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# On Asymptotically Wijsman Lacunary Statistical Convergence of Set Sequences in Ideal Context

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**Abstract.** In this paper, we introduce some definitions which are natural combination of the notions of asymptotic equivalence, statistical convergence, lacunary statistical convergence, Wijsman convergence and ideal. In addition, we also define the concept of asymptotically equivalent sequences of sets in the sense of Wijsman convergence and prove some interesting results related to these concepts.

## 1. Introduction

Marouf [28], peresented definitions for asymptotically equivalent and asymptotic regular matrices. Pobyvancts [41], introduced the concepts of asymptotically regular matrices, which preserve the asymptotic equivalence of two nonnegative numbers sequences.

Patterson [39], extend these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. Patterson and Savaş [40], introduced the concepts of an asymptotically lacunary statistical equivalent sequences of real numbers. In [24] Hazarika, introduced the notion of asymptotically ideal equivalent sequences and proved some interesting results. The concept of convergence of sequences of points has been extended by several authors to convergence of sequences of sets. The one of these such extensions considered in this paper is the concept of Wijsman convergence. The concept of Wijsman statistical convergence which is implementation of the concept of statistical convergence to sequences of sets presented by Nuray and Rhoades [38]. Similar to the concept, the concept of Wijsman lacunary statistical convergence presented Ulusu and Nuray [49]. For more details on convergence of sequences of sets, we refer to [1–5, 7, 21–23, 50–52].

The notion of statistical convergence which is a generalization of the usual concept of sequential limit was introduced by Fast [12] and Steinhaus [44] independently in the same year 1951. A lot of developments have been made in this area after the works of Šalát [42] and Fridy [14]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Fridy and Orhan [15] introduced the concept of lacunary statistical convergence. Mursaleen and Mohiuddine [33], introduced the concept of lacunary statistical convergence with respect to the intuitionistic fuzzy normed space. For more details related to these concepts, we refer to (see [6, 15, 16, 27]) and references therein.

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The notion of the ideal convergence is the dual (equivelant) to the notion of filter convergence introduced by Cartan in 1937 [9]. The notion of the filter convergence is a generalization of the classical notion of convergence of a sequence and it has been an important tool in general topology and functional analysis. Nowadays many authors use an equivalent dual notion of the ideal convergence. Kostyrko et al. [25] and Nuray and Ruckle [37] independently studied in detalis about the notion of ideal convergence which is based on the structure of the admissible ideal $I$ of subsets of natural numbers $\mathbb{N}$. Quite recently, Das et al.[10], unified these two approaches to indroduce new concepts $I$-statistical convergence, $I$-lacunary statistical convergence and investigated some of its consequences. The notion of lacunary ideal convergence of real sequences was introduced in [47, 48]. Hazarika [17, 18], was introduced the lacunary ideal convergent sequences of fuzzy real numbers and studied some basic properties of this notion. For more details related to the concept of ideals we refer to [8, 11, 19, 20, 26, 29, 34–36, 43, 45–48].

In this work, we define the notion of asymptotically lacunary statistical equivalent sequences of sets in sense of Wijsman and establish some basic results regarding the notions asymptotically lacunary statistical equivalent sequences of sets in the sense of Wijsman and asymptotically Wijsman lacunary statistical equivalent sequences of sets using the notion of ideal.

## 2. Definitions, Notations and Preliminaries

We will assume throughout this paper that the symbols $\mathbb{R}$ and $\mathbb{N}$ will denote the set of real and natural numbers, respectively. Throughout the paper, we shall also denote by $I$ an admissible ideal of subsets of $\mathbb{N}$, unless otherwise stated.

A family of sets $I \subset P(\mathbb{N})$ (power sets of $\mathbb{N}$) is called an *ideal* if and only if for each $A, B \in I$, we have $A \cup B \in I$ and for each $A \in I$ and each $B \subset A$, we have $B \in I$. A non-empty family of sets $\mathcal{F} \subset P(\mathbb{N})$ is a *filter* on $\mathbb{N}$ if and only if $\Phi \notin \mathcal{F}$, for each $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$ and each $A \in \mathcal{F}$ and each $A \subset B$, we have $B \in \mathcal{F}$. An ideal $I$ is called non-trivial ideal if $I \neq \Phi$ and $\mathbb{N} \notin I$. Clearly $I \subset P(\mathbb{N})$ is a non-trivial ideal if and only if $\mathcal{F} = \mathcal{F}(I) = \{\mathbb{N} - A : A \in I\}$ is a filter on $\mathbb{N}$. A non-trivial ideal $I \subset P(\mathbb{N})$ is called *admissible* if and only if $\{\{x\} : x \in \mathbb{N}\} \subset I$. A non-trivial ideal $I$ is *maximal* if there cannot exists any non-trivial ideal $J \neq I$ containing $I$ as a subset. Further details on ideals of $P(\mathbb{N})$ can be found in Kostyrko et al. [25]. Recall that a sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be $I$-convergent to a real number $\ell$ if $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \in I$ for every $\varepsilon > 0$ ([25]). In this case we write $I - \lim x_k = \ell$.

By a lacunary sequence $\theta = (k_r)$, where $k_0 = 0$ , we shall mean an increasing sequence of non-negative integers with $h_r : k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by $\theta$ will be denoted by $J_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be defined by $q_r$ for $r \neq 1$ (see [13]).

Now we recall the difinitions of statistical convergence, lacunary statistical convergence and Wijsman convergence.

**Definition 2.1.** A real or complex number sequence $x = (x_k)$ is said to be *statistically convergent* to $L$ if for every $\varepsilon > 0$

$$\lim_{n} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case, we write $S - \lim x = L$ or $x_k \to L(S)$ and $S$ denotes the set of all statistically convergent sequences.

**Definition 2.2** [15]. A sequence $x = (x_k)$ is said to be lacunary statistically convergent to the number $L$ if for every $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in J_r : \ |x_k - L| \geq \varepsilon\}| = 0.$$

By $S_\theta$, we denote the set of all lacunary statistically convergent sequences.

Let $(X, \rho)$ be a metric space. For any point $x \in X$ and any non-empty subset $A \subset X$, the distance from x to A is defined by

$$d(x, A) = \inf_{y\in A} \rho\,(x, y)\,.$$

**Definition 2.3** [2]. Let (X,$\rho$) be a metric space. For any non-empty closed subsets $A, A_k \subset X$ ($k \in \mathbb{N}$), we say that the sequence $(A_k)$ is Wijsman convergent to $A$ if $\lim_k d(x, A_k) = d(x, A)$ for each $x \in X$. In this case we write $W - \lim A_k = A$.

**Definition 2.4** [28]. Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1,$$

denoted by $x \sim y$.

**Definition 2.5** [39]. Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple $L$ provided that for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0,$$

denoted by $x \sim^{S^L} y$ and simply asymptotically statistical equivalent if $L = 1$. Patterson and Savaş [40] defined the notion asymptotically lacunary statistical equivalent sequences as follows:

**Definition 2.6.** Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary statistical equivalent of multiple $L$ provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in J_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0, \text{ uniformly on } m,$$

denoted by $x \sim^{S_\theta^L} y$ and simply asymptotically lacunary statistical equivalent if $L = 1$. If we take $\theta = (2^r)$, then we get the definition 2.2.

The concepts of Wijsman statistical convergence and boundedness for the sequence $(A_k)$ were given by Nuray and Rhoades [38] as follows:

**Definition 2.7.** Let $(X, \rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subset X$ ($k \in \mathbb{N}$), we say that the sequence $(A_k)$ is Wijsman statistical convergent to $A$ if the sequence $(d(x, A_k))$ is statistically convergent to $d(x, A)$, i.e., for $\varepsilon > 0$ and for each $x \in X$

$$\lim_n \frac{1}{n} |\{k \leq n : \ |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case, we write $st - \lim_k A_k = A$ or $A_k \to A\,(WS)$. The sequence $(A_k)$ is bounded if $\sup_k d(x, A_k) < \infty$ for each $x \in X$. The set of all bounded sequences of sets denoted by $L_\infty$.

In [50], Ulusu and Nuray defined the concepts of asymptotically equivalent, asymptotically statistical equivalent and asymptotically lacunary statistical equivalent sequences of sets as follows:

**Definition 2.8.** Let $(X, \rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences $(A_k)$ and $(B_k)$ are asymptotically equivalent (Wijsman sense) if for each $x \in X$,

$$\lim_k \frac{d(x, A_k)}{d(x, B_k)} = 1,$$

denoted by $(A_k) \sim (B_k)$.

**Definition 2.9.** Let $(X, \rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences $(A_k)$ and $(B_k)$ are asymptotically statistical equivalent (Wijsman sense) if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| = 0$$

denoted by $(A_k) \sim^{WS^L} (B_k)$ and simply asymptotically statistical equivalent (Wijsman sense) if $L = 1$.

**Definition 2.10.** Let $(X, \rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences $(A_k)$ and $(B_k)$ are asymptotically lacunary statistical equivalent (Wijsman sense) if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in J_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| = 0$$

denoted by $(A_k) \sim^{WS^L_\theta} (B_k)$ and simply asymptotically lacunary statistical equivalent (Wijsman sense) if $L = 1$.

**Definition 2.11** [38]. Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subset X$, we say $\{A_k\}$ is *Wijsman strongly almost convergent* to $A$ if for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_{k+m}) - d(x, A)| = 0 \text{ uniformly in } m.$$

## 3. Asymptotically Wijsman Lacunary Statistical Equivalent Sequences Using Ideals

In this section, we define the notions of Wijsman ideal convergence, Wijsman $I$-statistical convergence, asymptotically Wijsman $I$-equivalent, asymptotically Wijsman $I$-statistical equivalent, asymptotically Wijsman $I$-lacunary statistical equivalent and asymptotically Wijsman lacunary $I$-equivalent sequences of sets and obtain some analogous results in view of these definitions.

**Definition 3.1.** Let $I \subset P(\mathbb{N})$ be a non-trivial ideal. Let $(X, \rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $(A_k)$ is Wijsman $I$-convergent to $A$ if the sequence $(d(x, A_k))$ is $I$-convergent to $d(x, A)$, i.e., for each $\varepsilon > 0$ and for each $x \in X$,

$$\{k \in \mathbb{N} : |d(x, A_k) - d(x, A)| \geq \varepsilon\} \in I.$$

In this case we write $I_W - \lim A_k = A$.

**Lemma 3.2** [25]. Let $I \subset P(\mathbb{N})$ be an admissible ideal. Let $(X, \rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$. Sequential method $I_W$ is regular.

**Definition 3.3.** Let $I \subset P(\mathbb{N})$ be a non-trivial ideal. Let $(X, \rho)$ be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $(A_k)$ is Wijsman $I$-statistically convergent to $A$ if the sequence $(d(x, A_k))$ is Wijsman $I$-statistically convergent to $d(x, A)$, i.e., for each $\varepsilon > 0, \delta > 0$ and for each $x \in X$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \delta \right\} \in I.$$

In this case we write $I(S)_W - \lim A_k = A$.

**Definition 3.4.** Let $I \subset P(\mathbb{N})$ be a non-trivial ideal. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. Two sequences $(A_k)$ and $(B_k)$ are said to be asymptotically Wijsman $I$-equivalent of multiple $L$ provided that for every $\varepsilon > 0$

$$\left\{ k \in \mathbb{N} : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \in I,$$

denoted by $(A_k) \sim^{WI^L} (B_k)$ and simply asymptotically Wijsman $I$-equivalent if $L = 1$.

**Lemma 3.5.** Let $I \subset P(\mathbb{N})$ be an admissible ideal. Let $(X, \rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ and $(A_k), (B_k) \in L_\infty$ with $I_W - \lim_k A_k = \{0\} = I_W - \lim_k B_k$ such that $(A_k) \sim^{WI^L} (B_k)$. Then there exists a sequence $(C_k) \in L_\infty$ with $I_W - \lim_k C_k = \{0\}$ such that $(A_k) \sim^{WI^L} (C_k) \sim^{WI^L} (B_k)$.

**Definition 3.6.** Let $I \subset P(\mathbb{N})$ be a non-trivial ideal. Let $(X, \rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. Two sequences $(A_k)$ and $(B_k)$

are said to be asymptotically Wijsman $I$-statistically equivalent of multiple $L$ provided that for every $\varepsilon > 0$ and for every $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in I,$$

denoted by $(A_k) \sim^{WI(S)^L} (B_k)$ and simply asymptotically Wijsman $I$-statistical equivalent if $L = 1$.

**Definition 3.7.** Let $I \subset P(\mathbb{N})$ be a non-trivial ideal. Let $(X, \rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences $(A_k)$ and $(B_k)$ are Cesaro asymptotically Wijsman $I$-equivalent (or $I(\sigma_1)$-equivalent) of multiple $L$ provided that for every $\delta > 0$ and for each $x \in X$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left(\frac{d(x,A_k)}{d(x,B_k)} - L\right) \geq \delta\right\} \in I$$

denoted by $(A_k) \sim^{WI(\sigma_1)^L} (B_k)$ and simply asymptotically Wijsman $I(\sigma_1)$-equivalent if $L = 1$.

**Definition 3.8.** Let $I \subset P(\mathbb{N})$ be a non-trivial ideal. Let $(X, \rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences $(A_k)$ and $(B_k)$ are strongly Cesaro asymptotically Wijsman $I$-equivalent (or $I(|\sigma_1|)$-equivalent) of multiple $L$ provided that for every $\delta > 0$ and for each $x \in X$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \delta\right\} \in I$$

denoted by $(A_k) \sim^{WI(|\sigma_1|)^L} (B_k)$ and simply strongly Cesaro asymptotically Wijsman $I(|\sigma_1|)$-equivalent if $L = 1$.

**Definition 3.9.** Let $(X, \rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences $(A_k)$ and $(B_k)$ are strongly asymptotically Wijsman lacunary equivalent if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_r \frac{1}{h_r}\sum_{k \in J_r}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| = 0$$

denoted by $(A_k) \sim^{W[N_\theta]^L} (B_k)$ and simply strongly asymptotically Wijsman lacunary equivalent if $L = 1$.

**Definition 3.10.** Let $I \subset P(\mathbb{N})$ be a non-trivial ideal. Let $(X, \rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences $(A_k)$ and $(B_k)$ are asymptotically Wijsman $I$-lacunary equivalent (or $I(N_\theta)$-equivalent) of multiple $L$ provided that for every $\delta > 0$ and for each $x \in X$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in J_r}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \delta\right\} \in I$$

denoted by $(A_k) \sim^{WI(N_\theta)^L} (B_k)$ and simply asymptotically Wijsman $I(N_\theta)$-equivalent if $L = 1$.

**Definition 3.11.** Let $I \subset P(\mathbb{N})$ be a non-trivial ideal. Let $(X, \rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences $(A_k)$ and $(B_k)$ are asymptotically Wijsman $I$-lacunary statistically equivalent (or $I(S_\theta)$-equivalent) of multiple $L$ provided that for every $\varepsilon > 0$, for every $\delta > 0$ and for each $x \in X$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in J_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in I$$

denoted by $(A_k) \sim^{WI(S_\theta)^L} (B_k)$ and simply asymptotically Wijsman $I(S_\theta)$-equivalent if $L = 1$.

**Theorem 3.12.** Let $(X,\rho)$ be a metric space and $A_k, B_k$ be two non-empty closed subsets of $X$ $(k \in \mathbb{N})$. If $(A_k), (B_k) \in L_\infty$ and $(A_k) \sim^{WI(S)^L} (B_k)$. Then $(A_k) \sim^{WI(\sigma_1)^L} (B_k)$.

**Proof.** Suppose that $(A_k), (B_k) \in L_\infty$ and $(A_k) \sim^{WI(S)^L} (B_k)$. Then we can assume that

$$\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \leq M \text{ for almost all } k.$$

Let $\varepsilon > 0$ be given. Then, we have

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{k=1}^{n}\left(\frac{d(x,A_k)}{d(x,B_k)} - L\right)\right| \\
&\leq \frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \\
&\leq \frac{1}{n}\sum_{\substack{k \\ \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon}}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| + \frac{1}{n}\sum_{\substack{k \\ \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| < \varepsilon}}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \\
&\leq M \cdot \frac{1}{n}\left|\left\{k \leq n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| + \frac{1}{n}\cdot n \cdot \varepsilon.
\end{aligned}$$

Consequently for any $\delta > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left(\frac{d(x,A_k)}{d(x,B_k)} - L\right) \geq \delta\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \geq \frac{\delta}{M}\right\} \in I.$$

This shows that $(A_k) \sim^{WI(\sigma_1)^L} (B_k)$.

**Corollary 3.13.** Let $(X,\rho)$ be a metric space and $A_k, B_k$ be two non-empty closed subsets of $X$ $(k \in \mathbb{N})$. If $(A_k), (B_k) \in L_\infty$ and $(A_k) \sim^{WI(S)^L} (B_k)$. Then $(A_k) \sim^{WI(|\sigma_1|)^L} (B_k)$.

**Theorem 3.14.** Let $(X,\rho)$ be a metric space and $A_k, B_k$ be two non-empty closed subsets of $X$ $(k \in \mathbb{N})$. Then

(a) $(A_k) \sim^{WI(N_\theta)^L} (B_k) \Rightarrow (A_k) \sim^{WI(S_\theta)^L} (B_k)$.
(b) $WI(N_\theta)^L$ is a proper subset of $WI(S_\theta)^L$.
(c) Let $(A_k), (B_k) \in L_\infty$ and $(A_k) \sim^{WI(S_\theta)^L} (B_k)$, then $(A_k) \sim^{WI(N_\theta)^L} (B_k)$.
(d) $WI(S_\theta)^L \cap L_\infty = WI(N_\theta)^L \cap L_\infty$.

**Proof.** (a) Let $\varepsilon > 0$ and $(A_k) \sim^{WI(N_\theta)^L} (B_k)$. Then we can write

$$\sum_{k \in J_r}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \sum_{\substack{k \in J_r \\ \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon}}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\left|\left\{k \in J_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right|.$$

It follows that

$$\frac{1}{\varepsilon \cdot h_r}\sum_{k \in J_r}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \frac{1}{h_r}\left|\left\{k \in J_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right|.$$

Thus for any $\delta > 0$, we have

$$\frac{1}{h_r}\left|\left\{k \in J_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \geq \delta$$

which implies that

$$\frac{1}{h_r}\sum_{k\in J_r}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq \varepsilon\delta.$$

Therefore, we obtain

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\left|\left\{k\in J_r:\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\subset\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in J_r}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\delta\right\}.$$

Since $(A_k)\sim^{WI(N_\theta)^L}(B_k)$, so that

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in J_r}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\delta\right\}\in I$$

which implies that

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\left|\left\{k\in J_r:\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\in I.$$

This shows that $(A_k)\sim^{WI(S_\theta)^L}(B_k)$.

(b) Suppose that $WI(N_\theta)^L\subset WI(S_\theta)^L$. Let $(A_k)$ and $(B_k)$ be two sequences defined as follows:

$$A_k=\begin{cases}\{k\}, & \text{if } k_{r-1}<k\leq k_{r-1}+[\sqrt{h_r}], r=1,2,3,\cdots;\\ \{0\}, & \text{otherwise}\end{cases}$$

and

$$B_k=\{1\} \text{ for all } k\in\mathbb{N}.$$

It is clear that $(A_k)\notin L_\infty$ and for $\varepsilon>0$ and for each $x\in X$,

$$\frac{1}{h_r}\left|\left\{k\in J_r:\ \left|\frac{d(x,A_k)}{d(x,B_k)}-1\right|\geq\varepsilon\right\}\right|\leq\frac{[\sqrt{h_r}]}{h_r}\text{ and }\frac{[\sqrt{h_r}]}{h_r}\to 0\text{ as }r\to\infty. \tag{1}$$

This implies that

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\left|\left\{k\in J_r:\ \left|\frac{d(x,A_k)}{d(x,B_k)}-1\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\subseteq\left\{r\in\mathbb{N}:\frac{[\sqrt{h_r}]}{h_r}\geq\delta\right\}.$$

By virtue of last part of (3.1), the set on the right side is a finite set and so it belongs to $I$. Consequently, we have

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\left|\left\{k\in J_r:\ \left|\frac{d(x,A_k)}{d(x,B_k)}-1\right|\geq\varepsilon\right\}\right|\geq\delta\right\}\in I.$$

Therefore $(A_k)\sim^{WI(S_\theta)^1}(B_k)$.

On the other hand we shall show that $(A_k)\sim^{WI(N_\theta)^1}(B_k)$ is not satisfied. Suppose that $(A_k)\sim^{WI(N_\theta)^1}(B_k)$. Then for every $\delta>0$, we have

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in J_r}\left|\frac{d(x,A_k)}{d(x,B_k)}-1\right|\geq\delta\right\}\in I. \tag{2}$$

Now,

$$\lim_r \frac{1}{h_r}\sum_{k\in J_r}\left|\frac{d(x,A_k)}{d(x,B_k)}-1\right| = \frac{1}{h_r}\left(\frac{[\sqrt{h_r}]([\sqrt{h_r}]-1)}{2}\right) \to \frac{1}{2} \text{ as } r\to\infty.$$

It follows that for the particular choice $\delta = \frac{1}{4}$,

$$\left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\left|\frac{d(x,A_k)}{d(x,B_k)}-1\right| \geq \frac{1}{4}\right\} = \left\{r\in\mathbb{N} : \left(\frac{[\sqrt{h_r}]([\sqrt{h_r}]-1)}{h_r}\right) \geq \frac{1}{2}\right\}$$

$$= \{m, m+1, m+2, \cdots\}$$

for some $m\in\mathbb{N}$ which belongs to $\mathcal{F}$ as $I$ is admissible. This contradicts (3.2) for the choice $\delta = \frac{1}{4}$. Therefore $(A_k) \nsim^{WI(N_\theta)^1} (B_k)$.

(c) Suppose that $(A_k) \sim^{WI(S_\theta)^L} (B_k)$ and $(A_k),(B_k)\in L_\infty$. We assume that $\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \leq M$ for each $x\in X$ and for all $k\in\mathbb{N}$. Given $\varepsilon>0$, we get

$$\frac{1}{h_r}\sum_{k\in J_r}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right| = \frac{1}{h_r}\sum_{\substack{k\in J_r\\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon}}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right| + \frac{1}{h_r}\sum_{\substack{k\in J_r\\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|<\varepsilon}}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|$$

$$\leq \frac{M}{h_r}\left|\left\{k\in J_r : \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right| + \varepsilon.$$

If we put

$$A(\varepsilon) = \left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}$$

and

$$B(\varepsilon) = \left\{r\in\mathbb{N} : \frac{1}{h_r}\left|\left\{k\in J_r : \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|\geq\frac{\varepsilon}{M}\right\},$$

then we have $A(\varepsilon)\subset B(\varepsilon)$ and so $A(\varepsilon)\in I$. This shows that $(A_k) \sim^{WI(N_\theta)^L} (B_k)$.

(d) It follows from (a), (b) and (c).

**Theorem 3.15.** Suppose for given $\delta>0$ and every $\varepsilon>0$

$$\left\{n\in\mathbb{N} : \frac{1}{n}\left|\left\{0\leq k\leq n-1 : \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right| < \delta\right\}\in\mathcal{F}$$

then $(A_k) \sim^{WI(S)^L} (B_k)$.

**Proof.** Let $\delta>0$ be given. For every $\varepsilon>0$, choose $n_1$ such that

$$\frac{1}{n}\left|\left\{0\leq k\leq n-1 : \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right| < \frac{\delta}{2} \quad \text{for all } n\geq n_1. \tag{3}$$

It is sufficient to show that there exists $n_2$ such that for $n\geq n_1$

$$\frac{1}{n}\left|\left\{0\leq k\leq n-1 : \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right| < \delta. \tag{4}$$

Let $n_0 = \max\{n_1, n_2\}$. The relation (3.3) will be true for $n > n_0$. If $m_0$ chosen fixed, then we get

$$\left|\left\{0 \leq k \leq m_0 - 1 : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| = M.$$

Now, for $n > m_0$, we have

$$\begin{aligned}
&\frac{1}{n}\left|\left\{0 \leq k \leq n-1 : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&\leq \frac{1}{n}\left|\left\{0 \leq k \leq m_0 - 1 : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| + \frac{1}{n}\left|\left\{m_0 \leq k \leq n-1 : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&\leq \frac{M}{n} + \frac{1}{n}\left|\left\{m_0 \leq k \leq n-1 : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \leq \frac{M}{n} + \frac{\delta}{2}.
\end{aligned}$$

Thus for sufficiently large $n$

$$\frac{1}{n}\left|\left\{m_0 \leq k \leq n-1 : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \leq \frac{M}{n} + \frac{\delta}{2} < \delta.$$

This established the result.

**Theorem 3.16.** Let $(X,\rho)$ be a metric space and $A_k, B_k$ be two non-empty closed subsets of $X$ ($k \in \mathbb{N}$). Let $\theta = (k_r)$ be a lacunary sequence with $\liminf_r q_r > 1$. Then $(A_k) \sim^{WI(S)^L} (B_k) \Rightarrow (A_k) \sim^{WI(S_\theta)^L} (B_k)$.

**Proof.** Suppose that $\liminf_r q_r > 1$, then there exists a $\alpha > 0$ such that $q_r \geq 1 + \alpha$ for sufficiently large $r$. Then, we have

$$\frac{h_r}{k_r} \geq \frac{\alpha}{1+\alpha}.$$

If $(A_k) \sim^{WI(S^L)} (B_k)$ then for every $\varepsilon > 0$ and for sufficiently large $r$, we have

$$\begin{aligned}
\frac{1}{k_r}\left|\left\{k \leq k_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| &\geq \frac{1}{k_r}\left|\left\{k \in J_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&\geq \frac{\alpha}{1+\alpha} \cdot \frac{1}{h_r}\left|\left\{k \in J_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right|.
\end{aligned}$$

Therefore, for any $\delta > 0$, we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in J_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{k_r}\left|\left\{k \leq k_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \geq \frac{\alpha\delta}{1+\alpha}\right\} \in I$$

This completes the proof.

**Theorem 3.17.** Let $I = I_{fin} = \{A \subset \mathbb{N} : A \text{ is a finite set}\}$ be a non-trivial ideal. Let $(X,\rho)$ be a metric space and $A_k, B_k$ be two non-empty closed subsets of $X$ ($k \in \mathbb{N}$). Let $\theta = (k_r)$ be a lacunary sequence with $\limsup_r q_r < \infty$. Then $(A_k) \sim^{WI(S_\theta)^L} (B_k) \Rightarrow (A_k) \sim^{WI(S)^L} (B_k)$.

**Proof.** If $\limsup_r q_r < \infty$. Then there exists an $K > 0$ such that $q_r < K$ for all $r \geq 1$. Let $(A_k) \sim^{WS^L_{\theta,\sigma}} (B_k)$ and $\delta > 0$. Then there exists $B > 0$ and $\varepsilon > 0$ such that for every $j \geq B$

$$M_j = \frac{1}{h_j}\left|\left\{k \in J_j : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| < \delta.$$

Also we can find $A > 0$ such that $M_j < A$ for all $j = 1, 2, 3, \cdots$. Now, let $i$ be an integer satisfying $k_{r-1} < i \leq k_r$, where $r > B$. Then, we can write

$$\begin{aligned}
&\frac{1}{n}\left|\left\{k \leq i : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&\leq \frac{1}{k_{r-1}}\left|\left\{k \leq k_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&= \frac{1}{k_{r-1}}\left|\left\{k \in J_1 : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| + \frac{1}{k_{r-1}}\left|\left\{k \in J_2 : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&\quad + \cdots + \frac{1}{k_{r-1}}\left|\left\{k \in J_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&= \frac{k_1}{k_{r-1}k_1}\left|\left\{k \in J_1 : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| + \frac{k_2 - k_1}{k_{r-1}(k_2 - k_1)}\left|\left\{k \in J_2 : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&\quad + \cdots + \frac{k_B - k_{B-1}}{k_{r-1}(k_B - k_{B-1})}\left|\left\{k \in J_B : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&\quad + \cdots + \frac{k_r - k_{r-1}}{k_{r-1}(k_r - k_{r-1})}\left|\left\{k \in J_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \\
&= \frac{k_1}{k_{r-1}M_1} + \frac{k_2 - k_1}{k_{r-1}}M_2 + \cdots + \frac{k_B - k_{B-1}}{k_{r-1}}M_B + \frac{k_{B+1} - k_B}{k_{r-1}}M_{B+1} + \cdots + \frac{k_r - k_{r-1}}{k_{r-1}}M_r \\
&\leq \left\{\sup_{i\geq 1} M_i\right\}\frac{k_B}{k_{r-1}} + \left\{\sup_{i\geq B} M_i\right\}\frac{k_r - k_B}{k_{r-1}} \leq A\frac{k_B}{k_{r-1}} + \delta K.
\end{aligned}$$

This completes the proof of the theorem.

**Definition 3.18.** Let $p \in (0, \infty)$. Let $(X, \rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences $(A_k)$ and $(B_k)$ are strongly asymptotically Wijsman lacunary $p$-equivalent if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_r \frac{1}{h_r}\sum_{k\in J_r}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^p = 0$$

denoted by $(A_k) \sim^{W[N_\theta]_p^L} (B_k)$ and simply strongly asymptotically Wijsman lacunary $p$-equivalent if $L = 1$.

**Definition 3.19.** Let $p \in (0, \infty)$. Let $(X, \rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences $(A_k)$ and $(B_k)$ are asymptotically Wijsman lacunary $p$-statistically equivalent if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_r \frac{1}{h_r}\left|\left\{k \in J_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^p \geq \varepsilon\right\}\right| = 0$$

denoted by $(A_k) \sim^{WS_{\theta_p}^L} (B_k)$ and simply asymptotically Wijsman lacunary $p$-statistical equivalent if $L = 1$.

**Theorem 3.20.** Let $(X, \rho)$ be a metric space and $A_k, B_k$ be two non-empty closed subsets of $X$ $(k \in \mathbb{N})$. Then

(a) $(A_k) \sim^{W[N_\theta]_p^L} (B_k) \Rightarrow (A_k) \sim^{WS_{\theta_p}^L} (B_k)$.

(b) $W[N_\theta]_p^L$ is a proper subset of $WS_{\theta_p}^L$.

(c) Let $(A_k), (B_k) \in L_\infty$ and $(A_k) \sim^{WS^L_{\theta_p}} (B_k)$, then $(A_k) \sim^{W[N_\theta]^L_p} (B_k)$.

(d) $WS^L_{\theta_p} \cap L_\infty = W[N_\theta]^L_p \cap L_\infty$.

The proof of the above theorem is similar to Theorem 3.14 for $I = I_{fin}$.

**Definition 3.21.** Let $p \in (0,\infty)$. Let $(X,\rho)$ be a metric space. For any two non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$. We say that the sequences $(A_k)$ and $(B_k)$ are strongly asymptotically Wijsman $I$-lacunary $p$-equivalent if for every $\varepsilon > 0$ and for each $x \in X$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^p \geq \varepsilon\right\} \in I,$$

denoted by $(A_k) \sim^{WI(N_\theta)^L_p} (B_k)$ and simply strongly asymptotically Wijsman $I$-lacunary $p$-equivalent if $L = 1$.

**Definition 3.22.** Let $p \in (0,\infty)$. Let $(X,\rho)$ be a metric space. For any two non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$. We say that the sequences $(A_k)$ and $(B_k)$ are asymptotically Wijsman $I$-lacunary $p$-statistically equivalent if for every $\varepsilon > 0$, for every $\delta > 0$ and for each $x \in X$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in J_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^p \geq \varepsilon\right\}\right| \geq \delta\right\} \in I,$$

denoted by $(A_k) \sim^{WI(S_{\theta_p})^L} (B_k)$ and simply asymptotically Wijsman $I$-lacunary $p$-statistical equivalent if $L = 1$.

**Theorem 3.23.** Let $(X,\rho)$ be a metric space and $A_k, B_k$ be non-empty closed subsets X ($k \in \mathbb{N}$). Then

(a) $(A_k) \sim^{WI(N_{\theta_p})^L} (B_k) \Rightarrow (A_k) \sim^{WI(S_{\theta_p})^L} (B_k)$.

(b) $WI(N_{\theta_p})^L$ is a proper subset of $WI(S_{\theta_p})^L$.

(c) Let $(A_k), (B_k) \in L_\infty$ and $(A_k) \sim^{WI(S_{\theta_p})^L} (B_k)$, then $(A_k) \sim^{WI(N_{\theta_p})^L} (B_k)$.

(d) $WI(S_{\theta_p})^L \cap L_\infty = WI(N_{\theta_p})^L \cap L_\infty$.

The proof of this theorem follows from Theorem 3.14 and Theorem 3.20.

**Definition 3.24.** Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$. We say that the sequences $(A_k)$ and $(B_k)$ are asymptotically Wijsman strongly almost equivalent if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_{k+m})}{d(x,B_{k+m})} - L\right| = 0 \text{ uniformly in } m$$

denoted by $(A_k) \sim^{W|AC|^L} (B_k)$ and simply asymptotically Wijsman strongly almost equivalent if $L = 1$.

**Definition 3.25.** Let $\theta = (k_r)$ be a lacunary sequence. Let $(X,\rho)$ be a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$. We say that the sequences $(A_k)$ and $(B_k)$ are asymptotically Wijsman lacunary strongly almost equivalent if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_r \frac{1}{h_r}\sum_{k\in J_r}\left|\frac{d(x,A_{k+m})}{d(x,B_{k+m})} - L\right| = 0 \quad \text{uniformly in } m,$$

denoted by $(A_k) \sim^{W|AC|^L_\theta} (B_k)$ and simply asymptotically Wijsman lacunary strongly almost equivalent if $L = 1$.

**Theorem 3.26.** Let $\theta = (k_r)$ be a lacunary sequence. Let $(X,\rho)$ be a metric space and $A_k, B_k$ be two non-empty closed subsets of $X$ ($k \in \mathbb{N}$). Then $(A_k) \sim^{W|AC|^L} (B_k) \Rightarrow (A_k) \sim^{W[N_\theta]^L} (B_k)$ but converse need not be true.

**Proof.** If $(A_k) \sim^{W|AC|^L} (B_k)$ and $\varepsilon > 0$ then there exists a $n_0 > 0$ such that

$$\frac{1}{n}\sum_{i=m+1}^{m+n}\left|\frac{d(x,A_i)}{d(x,B_i)} - L\right| < \varepsilon$$

for every $n > n_0$ amd $m = 1,2,3,\cdots$. Since $\theta$ is a lacunary sequence we can choose $M > 0$ such that $r \geq M$ implies that $h_r > n_0$, respectively

$$\frac{1}{h_r}\sum_{i\in J_r}\left|\frac{d(x,A_i)}{d(x,B_i)} - L\right| < \varepsilon.$$

Hence $(A_k) \sim^{W[N_\theta]^L} (B_k)$.

In general, the converse implication is not true. Let us consider $L = 2$ and the two sequences $(A_i)$ and $(B_i)$ defined as follow:

$$A_i = \begin{cases} \{3\}, \text{ if } k_{r-1} < i \leq k_{r-1} + [\sqrt{h_r}], r = 1,2,3,\cdots; \\ \qquad\qquad \{2\} \qquad \text{otherwise} \end{cases}$$

and

$$B_i = \{1\} \text{ for all } i \in \mathbb{N}.$$

Then it is easy to prove that $(A_k) \sim^{W[N_\theta]^L} (B_k)$ and $(A_k) \nsim^{W|AC|^L} (B_k)$.

**Theorem 3.27.** Let $\theta = (k_r)$ be a lacunary sequence. Let $(X,\rho)$ be a metric space and $A_k, B_k$ be non-empty closed subsets X $(k \in \mathbb{N})$. Then the following relations are valid:

(a) If $\liminf_r q_r > 1$ then $(A_k) \sim^{W|AC|^L} (B_k) \Rightarrow (A_k) \sim^{W|AC|^L_\theta} (B_k)$.
(b) If $\limsup_r q_r < \infty$ then $(A_k) \sim^{W|AC|^L_\theta} (B_k) \Rightarrow (A_k) \sim^{W|AC|^L} (B_k)$.
(c) If $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$ then $(A_k) \sim^{W|AC|^L} (B_k) \Leftrightarrow (A_k) \sim^{W|AC|^L_\theta} (B_k)$.

The proof of the above theorem is similar to Theorem 3.16 and Theorem 3.17 for $I = I_{fin}$.

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