+2}\Big[\gamma_{n+3} - \gamma_{n+5} - t_{n+5} + t_{n+6} - s_{n+5}(\beta_{n+4} - \beta_{n+5} - s_{n+5} + s_{n+6})\Big]\\&+ (\beta_{n+2} - \beta_{n+5} - s_{n+5} + s_{n+6})[\gamma_{n+5} + t_{n+5} - t_{n+6} + s_{n+5}(\beta_{n+4} - \beta_{n+5} - s_{n+5} + s_{n+6})].\end{aligned}$$

Using (2.18) in the expression $\frac{s_{n+4}\gamma_{n+2}\gamma_{n+3}}{r_{n+4}}$ for $n \geq 0$, the last equation becomes

$$\begin{aligned}&\frac{\gamma_{n+1}\gamma_{n+2}\gamma_{n+3}}{r_{n+3}} + (s_{n+4} - \beta_{n+3})\frac{s_{n+3}}{r_{n+3}}\gamma_{n+1}\gamma_{n+2} + \frac{t_{n+4}}{r_{n+4}}\gamma_{n+2}\Big[s_{n+4}(s_{n+4} - s_{n+5} - \beta_{n+3} - \beta_{n+2} + \beta_{n+4}) + \beta_{n+3}\beta_{n+2}\\&-\gamma_{n+3} - \gamma_{n+4} + t_{n+5}\Big] = \frac{\gamma_{n+2}\gamma_{n+3}\gamma_{n+4}}{r_{n+4}} + (s_{n+5} - \beta_{n+4})\frac{s_{n+4}}{r_{n+4}}\gamma_{n+2}\gamma_{n+3}\\&+ \frac{t_{n+4}}{r_{n+4}}\gamma_{n+2}\Big[s_{n+5}(s_{n+5} - s_{n+6} - \beta_{n+4} - \beta_{n+3} + \beta_{n+5}) + \beta_{n+4}\beta_{n+3} - \gamma_{n+4} - \gamma_{n+5} + t_{n+6}\Big] + \gamma_{n+2}(s_{n+4} - \beta_{n+3})\\&+ (\beta_{n+2} - \beta_{n+5} + \beta_{n+3} - s_{n+5} + s_{n+6} - s_{n+4})\Big[\gamma_{n+5} + t_{n+5} - t_{n+6} + s_{n+5}(\beta_{n+4} - \beta_{n+5} - s_{n+5} + s_{n+6})\Big].\end{aligned}$$

Using (2.17), for $n + 1$ instead of n, in the expression $\frac{t_{n+4}}{r_{n+4}}\gamma_{n+2}$ which appears in the right hand side of the above formula, we obtain

$$C_{n+2} = C_{n+1}, \quad n \geq 0. \tag{2.20}$$

Hence (2.15).

• Next we show that (2.13)-(2.15) $\Rightarrow$ (2.9)-(2.11).
Notice first that (2.13)-(2.15) are equivalent to (2.17), (2.18) and (2.20).
From (2.17), we can derive (2.9).

Taking into account the new expression of $\frac{s_{n+3}\gamma_{n+2}\gamma_{n+1}}{r_{n+3}}$ obtained from (2.18) and $\frac{t_{n+5}\gamma_{n+3}}{r_{n+5}}$ obtained from (2.17) written for $n + 1$ instead of n, we can reformulate (2.20)

$$\frac{\gamma_{n+2}\gamma_{n+3}}{r_{n+3}}\Big(\gamma_{n+1} + \frac{r_{n+3}}{r_{n+4}}[-s_{n+4}(s_{n+5} - \beta_{n+4} - s_{n+4} - \beta_{n+3}) - t_{n+4} + t_{n+5} - \gamma_{n+4}]\Big) = 0.$$

Then, we deduce (2.10).

Taking into account the new expression of $\frac{t_{n+4}}{r_{n+4}}\gamma_{n+2}$ obtained from (2.17), the (2.18) reads as

$$\begin{aligned}&\frac{\gamma_{n+2}}{r_{n+4}}\Big[\frac{r_{n+4}}{r_{n+3}}s_{n+3}\gamma_{n+1} - s_{n+4}\gamma_{n+3} - t_{n+4}(s_{n+5} - \beta_{n+4} + \beta_{n+2} - s_{n+4}) - r_{n+4}\Big]\\&= -\gamma_{n+5} - t_{n+5} + t_{n+4} - s_{n+5}(\beta_{n+4} - \beta_{n+5} - s_{n+5} + s_{n+6}).\end{aligned}$$

From (2.4) and (2.6), we have

$$\frac{\gamma_{n+2}}{r_{n+4}}\Big[s_{n+3}\tilde{\gamma}_{n+4} - s_{n+4}\gamma_{n+3} - t_{n+4}(s_{n+5} - \beta_{n+4} + \beta_{n+2} - s_{n+4}) - r_{n+4} + r_{n+5}\Big] = 0,$$

therefore (2.11) holds.

To conclude the proof, it remains to deduce the relation between the functionals u and v in terms of the constants $a,\ b$ and c. If we expand the linear functional u in the dual basis $\{\frac{Q_j v}{\langle v,\ Q_j^2\rangle}\}_{j\geq 0}$ of the polynomials $\{Q_j\}_{j\geq 0}$ (see [15]) and taking into account (2.1), then

$$u = \sum_{j=0}^{3} \frac{\langle u,\ Q_j\rangle}{\langle v,\ Q_j^2\rangle} Q_j v = \left(\frac{r_3}{\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3} Q_3 + \frac{t_2}{\tilde{\gamma}_1\tilde{\gamma}_2} Q_2 + \frac{s_1}{\tilde{\gamma}_1} Q_1 + 1\right) v.$$

Introducing the polynomials $Q_3,\ Q_2$ and Q_1 given by (2.1) and the explicit expression of the polynomials $P_1,\ P_2$ and P_3 given by recurrence relation, we obtain

$$\begin{aligned} u = \frac{r_3}{\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3}\Big[& x^3 + \Big(\frac{t_2\tilde{\gamma}_3}{r_3} - \beta_0 - \beta_1 - \beta_2 + s_3\Big)x^2 + \Big(\frac{s_1\tilde{\gamma}_2\tilde{\gamma}_3}{r_3} - (\beta_0 + \beta_1 - s_2)\frac{t_2}{r_3}\tilde{\gamma}_3 + \beta_1\beta_2 \\ & + \beta_0(\beta_1 + \beta_2) - \gamma_1 - \gamma_2 - s_3(\beta_0 + \beta_1) + t_3\Big)x + \frac{\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3}{r_3} + (s_1 - \beta_0)\frac{s_1}{r_3}\tilde{\gamma}_2\tilde{\gamma}_3 \\ & + (\beta_0\beta_1 - \gamma_1 + t_2 - s_2\beta_0)\frac{t_2}{r_3}\tilde{\gamma}_3 - \beta_0\beta_1\beta_2 + \gamma_1\beta_2 + \gamma_2\beta_0 + s_3(\beta_0\beta_1 - \gamma_1) - t_3\beta_0 + r_3\Big]v. \end{aligned}$$

Taking into account, (2.2) where $n = 2,\ 3$, (2.3) for $n = 3$ and (2.6) with $n = 1,\ 2,\ 3$, we get

$$\frac{t_2\tilde{\gamma}_3}{r_3} - \beta_0 - \beta_1 - \beta_2 + s_3 = A_1, \tag{2.21}$$

$$\frac{s_1\tilde{\gamma}_2\tilde{\gamma}_3}{r_3} - (\beta_0 + \beta_1 - s_2)\frac{t_2}{r_3}\tilde{\gamma}_3 + \beta_1\beta_2 + \beta_0(\beta_1 + \beta_2) - \gamma_1 - \gamma_2 - s_3(\beta_0 + \beta_1) + t_3 = B_1, \tag{2.22}$$

$$\begin{aligned} & \frac{\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3}{r_3} + (s_1 - \beta_0)\frac{s_1}{r_3}\tilde{\gamma}_2\tilde{\gamma}_3 + (\beta_0\beta_1 - \gamma_1 + t_2 - s_2\beta_0)\frac{t_2}{r_3}\tilde{\gamma}_3 - \beta_0\beta_1\beta_2 + \gamma_1\beta_2 + \gamma_2\beta_0 \\ & + s_3(\beta_0\beta_1 - \gamma_1) - t_3\beta_0 + r_3 = C_1. \end{aligned} \tag{2.23}$$

Then $-ku = (x^3 + A_1x^2 + B_1x + C_1)v = (x^3 + ax^2 + bx + c)v, \quad \text{with } k = -\dfrac{\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3}{r_3}$. □

Remark 2. The converse problem, i.e. the analysis of the regularity of a linear functional v such that there exists a polynomial $q(x)$ such that $q(x)v = -ku,\ k \in \mathbb{C} - \{0\}$, has been studied by many authors. In particular, in [10], [11] and [12] the cases $q(x) = x^4$ and $q(x) = x^3$ have been deeply analyzed.

3. Reducible Cases

The next Theorem will play an important role in the sequel.

Theorem 3.1. *[9] Let $\{S_n\}_{n\geq 0}$, be a SMOP with respect to a linear functional w, $\{\mu_n\}_{n\geq 1}$ a sequence of complex parameters and $\{Z_n\}_{\geq 0}$ a simple set of monic polynomials, such that*

$$Z_n = S_n + \mu_n S_{n-1},\ n \geq 1, \quad \textit{with } \mu_n \neq 0. \tag{3.1}$$

Suppose also that $(\varepsilon_n,\ \rho_n)_{n\geq 0}$ is the set of parameters of the recurrence relation of the sequence $\{S_n\}_{n\geq 0}$. Then, $\{Z_n\}_{n\geq 0}$ is an SMOP with respect to a linear functional ϑ if and only there exist complex numbers $x_1 \neq \varepsilon_0 - \mu_1$ such that, for $n \geq 1$,

$$\varepsilon_n - \mu_{n+1} - \frac{\rho_n}{\mu_n} = x_1,\ n \geq 1. \tag{3.2}$$

Furthermore,

$$(x - x_1)\vartheta = (\varepsilon_0 - x_1 - \mu_1)w. \tag{3.3}$$

Notice that if $\{Q_n\}_{n\geq 0}$ satisfies (2.1), then the polynomials Q_n cannot be represented as a linear combination of the at most three consecutive polynomials P_n, P_{n-1} and P_{n-2} . A natural question arises: Can the SMOP $\{Q_n\}_{n\geq 0}$ be generated from $\{P_n\}_{n\geq 0}$ in two or three steps with the help of some intermediate SMOPs? The interest of this question is to simplify the computation of the parameters s_n, t_n and r_n, in each case via one of the other parameters noted a_n, b_n and c_n.

From now on, let $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be two SMOPs with respect to the regular linear functionals u and v, respectively which are related by (2.1).

3.1. The split of a 1-4 relation in three 1-2 relations.

Proposition 3.2. *Let $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ be three sequences of nonzero complex numbers. The representation (2.1) can be written as, for $n \geq 1$*

$$\begin{aligned} Q_n &= \tilde{R}_n + c_n\tilde{R}_{n-1}, \\ \tilde{R}_n &= R_n + b_nR_{n-1}, \\ R_n &= P_n + a_nP_{n-1}, \end{aligned} \tag{3.4}$$

where , $\{R_n\}_{n\geq 0}$ and $\{\tilde{R}_n\}_{n\geq 0}$ are two SMOPs, if and only if there exist two complex numbers $\alpha,\ \beta$, such that

$$\begin{aligned} D_n :&\ = \beta_n - a_{n+1} - \frac{\gamma_n}{a_n} = \alpha,\ n \geq 1, \\ E_n :&\ = \beta_n + a_n - a_{n+1} - b_{n+1} - \frac{\gamma_n + a_n(\beta_{n-1} - \beta_n - a_n + a_{n+1})}{b_n} = \beta,\ n \geq 1, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &a_{n+1} = s_{n+1} - \frac{t_{n+1}}{a_n} + \frac{(s_{n+1} - a_{n+1} - b_{n+1})b_n}{a_n},\ n \geq 1, \\ &b_{n+2} = s_{n+2} - a_{n+2} - \frac{r_{n+2}}{b_{n+1}a_n},\ n \geq 1, \\ &c_n = s_n - a_n - b_n,\ n \geq 1, \end{aligned} \tag{3.6}$$

with $a_1 \neq s_1 - b_1$, $a_2 \neq s_2 - b_2$.
Under such conditions, α and β are two of the zeros of $q(x) := x^3 + ax^2 + bx + c$.
Furthermore, $\frac{q(x)}{x-\alpha}v$ and $\frac{q(x)}{(x-\alpha)(x-\beta)}v$ are the linear functionals respect to which $\{R_n\}_{n\geq 0}$ and $\{\tilde{R}_n\}_{n\geq 0}$ are orthogonal, respectively.

Proof. From Theorem 3.1 and the two first equations of (3.4), we have (3.5) and

$$(x - \alpha)w_1 = -\lambda u, \quad (x - \beta)w_2 = -\varepsilon w_1, \tag{3.7}$$

where α, β , λ and ε are certain complex numbers, λ, $\varepsilon \neq 0$, w_1 and w_2 are the linear functionals with respect to which $\{R_n\}_{n\geq 0}$ and $\{\tilde{R}_n\}_{n\geq 0}$ are orthogonal, respectively. Substituting R_n and $\tilde{R}_n$ in (2.1), we get

$$\begin{aligned} &Q_1 = \tilde{R}_1 + s_1 - a_1 - b_1, \\ &Q_2 = \tilde{R}_2 + (s_2 - a_2 - b_2)\tilde{R}_1 + [t_2 - a_1(s_2 - a_2) - b_1(s_2 - a_2 - b_2)], \\ &Q_n = \tilde{R}_n + (s_n - a_n - b_n)\tilde{R}_{n-1} + [t_n - a_{n-1}(s_n - a_n) - b_{n-1}(s_n - a_n - b_n)]R_{n-2} \\ &\quad + [r_n - a_{n-2}[t_n - a_{n-1}(s_n - a_n)]P_{n-3},\ n \geq 3, \end{aligned}$$

then we also have $t_n = (s_n - a_n - b_n)b_{n-1} + a_{n-1}(s_n - a_n)$ for all $n \geq 2$ and $r_n = a_{n-2}(s_n - a_n - b_n)b_{n-1}$ for all $n \geq 3$, thus, $s_n \neq a_n + b_n$ for every $n \geq 3$. Hence (3.6) follows and, furthermore $s_1 \neq a_1 + b_1$, $s_2 \neq a_2 + b_2$ hold.

Moreover, since $\{\tilde{R}_n\}_{n\geq 0}$ is an SMOP with respect to w_2, being $\{Q_n\}_{n\geq 0}$ an SMOP with respect to v, then using Theorem 3.1, we find

$$(x-\gamma)v = -\mu w_2, \tag{3.8}$$

where γ and μ are complex numbers, $\mu \neq 0$. Thus,

$$q(x)v = -\mu\lambda\varepsilon(x-\alpha)(x-\beta)(x-\gamma)v.$$

Since v is regular this gives $k = \mu\lambda\varepsilon$ and α, β and γ are the zeros of $q(x)$.

Conversely, from Theorem 3.1, (3.5) implies that the sequence $\{R_n\}_{n\geq 0}$ defined by $R_n = P_n + a_n P_{n-1},\ n \geq 1$, and $\tilde{R}_n = R_n + b_n R_{n-1},\ n \geq 1$, are SMOPs with respect to the linear functionals w_1 and w_2 such that $(x-\alpha)w_1 = ku$ and $(x-\beta)w_2 = k'w_1$ where $k,\ k' \in \mathbb{C} - \{0\}$, respectively.

We have $s_n \neq a_n + b_n,\ n \geq 1$. Taking $t_n = (s_n - a_n - b_n)b_{n-1} + a_{n-1}(s_n - a_n),\ n \geq 2$, and $r_n = a_{n-2}(s_n - a_n - b_n)b_{n-1},\ n \geq 3$, we obtain

$$\begin{aligned}
\tilde{R}_n &= P_n + (b_n + a_n)P_{n-1} + b_n a_{n-1} P_{n-2},\ n \geq 2,\\
Q_n &= P_n + s_n P_{n-1} + ((s_n - a_n - b_n)b_{n-1} + a_{n-1}(s_n - a_n))P_{n-2} + a_{n-2}(s_n - a_n - b_n)b_{n-1}P_{n-3},\ n \geq 3,\\
Q_n &= \tilde{R}_n + (s_n - a_n - b_n)\tilde{R}_{n-1},\ n \geq 1.
\end{aligned}$$

□

A matrix interpretation. If w_1, w_2 and v denote the corresponding linear functionals for $\{R_n\}_{n\geq 0}, \{\tilde{R}_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, respectively, defined by (3.4), (3.7) and (3.8), then it is well known (see [7]) that

$$\begin{aligned}
(x-\alpha)P_n &= R_{n+1} + d_n R_n,\\
(x-\beta)R_n &= \tilde{R}_{n+1} + d'_n \tilde{R}_n,\\
(x-\gamma)\tilde{R}_n &= Q_{n+1} + d''_n Q_n,\ n \geq 0,
\end{aligned} \tag{3.9}$$

with $d_n d'_n d''_n \neq 0$.

Lets $\mathcal{P} = (P_0, P_1, ...)^T$, $\mathcal{R} = (R_0, R_1, ...)^T$, $\tilde{\mathcal{R}} = (\tilde{R}_0, \tilde{R}_1, ...)^T$, and $\mathcal{Q} = (Q_0, Q_1, ...)^T$ and $J_{\mathcal{P}},\ J_{\mathcal{R}},\ J_{\tilde{\mathcal{R}}}$ and $J_{\mathcal{Q}}$ the corresponding monic Jacobi matrices. Then, the recurrence relations for such SMOPs read

$$x\mathcal{P} = J_{\mathcal{P}}\mathcal{P}, \qquad x\mathcal{R} = J_{\mathcal{R}}\mathcal{R}, \qquad x\tilde{\mathcal{R}} = J_{\tilde{\mathcal{R}}}\mathcal{H}, \qquad x\mathcal{Q} = J_{\mathcal{Q}}\mathcal{Q}. \tag{3.10}$$

On the other hand , from (3.4) and (3.9) we have the matrix representations

$$\begin{aligned}
\mathcal{R} &= L_1\mathcal{P}, & (x-\alpha)\mathcal{P} &= U_1\mathcal{R},\\
\tilde{\mathcal{R}} &= L_2\mathcal{R}, & (x-\beta)\mathcal{R} &= U_2\mathcal{H},\\
\mathcal{Q} &= L_3\tilde{\mathcal{R}}, & (x-\gamma)\tilde{\mathcal{R}} &= U_3\mathcal{Q},
\end{aligned} \tag{3.11}$$

where $L_1,\ L_2$ and L_3 are three lower bidiagonal matrices with 1 as entries in the diagonal and $U_1,\ U_2$ and U_3 are upper bidiagonal matrices with 1 as entries in the upper diagonal given explicitly by

$$L_1 = \begin{pmatrix} 1 & & & & \\ a_1 & 1 & & & \\ & a_2 & 1 & & \\ & & a_3 & \ddots & \\ & & & \ddots & \end{pmatrix},\ L_2 = \begin{pmatrix} 1 & & & & \\ b_1 & 1 & & & \\ & b_2 & 1 & & \\ & & b_3 & \ddots & \\ & & & \ddots & \end{pmatrix},\ L_3 = \begin{pmatrix} 1 & & & \\ s_1 - a_1 - b_1 & 1 & & \\ & s_2 - a_2 - b_2 & 1 & \\ & & \ddots & \ddots \\ & & & \ddots \end{pmatrix},$$

$$U_1 = \begin{pmatrix} d_0 & 1 & & & \\ & d_1 & 1 & & \\ & & d_2 & 1 & \\ & & & d_3 & \ddots \\ & & & & \ddots \end{pmatrix},\ U_2 = \begin{pmatrix} d'_0 & 1 & & & \\ & d'_1 & 1 & & \\ & & d'_2 & 1 & \\ & & & d'_3 & \ddots \\ & & & & \ddots \end{pmatrix} \text{ and } U_3 = \begin{pmatrix} d''_0 & 1 & & & \\ & d''_1 & 1 & & \\ & & d''_2 & 1 & \\ & & & d''_3 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

Notice that from (3.10) and (3.11), we get

$$J_{\mathcal{P}} - \alpha I = U_1 L_1, \tag{3.12}$$
$$J_{\mathcal{R}} - \alpha I = L_1 U_1, \tag{3.13}$$
$$J_{\mathcal{R}} - \beta I = U_2 L_2, \tag{3.14}$$
$$J_{\tilde{\mathcal{R}}} - \beta I = L_2 U_2, \tag{3.15}$$
$$J_{\tilde{\mathcal{R}}} - \gamma I = U_3 L_3, \tag{3.16}$$
$$J_{\mathcal{Q}} - \gamma I = L_3 U_3. \tag{3.17}$$

As a consequence we can summarize the process as follows.
Step 1. Given $J_{\mathcal{P}}$, from L_1 and (3.12) we get U_1.
Step 2. From (3.13) we get $J_{\mathcal{R}}$.
Step 3. Given $J_{\mathcal{R}}$, from L_2 and (3.14) we get U_2.
Step 4. From (3.15) we get $J_{\tilde{\mathcal{R}}}$.
Step 5. Given $J_{\tilde{\mathcal{R}}}$, from L_3 and (3.16) we get U_3.
Step 6. From (3.17) we get $J_{\mathcal{Q}}$.

Notice that this is essentially the iteration of canonical Geronimus transformations (see [18]).

3.2. The split of a 1-4 relation in two relations of types 1-2 and 1-3 .

We have to consider two subcases:

3.2.1. 1-2 relation and then 1-3 relation.

Proposition 3.3. *Let $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ be sequences of complex numbers, $a_n \neq 0$, $n \geq 1$ and $c_n \neq 0$, $n \geq 2$. The representation* (2.1) *can be written as*

$$\begin{aligned} R_n &= P_n + a_n P_{n-1}, \quad n \geq 1, \\ Q_n &= R_n + b_n R_{n-1} + c_n R_{n-2}, \quad n \geq 2, \end{aligned} \tag{3.18}$$

where $\{R_n\}_{n\geq 0}$ is an SMOP if and only if there exists a complex number α such that

$$D_n := \beta_n - a_{n+1} - \frac{\gamma_n}{a_n} = \alpha, \; n \geq 1, \tag{3.19}$$

and

$$\begin{aligned} & t_{n+2} = a_{n+1}(s_{n+2} - a_{n+2}) + \tfrac{r_{n+2}}{a_n}, \; n \geq 1, \\ & b_n = s_n - a_n, \; n \geq 1, \\ & c_n = t_n - (s_n - a_n)a_{n-1}, \; n \geq 2, \end{aligned} \tag{3.20}$$

with $t_2 \neq a_1(s_2 - a_2)$.
Furthermore, α is a zero of $q(x)$ and $\frac{q(x)}{x-\alpha}v$ is the corresponding linear functional of the SMOP $\{R_n\}_{n\geq 0}$.

Proof. From Theorem 3.1 and the first equation of (3.18), we have (3.19) and

$$(x - \alpha)w = -\lambda u, \tag{3.21}$$

where α and λ are certain complex numbers, $\lambda \neq 0$, and w is the linear functional with respect to which $\{R_n\}_{n\geq 0}$ is orthogonal. Replacing R_n in (2.1), we get

$$\begin{aligned} Q_1 &= R_1 + s_1 - a_1, \\ Q_2 &= R_2 + (s_2 - a_2)R_1 + t_2 - (s_2 - a_2)a_1, \\ Q_n &= R_n + (s_n - a_n)R_{n-1} + [t_n - (s_n - a_n)a_{n-1}]R_{n-2} + [r_n - (t_n - (s_n - a_n)a_{n-1})a_{n-2}]P_{n-3}, \; n \geq 3. \end{aligned}$$

Then we have $r_n - (t_n - (s_n - a_n)a_{n-1})a_{n-2} = 0$ for all $n \geq 3$ and the conditions $t_n \neq a_{n-1}(s_n - a_n)$ for each $n \geq 3$. Hence (3.20) follows and, furthermore $t_2 \neq (s_2 - a_2)a_1$ holds. Moreover, since $\{R_n\}_{n\geq 0}$ is an SMOP with respect to w, being $\{Q_n\}_{n\geq 0}$ an SMOP with respect to v, then using Theorem 2.2 in [2] we find

$$(x^2 + \beta x + \gamma)v = \mu w, \tag{3.22}$$

where β, γ and μ are complex numbers, $\mu \neq 0$. Thus,

$$q(x)v = \frac{k}{\mu\lambda}(x^2 + \beta x + \gamma)(x - \alpha)v.$$

Since v is regular, this gives $k = \mu\lambda$ and α is a zero of $q(x)$.

Conversely, given $\{a_n\}_{n\geq 1}$ in the above conditions, from Theorem 3.1, (3.19) implies that the sequence $\{R_n\}_{n\geq 0}$ defined by $R_n = P_n + a_n P_{n-1}$, $n \geq 1$, is an SMOP with respect to a linear functional w such that $(x - \alpha)w = ku$ where $k \in \mathbb{C} - \{0\}$.

Taking $r_n = (t_n - (s_n - a_n)a_{n-1})a_{n-2}$, $n \geq 3$, we have $t_n \neq (s_n - a_n)a_{n-1}$, $n \geq 2$. So, we can write

$$\begin{aligned} Q_n &= P_n + s_n P_{n-1} + t_n P_{n-2} + (t_n - (s_n - a_n)a_{n-1})a_{n-2}P_{n-3} \\ &= R_n + (s_n - a_n)R_{n-1} + (t_n - (s_n - a_n)a_{n-1})R_{n-2}, \ n \geq 2. \end{aligned}$$

□

A matrix interpretation. In the sequel, we present a matrix interpretation of these results in terms of the monic Jacobi matrices associated with the SMOPs $\{P_n\}_{n\geq 0}$, $\{R_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, respectively.

Let $\mathcal{P} = (P_0, P_1, ...)^T$, $\mathcal{R} = (R_0, R_1, ...)^T$ and $\mathcal{Q} = (Q_0, Q_1, ...)^T$ be the column vectors associated with these orthogonal families, and $J_{\mathcal{P}}$, $J_{\mathcal{R}}$ and $J_{\mathcal{Q}}$ the corresponding monic Jacobi matrices. Then, the recurrence relations for such SMOPs read $x\mathcal{P} = J_{\mathcal{P}}\mathcal{P}$, $x\mathcal{R} = J_{\mathcal{R}}\mathcal{R}$ and $x\mathcal{Q} = J_{\mathcal{Q}}\mathcal{Q}$.

If w denotes the corresponding linear functional for $\{R_n\}_{n\geq 0}$, given by (3.21), then it is well known (see [7]) that

$$(x - \alpha)P_n = R_{n+1} + d_n R_n, \ n \geq 0, \quad \text{with, } d_n \neq 0.$$

Then, from the first equation of (3.18), we get

$$\mathcal{R} = L\mathcal{P}, \qquad (x - \alpha)\mathcal{P} = U\mathcal{R}, \tag{3.23}$$

where L is a lower bidiagonal matrix with 1 as diagonal entries and U is an upper bidiagonal matrix with 1 as entries in the upper diagonal given explicitly by

$$L = \begin{pmatrix} 1 & & & & \\ a_1 & 1 & & & \\ & a_2 & 1 & & \\ & & a_3 & 1 & \\ & & & \ddots & \ddots \end{pmatrix} \text{ and } U = \begin{pmatrix} d_0 & 1 & & & \\ & d_1 & 1 & & \\ & & d_2 & 1 & \\ & & & d_3 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

Thus, we get

$$J_{\mathcal{P}} - \alpha I = UL \tag{3.24}$$

and

$$J_{\mathcal{R}} - \alpha I = LU. \tag{3.25}$$

The previous process is known as Darboux transformation and $J_{\mathcal{R}}$ is said to be the Darboux transform of $J_{\mathcal{P}}$ (see[5]).

On the other hand, from (3.22) and the classical Christoffel formula (see [7]) we can express $(x^2+\beta x+\gamma)\mathcal{R}$ using the matrix representation

$$(x^2+\beta x+\gamma)\mathcal{R}=\mathcal{N}\mathcal{Q},$$

where $\mathcal{N}$ is a banded upper triangular matrix such that $n_{k,k+2}=1$ and $n_{k,j}=0$ for $j-k>2$. Next, we will describe a method to find the matrix $J_{\mathcal{Q}}$ using the matrix $J_{\mathcal{R}}$ and the polynomial $x^2+\beta x+\gamma$. From the first equation of (3.18), we may write $\mathcal{Q}=\mathcal{M}\mathcal{R}$ where

$$\mathcal{M}=\begin{pmatrix} 1 & 0 & 0 & \dots & \dots & \dots & \dots & \dots \\ b_1 & 1 & 0 & 0 & \dots & \dots & \dots & \dots \\ c_2 & b_2 & \ddots & \ddots & 0 & \dots & \dots & \dots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \dots & \dots \\ \vdots & \ddots & c_n & b_n & 1 & 0 & 0 & \dots \\ 0 & \dots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

with, $b_n=s_n-a_n,\ n\geq 1$ and $c_n=t_n-(s_n-a_n)a_{n-1},\ n\geq 2$, then $x\mathcal{M}\mathcal{R}=J_{\mathcal{Q}}\mathcal{M}\mathcal{R}$ and, as a consequence, $J_{\mathcal{R}}\mathcal{R}=\mathcal{M}^{-1}J_{\mathcal{Q}}\mathcal{M}\mathcal{R}$. Thus, we get

$$\mathcal{M}J_{\mathcal{R}}=J_{\mathcal{Q}}\mathcal{M}.$$

Thus $(x^2+\beta x+\gamma)\mathcal{R}=\mathcal{N}\mathcal{M}\mathcal{R}$, and then

$$J_{\mathcal{R}}^2+\beta J_{\mathcal{R}}+\gamma I=\mathcal{N}\mathcal{M}. \tag{3.26}$$

But, from $(x^2+\beta x+\gamma)\mathcal{Q}=\mathcal{M}\mathcal{N}\mathcal{Q}$, we get

$$J_{\mathcal{Q}}^2+\beta J_{\mathcal{Q}}+\gamma I=\mathcal{M}\mathcal{N}. \tag{3.27}$$

As a conclusion, we can summarize our process as follows.

Step 1. Given $J_{\mathcal{P}}$, from L and (3.24) we get U.
Step 2. From (3.25) we get $J_{\mathcal{R}}$.
Step 3. Given $J_{\mathcal{R}}$, we find the polynomial matrix $J_{\mathcal{R}}^2+\beta J_{\mathcal{R}}+\gamma I$.
Step 4. From $\mathcal{M}$ and (3.26) we find $\mathcal{N}$.
Step 5. From (3.27) we obtain the polynomial matrix $J_{\mathcal{Q}}^2+\beta J_{\mathcal{Q}}+\gamma I$.
Step 6. Taking into account that $J_{\mathcal{Q}}$ is a tridiagonal matrix, from step 3 we can deduce $J_{\mathcal{Q}}$, since $(J_{\mathcal{Q}}+\frac{\beta}{2}I)^2=\mathcal{M}\mathcal{N}-(\gamma-\frac{\beta^2}{4})I$.

3.2.2. 1-3 relation and then 1-2 relation.

Proposition 3.4. *Given three sequences of complex numbers $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$, $b_n\neq 0,\ n\geq 2$ and $c_n\neq 0,\ n\geq 1$, then (2.1) can be written as*

$$\begin{aligned} R_n &= P_n+a_nP_{n-1}+b_nP_{n-2},\ n\geq 2,\\ Q_n &= R_n+c_nR_{n-1},\ n\geq 1, \end{aligned} \tag{3.28}$$

where $\{R_n\}_{n\geq 0}$ is a SMOP, if and only if $\gamma_i+b_i-b_{i+1}+a_i(\beta_{i-1}-\beta_i-a_i+a_{i+1})\neq 0$, for $i=1,\ 2$ and there exist two complex numbers $\alpha,\ \beta$, such that

$$\begin{aligned} D_n:\quad & \frac{a_n}{b_{n+1}}[\gamma_{n+1}+b_{n+1}-b_{n+2}+a_{n+1}(\beta_n-\beta_{n+1}-a_{n+1}+a_{n+2})]+a_{n+1}-\beta_{n-1}-\beta_n\\ & =\alpha,\ n\geq 1,\\ E_n:\quad & \frac{1}{b_{n+1}}[\gamma_{n+1}-b_{n+2}+a_{n+1}(\beta_n-\beta_{n+1}-a_{n+1}+a_{n+2})][\gamma_n+b_n-b_{n+1}+a_n(a_{n+1}-\beta_n)]\\ & +b_n-\gamma_{n-1}+(a_{n+1}-\beta_n)(a_n-\beta_{n-1})=\beta,\ n\geq 1, \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} &a_{n+1} = s_{n+1} - \tfrac{r_{n+1}}{b_n}, \quad n \geq 2, \\ &b_{n+1} = t_{n+1} - (a_{n+1} - s_{n+1})a_n, \quad n \geq 1, \\ &c_n = s_n - a_n, \ n \geq 1, \end{aligned} \tag{3.30}$$

with $a_1 \neq s_1$, $a_2 \neq s_2$.
In this case $q(x) = (x - x_1)(x^2 + \alpha x + \beta)$ *where* $x_1 \in \mathbb{C}$ *and* $(x - x_1)v$ *is the linear functional associated with the SMOP* $\{R_n\}_{n\geq 0}$.

Proof. From Theorem 2.2 in [2], we have (3.29) and

$$(x^2 + \alpha x + \beta)w = \lambda u, \tag{3.31}$$

where α, β and λ are certain complex numbers, $\lambda \neq 0$, and w is the regular functionals with respect to which $\{R_n\}_{n\geq 0}$ is orthogonal. Replacing R_n in (2.1), we get

$$\begin{aligned} &Q_1 = R_1 + s_1 - a_1, \\ &Q_2 = R_2 + (s_2 - a_2)R_1 + t_2 - b_2 - (s_2 - a_2)a_1, \\ &Q_n = R_n + (s_n - a_n)R_{n-1} + [t_n - b_n - (s_n - a_n)a_{n-1}]P_{n-2} + [r_n - (s_n - a_n)b_{n-1}]P_{n-3}, \ n \geq 3. \end{aligned}$$

Therefore $t_n - b_n + (s_n - a_n)a_{n-1} = 0$ for all $n \geq 2$ and $r_n - (s_n - a_n)b_{n-1} = 0$ for all $n \geq 3$. Then $a_n \neq s_n$, $n \geq 3$. Hence (3.30) follows and, furthermore, $a_1 \neq s_1$ and $a_2 \neq s_2$ hold. Using the second equation of (3.28) and Theorem 3.1, we have

$$(x - \gamma)v = -k_1 w, \tag{3.32}$$

where γ and k_1 are complex numbers, $k_1 \neq 0$. Thus, since by hypothesis we also have

$$q(x)v = \frac{k}{k_1\lambda}(x - \gamma)(x^2 + \alpha x + \beta)v.$$

This gives $k = k_1\lambda$ and then γ is one of the zeros of $q(x) := x^3 + ax^2 + bx + c$ because v is regular.

Conversely, given $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 2}$ in the above conditions, from Theorem 2.2 in [2], (3.29) implies that the sequence $\{R_n\}_{n\geq 0}$ defined by $R_0 = 1$, $R_1 = P_1 + a_1P_0$, $R_n = P_n + a_nP_{n-1} + b_nP_{n-2}$, $n \geq 2$, is an SMOP with respect to a regular linear functional w such that $(x^2 + \alpha x + \beta)w = ku$ where $k \in \mathbb{C} - \{0\}$.

Taking $t_n = b_n + (s_n - a_n)a_{n-1}$, $n \geq 2$ and $r_n = (s_n - a_n)b_{n-1}$, $n \geq 3$. So, we can write

$$Q_n = P_n + s_nP_{n-1} + (b_n + (s_n - a_n)a_{n-1})P_{n-2} + (s_n - a_n)b_{n-1}P_{n-3} = R_n + (s_n - a_n)R_{n-1}, \ n \geq 1.$$

□

When $\{P_n\}_{n\geq 0}$ is symmetric.

Assume that the sequence $\{P_n\}_{n\geq 0}$ is orthogonal with respect to a symmetric linear functional u (i.e. $(u)_{2n+1} = 0, n \geq 0$). Then $\beta_n = 0, n \geq 0$, and there exist two polynomial sequences $\{V_n\}_{n\geq 0}$ and $\{V_n^*\}_{n\geq 0}$ such that for all n, $P_{2n}(x) = V_n(x^2)$ and $P_{2n+1} = xV_n^*(x^2)$.

It is known (see [6]) that $\{V_n\}_{n\geq 0}$ and $\{V_n^*\}_{n\geq 0}$ are SMOPs with respect to the linear functionals σ_u and $x\sigma_u$ where $(\sigma_u, \ x^n) = (u, \ x^{2n})$, $n \geq 0$.

It's clear that the polynomials Q_n defined by (2.1) can not be symmetric because $r_n \neq 0$ for all $n \geq 3$. Suppose that the sequence $\{Q_n\}_{n\geq 0}$ is orthogonal with respect to a linear functional v such that xv is symmetric and regular, then v is said to be quasi-antisymmetric (for more information about these linear functionals please see [14] and [16]). From (2.16), we obtain $(ax + c)\sigma_{xv} = 0$ then $a = c = 0$ because σ_{xv} is regular. Therefore, the relation between the linear functionals u and v is $(x^2 + b)xv = -ku$. Noting $w = xv$, then $(x^2 + b)w = -ku$ and from Proposition 2.1 in [13], there exists a symmetric sequence $\{R_n\}_{n\geq 0}$ orthogonal with respect to w and satisfying (3.28). Thus $a_n = 0$. From Proposition 3.4, we obtain $b_n = t_n$ and $s_{n+1} = \frac{r_{n+1}}{t_n}$. Furthermore, there exist $\{G_n\}_{n\geq 0}$ and $\{G^*_n\}_{n\geq 0}$ SMOPs with respect to σ_{xv} and $x\sigma_{xv}$, respectively, satisfying

$$Q_{2n}(x) = G_n(x^2) + \theta_n x G^*_{n-1}(x^2), \quad Q_{2n+1}(x) = \lambda_n G_n(x^2) + xG^*_n(x^2),\ n \geq 0,$$

with $\theta_n \neq 0$ and $\lambda_n \neq 0$, $n \geq 0$. In this case, from (3.28), we have for $n \geq 0$, $Q_n = R_n + s_n R_{n-1}$, $\theta_n = s_{2n}$, $\lambda_n = s_{2n+1}$ and

$$R_{2n}(x) = G_n(x^2), \quad R_{2n+1}(x) = xG^*_n(x^2),$$

where

$$G_n(x) = V_n(x) + t_{2n}V_{n-1}(x), \quad G^*_n(x) = V^*_n(x) + t_{2n+1}V^*_{n-1}(x).$$

The coefficients s_n, t_{2n} and t_{2n+1} can be computed using Theorem 3.1.

Moreover, the parameters $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ of the recurrence relation of the sequence $\{Q_n\}_{n\geq 0}$ are defined by

$$\tilde{\beta}_n = s_n - s_{n+1},\ n \geq 0, \tag{3.33}$$

$$\tilde{\gamma}_n = -s_n^2,\ n \geq 1. \tag{3.34}$$

Indeed, taking $\beta_n = 0$ in (2.5) and $a = 0$ in (2.13) , we get (3.33) and

$$\frac{t_{n+2}}{r_{n+2}} = -s_{n+3}.$$

Using (2.3) for $n + 2$ instead of n, we obtain

$$\frac{1}{r_{n+2}}[t_{n+1}\tilde{\gamma}_{n+2} - r_{n+2}(s_{n+3} - s_{n+2})] = -s_{n+3},$$

and introducing $t_{n+1} = \frac{r_{n+2}}{s_{n+2}}$, that is, for $n \geq 1$

$$\tilde{\gamma}_{n+2} = -s_{n+2}^2. \tag{3.35}$$

From (2.23) and (3.35), for n=1, we obtain

$$-s_3^2\frac{\tilde{\gamma}_1\tilde{\gamma}_2}{r_3} - s_3^2 s_1^2\frac{\tilde{\gamma}_2}{r_3} - s_3^2(t_2 - \tilde{\gamma}_1)\frac{t_2}{r_3} - s_3\gamma_1 + r_3.$$

Inserting $t_2 = \frac{r_3}{s_3}$, we obtain

$$-\frac{s_3^2}{r_3}\tilde{\gamma}_2[\tilde{\gamma}_1 + s_1^2] = 0.$$

Then, we deduce $\tilde{\gamma}_1 = -s_1^2$.

Using (2.15) and (2.2)-(2.6), we get

$$\begin{aligned}&\frac{\tilde{\gamma}_2\tilde{\gamma}_3\tilde{\gamma}_4}{r_4} + (s_2 - \beta_1)\frac{s_2}{r_4}\tilde{\gamma}_3\tilde{\gamma}_4 - [s_3\beta_1 + \gamma_1 + \gamma_2 - t_3 - \beta_1\beta_2]\frac{t_3}{r_4}\tilde{\gamma}_4\\&+ (\beta_2 - \beta_0 + \beta_3 - s_4)\gamma_1 - (s_4 - \beta_3)\gamma_2 + \gamma_3\beta_1 - \beta_1 t_4 + \beta_1\beta_2 s_4 - \beta_1\beta_2\beta_3 + r_4 = C_2.\end{aligned}$$

From $\beta_n = 0$, $\tilde{\gamma}_3 = -s_3^2$, $\tilde{\gamma}_4 = -s_4^2$ and $t_3 = \frac{r_4}{s_4}$, we obtain

$$\frac{\tilde{\gamma}_2 s_3^2 s_4^2}{r_4} + \frac{s_2^2 s_3^2 s_4^2}{r_4} = 0.$$

Therefore $\tilde{\gamma}_2 = -s_2^2$.

A matrix interpretation. We will describe a method to find the matrix $J_{\mathcal{R}}$ using the matrix $J_{\mathcal{P}}$ and the polynomial $x^2 + \alpha x + \beta$.
Taking into account the first equation of (3.28) we may write $\mathcal{R} = \mathcal{M}\mathcal{P}$ where $M = (m_{k,j})$ is a banded lower triangular matrix such that $m_{k,k} = 1$, and $m_{k,j} = 0$ for $k - j > 2$,

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 & \dots & \dots & \dots \\ a_1 & 1 & 0 & 0 & \dots & \dots \\ b_2 & a_2 & 1 & \ddots & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & b_n & a_n & 1 & 0 \\ 0 & \dots & 0 & \ddots & \ddots & \ddots \end{pmatrix},$$

then $x\mathcal{M}\mathcal{P} = J_{\mathcal{R}}\mathcal{M}\mathcal{P}$ and, as a consequence, $J_{\mathcal{P}}\mathcal{P} = \mathcal{M}^{-1}J_{\mathcal{R}}\mathcal{M}\mathcal{P}$. Thus, we get

$$\mathcal{M}J_{\mathcal{P}} = J_{\mathcal{R}}\mathcal{M}.$$

On the other hand, from (3.31) and the classical Christoffel formula (see [7]) we can express $(x^2 + \alpha x + \beta)\mathcal{P}$ using the matrix representation

$$(x^2 + \alpha x + \beta)\mathcal{P} = \mathcal{N}\mathcal{R},$$

where $\mathcal{N}$ is a banded upper triangular matrix such that $n_{k,k+2} = 1$ and $n_{k,j} = 0$ for $j - k > 2$.
Thus $(x^2 + \alpha x + \beta)\mathcal{P} = \mathcal{N}\mathcal{M}\mathcal{P}$, and then

$$J_{\mathcal{P}}^2 + \alpha J_{\mathcal{P}} + \beta I = \mathcal{N}\mathcal{M}. \tag{3.36}$$

But, from $(x^2 + \alpha x + \beta)\mathcal{R} = \mathcal{M}\mathcal{N}\mathcal{R}$, we get

$$J_{\mathcal{R}}^2 + \alpha J_{\mathcal{R}} + \beta I = \mathcal{M}\mathcal{N}. \tag{3.37}$$

By (3.32), it is well known (see [7]) that

$$(x - \gamma)R_n = Q_{n+1} + d_n Q_n, \; n \geq 0, \quad \text{with, } d_n \neq 0.$$

Then, from the second equation of (3.28), we obtain

$$\mathcal{Q} = L\mathcal{R}, \qquad (x - \gamma)\mathcal{R} = U\mathcal{Q}, \tag{3.38}$$

where

$$L = \begin{pmatrix} 1 & & & & \\ c_1 & 1 & & & \\ & c_2 & 1 & & \\ & & c_3 & 1 & \\ & & & \ddots & \ddots \end{pmatrix} \text{ and } U = \begin{pmatrix} d_0 & 1 & & & \\ & d_1 & 1 & & \\ & & d_2 & 1 & \\ & & & d_3 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

Thus, we get

$$J_{\mathcal{R}} - \gamma I = UL \tag{3.39}$$

and

$$J_{\mathcal{Q}} - \gamma I = LU. \tag{3.40}$$

As a conclusion, we can summarize our process as follows.

Step 1. Given $J_{\mathcal{P}}$, we find the polynomial matrix $J^2_{\mathcal{P}} + \alpha J_{\mathcal{P}} + \beta I$.
Step 2. From $\mathcal{M}$ and (3.36) we find $\mathcal{N}$.
Step 3. From (3.37) we obtain the polynomial matrix $J^2_{\mathcal{R}} + \alpha J_{\mathcal{R}} + \beta I$.
Step 4. Taking into account that $J_{\mathcal{R}}$ is a tridiagonal matrix, from step 3 we can deduce $J_{\mathcal{R}}$, since $(J_{\mathcal{R}} + \frac{\alpha}{2}I)^2 = \mathcal{M}\mathcal{N} - (\beta - \frac{\alpha^2}{4})I$.
Step 5. Given $J_{\mathcal{R}}$, from L and (3.39) we get U.
Step 6. From (3.40) we get $J_{\mathcal{Q}}$.

4. Illustrative Examples

(1) Let $\{P_n = L_n^{(\alpha)}\}_{n\geq 0}$ be the sequence of monic Laguerre polynomials orthogonal with respect to the linear functional u defined by the weight function $x^\alpha e^{-x}\chi_{(0,+\infty)}$ with $\alpha > 1$. We can take the auxiliary polynomials $R_n(x) = L_n^{(\alpha-1)}(x)$ and $\tilde{R}_n(x) = L_n^{(\alpha-2)}(x)$ orthogonal, respectively with respect to w_1 and w_2. These polynomials satisfy $R_n(x) = L_n^{(\alpha)}(x) + nL_{n-1}^{(\alpha)}(x)$, and $\tilde{R}_n(x) = L_n^{(\alpha-1)}(x) + nL_{n-1}^{(\alpha-1)}(x)$ (see [6]). Furthermore, we have the following equations

$$xw_1 = \alpha u, \quad xw_2 = (\alpha - 1)w_1.$$

Then, the new sequence $\{Q_n\}_{n\geq 0}$ such that $Q_n(x) = \tilde{R}_n(x) + c_n\tilde{R}_{n-1}(x)$ is orthogonal with respect to the linear functional v satisfying $xv = (\alpha - 1 - c_1)w_2$. Thus

$$x^3 v = ku, \quad k = \alpha(\alpha - 1)(\alpha - 1 - c_1).$$

According to Proposition 3.2, the polynomials Q_n satisfy the relation (2.1) where $s_n = c_n + 2n$, $t_n = (2n - 2)s_n - 3n(n-1)$, $n \geq 1$ and $r_n = (n-1)(n-2)(s_n - 2n)$, $n \geq 1$. It is well known that the recurrence coefficients of $L_n^{(\alpha-2)}$, $\alpha > 1$, are $\beta_n = 2n + \alpha - 1$, $n \geq 0$ and $\gamma_n = n(n + \alpha - 2)$, $n \geq 1$ (see [6]) .

Using formula (3.2) for this case, having $x_1 = 0$ and $c_2 = 1 + \alpha - \frac{\alpha-1}{c_1}$, by induction we can derive that, for $\alpha > 1$, and $\alpha \neq 2$, the values of the parameters c_n in terms of c_1 are

$$c_n = n\frac{\Gamma(\alpha-1)(\alpha-1-c_1) + (c_1-1)\frac{\Gamma(n+\alpha-1)}{\Gamma(n+1)}}{\Gamma(\alpha-1)(\alpha-1-c_1) + (c_1-1)\frac{\Gamma(n-2+\alpha)}{\Gamma(n)}},\ n \geq 1, \tag{4.1}$$

and then v is regular if and only if

$$\Gamma(n)\Gamma(\alpha-1)(\alpha-1-c_1) + (c_1-1)\Gamma(n-2+\alpha) \neq 0,\ n \geq 1.$$

Notice that if $\alpha \in \mathbb{N} - \{2\}$ then c_n is a rational function of n, namely,

$$c_n = n\frac{\Gamma(\alpha-1)(\alpha-1-c_1) + (c_1-1)(\alpha+n-2)...(n+1)}{\Gamma(\alpha-1)(\alpha-1-c_1) + (c_1-1)(\alpha+n-3)...n},\ n \geq 1.$$

If $\alpha = 2$, then, by induction, we can also obtain, for $n \geq 2$

$$c_n = n\frac{(c_1-1)(1+\frac{1}{2}+...\frac{1}{n})+1}{(c_1-1)(1+\frac{1}{2}+...+\frac{1}{n-1})+1}, \tag{4.2}$$

and v is regular if and only if

$$(c_1-1)(1+\frac{1}{2}+...\frac{1}{n})+1 \neq 0,\ n \geq 1.$$

We have

$$v = (\alpha - c_1 - 1)x^{-1}w_2 + \delta_0. \tag{4.3}$$

In particular for $\alpha > 2$, we can write

$$(\alpha-2)v = (\alpha - c_1 - 1)w_3 + (c_1 - 1)\delta_0, \tag{4.4}$$

where w_3 is the corresponding linear functional for monic Laguerre polynomials $\{L_n^{(\alpha-3)}\}_{n\geq 0}$.

A matrix interpretation. If $P_n = L_n^{(\alpha)}$, $\alpha > 1$, and $a_n = b_n = n$, we obtain

$$(J_{\mathcal{P}})_{n+1} = \begin{pmatrix} \alpha+1 & 1 & 0 & \dots & 0 \\ \alpha+1 & \alpha+3 & \ddots & \ddots & \vdots \\ 0 & 2(\alpha+2) & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & n(n+\alpha) & 2n+\alpha+1 \end{pmatrix} \tag{4.5}$$

$$\text{and} \qquad (L_1)_{n+1} = (L_2)_{n+1} = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ & 2 & & & \\ & & \ddots & \ddots & \\ & & & n & 1 \end{pmatrix}. \tag{4.6}$$

From (3.12), we obtain

$$(U_1)_{n+1} = \begin{pmatrix} \alpha & 1 & & & \\ & \alpha+1 & 1 & & \\ & & \alpha+2 & 1 & \\ & & & \ddots & \ddots \\ & & & & \alpha+n \end{pmatrix},$$

thus,

$$(J_{\mathcal{R}})_{n+1} = (L_1)_{n+1}(U_1)_{n+1} = \begin{pmatrix} \alpha & 1 & & & \\ \alpha & \alpha+2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & n(n+\alpha-1) & \alpha+2n \end{pmatrix}.$$

Using (3.14), we obtain

$$(U_2)_{n+1} = \begin{pmatrix} \alpha-1 & 1 & & & \\ & \alpha & 1 & & \\ & & \alpha+1 & 1 & \\ & & & \ddots & \ddots \\ & & & & \alpha+n-1 \end{pmatrix}.$$

Then by (3.15), we get

$$(J_{\tilde{\mathcal{R}}})_{n+1} = (L_2)_{n+1}(U_2)_{n+1} = \begin{pmatrix} \alpha-1 & 1 & & & \\ \alpha-1 & \alpha+1 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & n(n+\alpha-2) & \alpha-1+2n \end{pmatrix}.$$

From (3.16), we have

$$(U_3)_{n+1} = \begin{pmatrix} \alpha-1-c_1 & 1 & & \\ & \alpha+1-c_2 & \ddots & \\ & & \ddots & 1 \\ & & & 2n+\alpha-1-c_{n+1} \end{pmatrix}.$$

With $c_n = s_n - 2n$ satisfies $c_{n+1} = 2n + \alpha - 1 - \frac{n(n+\alpha-2)}{c_n}$, $n \geq 1$. Then c_n is defined by (4.1) and (4.2).

Using (3.17), we get

$$(J_Q)_{n+1} = (L_3)_{n+1}(U_3)_{n+1} = \begin{pmatrix} \alpha-1-c_1 & 1 & & & \\ c_1(\alpha-1-c_1) & c_1+\frac{\alpha-1}{c_1} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & c_n(2n+\alpha-3-c_n) & c_n+\frac{n(n+\alpha-2)}{c_n} \end{pmatrix}. \tag{4.7}$$

(2) Let $\{P_n = \mathcal{U}_n\}_{n\geq 0}$ be the sequence of monic Chebyshev polynomials of the second kind, orthogonal with respect to the linear functional $u = \mathcal{U}$ defined by the weight function $(1-x^2)^{1/2}\chi_{(-1,1)}(x)$ with the recurrence coefficients $\beta_n^u = 0$, $n \geq 0$, and $\gamma_n^u = \frac{1}{4}$, $n \geq 1$. Consider the SMOP $\{R_n = T_n\}_{n\geq 0}$ orthogonal with respect to the Chebyshev linear functional of first kind $w = \mathcal{T}$. We have (see [6])

$$R_n = \mathcal{U}_n - \frac{1}{4}\mathcal{U}_{n-2},\ n \geq 2,$$

and $(x^2-1)w = -\frac{1}{2}u$. The new polynomials Q_n, such that $Q_n = R_n + s_nR_{n-1}$, $n \geq 1$, satisfy the relation (2.1) with $t_n = -\frac{1}{4}$ and $r_n = -\frac{1}{4}s_n$. Thus $xv = -2w$, and v is quasi-antisymmetric. It is well known that the recurrence coefficient of R_n are $\beta_n^w = 0$, $n \geq 0$, $\gamma_n^w = \frac{1}{4}$, $n \geq 2$, and $\gamma_1^w = \frac{1}{2}$ (see [6]).
Using Theorem 3.1 for this case, since $x_1 = 0$, by induction, the values of the parameters s_n, $n \geq 2$, are

$$\begin{aligned} \beta_0 - x_1 - s_1 &= -2, \text{ i.e. } s_1 = 2, \\ s_{2n} &= -\frac{1}{2s_1} = -\frac{1}{4},\ n \geq 1, \\ s_{2n+1} &= \frac{s_1}{2} = 1,\ n \geq 1. \end{aligned} \tag{4.8}$$

Then

$$\begin{gathered} r_{2n} = \frac{1}{16},\quad r_{2n+1} = -\frac{1}{4},\ n \geq 1, \\ \tilde{\beta}_0 = -s_1 = -2,\quad \tilde{\beta}_1 = s_1 - s_2 = \frac{9}{4},\quad \tilde{\beta}_{2n} = -\frac{5}{4},\quad \tilde{\beta}_{2n+1} = \frac{5}{4},\ n \geq 1. \end{gathered} \tag{4.9}$$

From (3.33)-(3.34), and (4.9), we get

$$\tilde{\gamma}_1 = -s_1^2 = -4,\quad \tilde{\gamma}_{2n} = -s_{2n}^2 = -\frac{1}{16},\quad \tilde{\gamma}_{2n+1} = -s_{2n+1}^2 = -1,\ n \geq 1. \tag{4.10}$$

The regular linear functional v is given by

$$v = -2x^{-1}w + \delta_0.$$

A matrix interpretation. From Proposition 3.4 where $P_n = \mathcal{U}_n$, and $R_n = T_n$, we have $a_n = 0$, $n \geq 0$, and $b_n = -\frac{1}{4}$, $n \geq 2$. Then, the polynomials $\{Q_n\}_{n\geq 0}$ satisfy the relation (2.1) with, $t_n = -\frac{1}{4}$, $n \geq 2$, and

$r_n = -\frac{1}{4}s_n,\ n \geq 3$. Therefore

$$(L)_{n+1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ s_1 & 1 & \ddots & \ddots & \vdots \\ 0 & s_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & s_n & 1 \end{pmatrix}.$$

From the above results, we have

$$(J_{\mathcal{P}})_{n+1} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1/4 & 0 & \ddots & \ddots & \vdots \\ 0 & 1/4 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 1/4 & 0 \end{pmatrix}, \quad (J_{\mathcal{R}})_{n+1} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1/2 & 0 & \ddots & \ddots & \vdots \\ 0 & 1/4 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 1/4 & 0 \end{pmatrix}.$$

Then

$$[(J_{\mathcal{P}})_{n+1}]^2 - I = \begin{pmatrix} -3/4 & 0 & 1 & \dots & 0 \\ 0 & -1/2 & \ddots & \ddots & \vdots \\ 1/16 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 1/16 & 0 & -1/2 \end{pmatrix}.$$

From (3.36), we obtain

$$(N)_{n+1} = \begin{pmatrix} -1/2 & 0 & 1 & \dots & 0 \\ 0 & -1/4 & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & -1/4 \end{pmatrix}.$$

From (3.39), we get

$$(U)_{n+1} = \begin{pmatrix} -s_1 & 1 & & & \\ & -s_2 & 1 & & \\ & & -s_3 & 1 & \\ & & & \ddots & \ddots \\ & & & & -s_{n+1} \end{pmatrix},$$

with s_n satisfis $s_2 = -\frac{1}{2s_1}$, and $s_{n+1} = \frac{1}{4s_n},\ n \geq 2$. Then s_n is again defined by (4.8).
Using again (3.40), we get

$$(J_{\mathcal{Q}})_{n+1} = (L)_{n+1}(U)_{n+1} = \begin{pmatrix} -s_1 & 1 & 0 & \dots & 0 \\ -s_1^2 & s_1 - s_2 & \ddots & \ddots & \vdots \\ 0 & -s_2^2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & -s_n^2 & s_n - s_{n+1} \end{pmatrix}.$$

(3) Let $\{P_n = B_n^{\alpha}\}_{n\geq 0}$ be the sequence of monic Bessel polynomials orthogonal with respect to the linear functional $u = \mathcal{B}^{\alpha}$ defined by the weight function $x^{2(\alpha-1)}e^{-\frac{2}{x}}\displaystyle\int_x^{+\infty}\epsilon^{-2\alpha}e^{\frac{2}{\epsilon}-\epsilon^{\frac{1}{4}}}\sin(\epsilon^{\frac{1}{4}})d\epsilon\chi_{(0,+\infty)}$ with $\alpha > 1$ (see [15]). We can take the auxiliary polynomials $R_n = B_n^{\alpha-1}$ satisfying

$$R_n(x) = B_n^{\alpha}(x) + \frac{n}{(n-2+\alpha)(n+\alpha-1)}B_{n-1}^{\alpha}(x) + \frac{n(n-1)}{(2n+2\alpha-5)(n-2+\alpha)^2(2n+2\alpha-3)}B_{n-2}^{\alpha}(x)$$

orthogonal with respect to the linear functional $w = \mathcal{B}^{\alpha-1}$.
This linear functional verifies $x^2 w = \frac{4}{(2\alpha-1)(\alpha-1)}u$ (see[6]).
According to Proposition 3.4, the new polynomials Q_n such that

$$Q_n(x) = R_n(x) + c_n R_{n-1}(x),\ n \geq 1, \tag{4.11}$$

satisfy the relation (2.1) with
$s_n = c_n + \frac{n(n-1)}{(2n+2\alpha-5)(n-2+\alpha)^2(2n+2\alpha-3)},\ n \geq 1,$
$t_n = \frac{n-1}{(n-3+\alpha)(n+\alpha-2)}c_n + \frac{n(n-1)}{(2n+2\alpha-5)(n-2+\alpha)^2(2n+2\alpha-3)},\ n \geq 2,$
and $r_n = \frac{(n-1)(n-2)}{(2n+2\alpha-7)(n-3+\alpha)^2(2n+2\alpha-5)}c_n,\ n \geq 3.$
It is well known that the recurrence coefficients of $B_n^{\alpha-1}$ are

$$\begin{aligned} \beta_0^w &= -\frac{1}{\alpha-1},\ \beta_n^w = \frac{2-\alpha}{(n+\alpha-2)(n+\alpha-1)},\ n \geq 1, \\ \gamma_n^w &= -\frac{n(n+2\alpha-4)}{(2n+2\alpha-5)(n+\alpha-2)^2(2n+2\alpha-3)},\ n \geq 1. \end{aligned} \tag{4.12}$$

Using formula (3.2) for this case, having $x_1 = 0$, and taking into account (4.12), we can deduce by induction

$$c_n = -\frac{n+2\alpha-4}{(n+\alpha-2)(2n+2\alpha-5)}\frac{x_n}{x_{n-1}},\ n \geq 1, \tag{4.13}$$

where

$$\begin{cases} \alpha \neq \dfrac{3}{2}: & x_n = (\lambda - \dfrac{2}{2\alpha-3}) - \dfrac{(-1)^n\lambda\Gamma(2\alpha-3)\Gamma(n+1)}{\Gamma(n+2\alpha-3)},\ n \geq 0, \\ \alpha = \dfrac{3}{2}: & x_n = 1 + (-1)^n\dfrac{n\lambda}{2},\ n \geq 0, \end{cases} \tag{4.14}$$

with $\lambda = \frac{1}{\alpha-1} + c_1$.
The linear functional v is regular for every c_1 such that $x_n \neq 0,\ n \geq 0$, and it is given by

$$v = -\left(\frac{1}{\alpha-1} + c_1\right)x^{-1}w + \delta_0. \tag{4.15}$$

In particular for $\alpha > 2$, we can write

$$v = -\frac{(2\alpha-3)(\alpha-2)}{2}\left(\frac{1}{\alpha-1} + c_1\right)x\mathcal{B}^{\alpha-2} - \left(\frac{(2\alpha-3)}{2}\left(\frac{1}{\alpha-1} + c_1\right) - 1\right)\delta_0.$$

A matrix interpretation. We have $P_n = B_n^{\alpha},\ \alpha > 1,$ and $R_n = B_n^{\alpha-1}$, then

$$(J_{\mathcal{P}})_{n+1} = \begin{pmatrix} \beta_0 & 1 & 0 & \dots & 0 \\ \gamma_1 & \beta_1 & \ddots & \ddots & \vdots \\ 0 & \gamma_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \gamma_n & \beta_n \end{pmatrix},\quad (J_{\mathcal{R}})_{n+1} = \begin{pmatrix} \beta_0^w & 1 & 0 & \dots & 0 \\ \gamma_1^w & \beta_1^w & \ddots & \ddots & \vdots \\ 0 & \gamma_2^w & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \gamma_n^w & \beta_n^w \end{pmatrix},$$

where $\beta_n,\ \gamma_n$ and $\beta_n^w,\ \gamma_n^w$ are the recurrence coefficients of B_n^α and $B_n^{\alpha-1}$, respectively.
Thus, $\mathcal{R} = \mathcal{MP}$, where

$$(M)_{n+1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ a_1 & 1 & \ddots & \ddots & \vdots \\ b_2 & a_2 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & b_n & a_n & 1 \end{pmatrix},$$

with

$$a_n = \frac{n}{(n-2+\alpha)(n+\alpha-1)},\ n \geq 1,$$
$$b_n = \frac{n(n-1)}{(2n+2\alpha-5)(n-2+\alpha)^2(2n+2\alpha-3)},\ n \geq 2.$$

The polynomials $Q_n,\ n \geq 0$, satisfy $\mathcal{Q} = L\mathcal{R}$, with

$$(L)_{n+1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ c_1 & 1 & \ddots & \ddots & \vdots \\ 0 & c_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & c_n & 1 \end{pmatrix},$$

and c_n defined by (4.13)-(4.14).
Using (3.39) for this case, we obtain

$$(U)_{n+1} = \begin{pmatrix} d_0 & 1 & & & \\ & d_1 & 1 & & \\ & & d_2 & 1 & \\ & & & \ddots & \ddots \\ & & & & d_n \end{pmatrix},$$

where

$$d_0 = \beta_0^w - c_1 = -\frac{1}{\alpha-1} - c_1,$$
$$d_n = \beta_n^w - c_{n+1} = \frac{\gamma_n^w}{c_n} = \frac{n}{(n+\alpha-2)(2n+2\alpha-3)}\frac{x_{n-1}}{x_n},\ n \geq 1.$$

From (3.40), we get

$$(J_Q)_{n+1} = (L)_{n+1}(U)_{n+1} = \begin{pmatrix} d_0 & 1 & 0 & \dots & 0 \\ c_1 d_0 & c_1 + d_1 & \ddots & \ddots & \vdots \\ 0 & c_2 d_1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & c_n d_{n-1} & c_n + d_n \end{pmatrix}.$$

Then

$$
\begin{aligned}
\tilde{\beta}_0 &= d_0 = -\frac{1}{\alpha-1} - c_1,\\
\tilde{\beta}_{n+1} &= c_{n+1} + d_{n+1} = -\frac{n+2\alpha-3}{(n+\alpha-1)(2n+2\alpha-3)}\frac{x_{n+1}}{x_n} + \frac{n+1}{(n+\alpha-1)(2n+2\alpha-1)}\frac{x_n}{x_{n+1}},\ n \geq 0,\\
\tilde{\gamma}_1 &= -c_1\Big(\frac{1}{\alpha-1} + c_1\Big),\\
\tilde{\gamma}_{n+1} &= -\frac{n(n+2\alpha-3)}{(n+\alpha-1)(n+\alpha-2)(2n+2\alpha-3)^2}\frac{x_{n-1}x_{n+1}}{x_n^2},\ n \geq 1.
\end{aligned}
$$

Acknowledgments. The authors are so grateful to Professor F. Marcellán for his continued interest and helpful comments concerning the present work. Special thanks go to the referee for his valuable comments and useful suggestions and for his careful reading of the manuscript.

References

[1] M. Alfaro, A. Peña, J. Petronilho, M.L. Rezola, Orthogonal polynomials generated by a linear structure relation: Inverse problem. J. Math. Anal. Appl. 401 (2013) 182–197.
[2] M. Alfaro, A. Peña, M.L. Rezola, F. Marcellán, Orthogonal polynomials associated with an inverse quadratic spectral transform. Comput. Math. Appl. 61 (2011) 888–900.
[3] M. Alfaro, F. Marcellán, A. Peña, M.L. Rezola, When do linear combinations of orthogonal polynomials yield new sequences of orthogonal polynomials, J. Comput. Appl. Math. 233 (2010) 1446–1452.
[4] M. Alfaro, F. Marcellán, A. Peña, M.L. Rezola, On linearly related orthogonal polynomials and their functionals, J. Math. Anal. Appl. 287 (2003) 307–319.
[5] M. I. Bueno, F. Marcellán, Darboux transformation and perturbation of linear functionals. Linear Algebra Appl. 384 (2004) 215–242.
[6] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.
[7] W. Gautschi, An algorithmic implementation of the Generalized Christoffel Theorem, in: G. Hammerlin (Ed.), Numerical Integration, in: ISNM, vol. 57, Birkhauser Verlag, Basel, (1982), 89-106.
[8] F. Marcellán, S. Varma, On an inverse problem for a linear combination of orthogonal polynomials, J. Differ. Equ. Appl. 20 (2014), no. 4, 570–585.
[9] F. Marcellán, J. Petronilho, Orthogonal polynomials and coherent pairs: the classical case, Indag. Math. (NS) 6 (1995) 287–307.
[10] P. Maroni, I. Nicolau, On the inverse problem of the product of a form by a monomial: the case n=4: Part II. Integral Transforms Spec. Funct. 22 (2011), no. 3, 175–193.
[11] P. Maroni, I. Nicolau, On the inverse problem of the product of a form by a monomial: the case n=4. I. Integral Transforms Spec. Funct. 21 (2010), no. 1-2, 35–56.
[12] P. Maroni, I. Nicolau, On the inverse problem of the product of a form by a polynomial: The cubic case. Appl. Numer. Math., 45 (2003) 419–451.
[13] P. Maroni, Semi-classical character and finite type relations between polynomial sequences, Appl. Numer. Math. 31 (1999) 295–330.
[14] P. Maroni, Sur la décomposition quadratique d'une suite de polynômes orthogonaux II, Port. Math., 50 (1993), no. 3, 305–329.
[15] P. Maroni, Une théorie algébrique des polynômes orthogonaux, Application aux polynômes orthogonaux semi-classiques., in: Orthogonal Polynomials and Their Applications (Eric 1990), IMACS Ann. Comput. Appl. Math. 9, J.C. Baltzer, Basel (1991) 95–130.
[16] P. Maroni, Sur la suite de polynômes orthogonaux associé à la forme $u = \delta_c + \lambda(x-c)^{-1}L$, Period. Math. Hungar. 21 (1990), no. 3, 223-248.
[17] J. Petronilho, On the linear functionals associated to linearly related sequences of orthogonal polynomials, J. Math. Anal. Appl. 315 (2006), no. 2, 379–393.
[18] A. Zhedanov, Rational spectral transformations and orthogonal polynomials, J. Comput. Appl. Math. 85 (1997), no. 1, 67–86.