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Saddle Point Criteria in Semi-Infinite Minimax Fractional Programming under (Φ, ρ)-Invexity

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Abstract. Semi-infinite minimax fractional programming problems with both inequality and equality constraints are considered. The sets of parametric saddle point conditions are established for a new class of nonconvex differentiable semi-infinite minimax fractional programming problems under (Φ, ρ)-invexity assumptions. With the reference to the said concept of generalized convexity, we extend some results of saddle point criteria for a larger class of nonconvex semi-infinite minimax fractional programming problems in comparison to those ones previously established in the literature.

1. Introduction

A semi-infinite programming problem is a mathematical programming problem with a finite number of variables and infinitely many constraints. In recent years, semi-infinite optimization has become an active field of research in applied mathematics. This is due to the fact that this model naturally arises in an abundant number of applications in different fields of engineering, control theory, mathematics, economics and others. For a wealth of information pertaining to various aspects of semi-infinite programming, including areas of applications, optimality conditions, duality relations, and numerical algorithms, the reader is referred, for instance, to [13], [15], [17], [21], [22], [25], [28], [29], [30], [33], [34], [40], [43], [44], [48], [49], [52], [53], [54], [55], [56], [57], [58], and others.

In the context of nonlinear optimization theory, the characterization of a constrained optimum as a saddle point of the Lagrangian function is known to be heavily dependent upon convexity properties of the underlying extremum problem. To the best of our knowledge, there are only a very few works available dealing with saddle point criteria for semi-infinite programming problems. López and Vercher [39] gave characterizations of optimal solutions in the nondifferentiable convex semi-infinite programming problem related to a Lagrangian saddle point. Rückman and Shapiro [45] studied an augmented Lagrangian approach to semi-infinite problems and presented necessary and sufficient conditions for the equivalence of an optimal solution and a saddle point in the considered class of semi-infinite programming problems. Ito et al. [31] formulated convex semi-infinite programming problems in a functional analytic setting and, by using the saddle point condition, proved that the set of multipliers satisfying the Karush-Kuhn-Tucker necessary optimality conditions coincides with the set of solutions to the dual problem. Guerra-Vázquez et al. [23] applied two convexification procedures to the Lagrangian of a nonconvex semi-infinite

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programming problem. Under the reduction approach, they showed that, locally around a local minimizer, the considered semi-infinite programming problem can be transformed equivalently in such a way that the transformed Lagrangian fulfills saddle point optimality conditions. Recently, Antczak [9] considered a semi-infinite minimax fractional programming problem with both inequality and equality constraints. For the considered semi-infinite minimax fractional programming problem, he gave characterizations of an optimal solution by a saddle point of the scalar Lagrange function and the vector-valued Lagrange function defined for such an optimization problem. Further, Antczak established the equivalence between an optimal solution and a saddle point of the scalar Lagrange function and the vector-valued Lagrange function in the considered semi-infinite minmax fractional programming problem under several (p, r)-invexity assumptions.

Optimization problems with a finite number of constraints, in which both a minimization and a maximization process are performed, are known in the area of the mathematical programming as minimax (minmax) problems. Problems of this type arise frequently in the area of game theory, in economics, in best approximation theory, and in a great variety of situations involving optimal decision making under uncertainty. Minimax programming problems have been the subject of immense interest in the past few years. Minimax fractional programming refers to minimizing the maximum of fractional functions. During the last three decades, much attention has been paid to optimality conditions for minimax fractional programming problems (see, for instance, [1], [3], [7], [11], [19], [32], [35], [36], [37], [38], [46], [47], [51], [53], and others).

In this paper, we consider a class of more general minimax fractional programming problems than those ones mentioned above, that is, semi-infinite minimax fractional programming problems with both inequality and equality constraints. The main purpose of this paper is to use the concept of (Φ, ρ)-invexity to establish saddle point criteria for such a class of nonconvex semi-infinite minimax fractional programming problems. In the presence of both inequality and equality constraints, by making use of necessary optimality conditions, we present the characterization of an optimal solution as a saddle point of the Lagrangian function defined for the considered semi-infinite minimax fractional programming problem with (Φ, ρ)-invex functions (with respect to, not necessarily, the same ρ). However, a suitable assumption is also imposed on all parameters ρ, with respect to which the functions constituting the considered vector optimization problem are (Φ, ρ)-invex. We illustrate the results established in the paper by the example of a nonconvex semi-infinite minimax fractional programming problem with (Φ, ρ)-invex functions. Based on this example, we also show that the equivalence between an optimal solution and a saddle point of the classical Lagrange function in such a class of nonconvex semi-infinite minimax fractional programming problem does not hold under other generalized convexity notions previously defined in the literature.

2. The Concept of (Φ, ρ)-Invexity

The following convention for equalities and inequalities will be used in the paper.

For any $x = (x_1, x_2, ..., x_n)^T$, $y = (y_1, y_2, ..., y_n)^T$, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, ..., n$;

(ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, ..., n$;

(iii) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, ..., n$;

(iv) $x \geq y$ if and only if $x \geqq y$ and $x \neq y$.

In this section, we provide some definitions and some results that we shall use in the sequel. Further, for convenience of a common reader, we recall the definition of certain classes of generalized invex functions which we need to prove the main results in the paper. Namely, in [16], Caristi et al. introduced a new class of nonconvex scalar functions, called (Φ, ρ)-invex functions. Now, we give an extension and generalization of this definition to the vectorial case.

Let X be a nonempty subset of R^n and the function $f : (f_1, f_2, ..., f_k) : X \to R^k$ be differentiable at $u \in X$.

Definition 2.1. *If there exist a function $\Phi : X \times X \times R^{n+1} \to R$, where $\Phi(x, u, \cdot)$ is convex on R^{n+1}, $\Phi(x, u, (0, a)) \geqq 0$ for all $x \in X$ and every $a \in R_+$ and $\rho = (\rho_1, ..., \rho_k) \in R^k$ such that, the following inequalities*

$$f_i(x) - f_i(u) \geqq \Phi(x, u, (\nabla f_i(u), \rho_i)) \quad (>),\ i = 1, ..., k \tag{1}$$

hold for all $x \in X$, then f is said to be a (vector) (Φ, ρ)-invex ((vector) strictly (Φ, ρ)-invex) function at u on X. If the inequalities (1) are satisfied at each u, then f is said to be a (Φ, ρ)-invex (strictly (Φ, ρ)-invex) function on X.

Definition 2.2. *Each function f_i, $i = 1, ..., k$, satisfying the inequality (1) is said to be (Φ, ρ_i)-invex (strictly (Φ, ρ_i)-invex) at u on X. If the inequality (1) is satisfied at each u, then f_i is said to be a (Φ, ρ_i)-invex (strictly (Φ, ρ_i)-invex) function on X.*

Remark 2.3. *Note that the definition of a (Φ, ρ)-invex function generalizes and extends many generalized convexity and generalized invexity notions previously introduced in the literature. Indeed, there are the following special cases:*

i) *If $\Phi(x, u, (\nabla f_i(u), \rho_i)) = \nabla f_i(u)(x - u)$, $\rho_i = 0$, $i = 1, ..., k$, then we obtain the definition of a differentiable vector-valued convex function.*
ii) *If $\Phi(x, u, (\nabla f_i(u), \rho_i)) = \nabla f_i(u)\eta(x, u)$ for a certain mapping $\eta : X \times X \to R^n$ and, moreover, $\rho_i = 0$, $i = 1, ..., k$, then we obtain the definition of a differentiable vector-valued invex function (in the scalar case, $k = 1$, see, Hanson [26]; in the vectorial case, see, Egudo and Hanson [20]).*
iii) *If $\Phi(x, u, (\nabla f_i(u), \rho_i)) = \frac{1}{b_i(x,u)}\nabla f_i(u)\eta(x, u)$ for a certain mapping $\eta : X \times X \to R^n$, $b_i : X \times X \to R_+\backslash\{0\}$ and $\rho_i = 0$, $i = 1, ..., k$, then (Φ, ρ)-invexity reduces to the definition of a vector (b, η)-invex function (see, Bector [10]).*
iv) *If $\Phi(x, u, (\nabla f_i(u), \rho_i)) = \nabla f_i(u)(x - u) + \rho\|\theta(x, u)\|^2$, where $\theta : X \times X \to R^n$, $\theta(x, u) \neq 0$, whenever $x \neq u$, then (Φ, ρ)-invexity reduces to the definition of a vector ρ-convex function (see, in the scalar case, Vial [50]).*
v) *If $\Phi(x, u, (\nabla f_i(u), \rho_i)) = \nabla f_i(u)\eta(x, u) + \rho\|\theta(x, u)\|^2$ for a certain mapping $\eta : X \times X \to R^n$, where $\theta : X \times X \to R^n$, $\theta(x, u) \neq 0$, whenever $x \neq u$, then (Φ, ρ)-invexity reduces to the definition of a vector ρ-invex function (with respect to η and θ) (see, Craven [18]).*
vi) *If $\Phi(x, u, (\nabla f_i(u), \rho_i)) = F(x, u, \nabla f_i(u))$, where $F(x, u, \cdot)$ is a sublinear functional on R^n, then (Φ, ρ)-invexity notion reduces to the definition of F-convexity introduced by Hanson and Mond [27], and considered by Gulati and Islam [24] in a vectorial case.*
vii) *If $\Phi(x, u, (\nabla f_i(u), \rho_i)) = F(x, u, \nabla f_i(u)) + \rho d^2(x, u)$, where $F(x, u, \cdot)$ is a sublinear functional on R^n and $d : X \times X \to R$, then (Φ, ρ)-invexity notion reduces to the definition of (F, ρ)-convexity introduced by Preda [42] and considered by Ahmad [2].*
viii) *If $\Phi(x, u, (\nabla f_i(u), \rho_i)) = \frac{1}{b_i(x,u)}\left(F(x, u, \nabla f_i(u)) + \rho d^2(x, u)\right)$, where $F(x, u, \cdot)$ is a sublinear functional on R^n, $b_i : X \times X \to R_+\backslash\{0\}$ and $d : X \times X \to R$, then (Φ, ρ)-invexity notion reduces to the definition of (b, F, ρ)-convexity introduced by Pandian [41].*

3. Semi-Infinite Minimax Fractional Programming and Saddle Point Criteria

Now, we consider the following semi-infinite minimax fractional programming problem:

$$\varphi(x) = \min_{x \in X} \max_{1 \leqq i \leqq p} \frac{f_i(x)}{g_i(x)}$$

$$\begin{aligned} \text{subject to} \quad & G_j(x, t) \leqq 0 \text{ for all } t \in T_j,\ j = 1, ..., q, \qquad \text{(SMFP)} \\ & H_k(x, s) = 0 \text{ for all } s \in S_k,\ k = 1, ..., r, \\ & x \in X, \end{aligned}$$

where $f_i : X \to R$, $g_i : X \to R$, $i \in I = \{1, ..., p\}$, are real-valued functions defined on a nonempty convex open subset X of R^n such that, for each $i \in I$, $f_i(x) \geqq 0$, $g_i(x) > 0$ for all $x \in X$, T_j, $j = 1, ..., q$, and S_k, $k = 1, ..., r$, are compact subsets of a complete metric space, $x \to G_j(x, t)$ is a function on X for all $t \in T_j$, for each $k \in K = \{1, ..., r\}$, $x \to H_k(x, s)$, is a function on X for all $s \in S_k$, for each $j \in J$ and $k \in K$, $t \to G_j(x, t)$ and $s \to H_k(x, s)$ are continuous real-valued functions defined, respectively, on T_j and S_k for all $x \in X$ satisfying the constraints of problem (SMFP).

Let

$$D := \left\{x \in X : G_j(x,t) \leqq 0 \text{ for all } t \in T_j,\ j = 1,...,q,\ H_k(x,s) = 0 \text{ for all } s \in S_k,\ k \in K\right\}$$

be the set of all feasible solutions of (SMFP) and let $\widehat{T}_j(\overline{x})$ denote $\widehat{T}_j(\overline{x}) = \left\{t \in T_j : G_j(\overline{x},t) = 0\right\}$.

Definition 3.1. *The tangent cone to the feasible set D in problem (SMFP) at $\overline{x} \in D$ is the set*

$$T(D;\overline{x}) \equiv \left\{h \in R^n : h = \lim_{n\to\infty} t_n (x^n - \overline{x}) \text{ such that } x^n \in D,\ \lim_{n\to\infty} x^n = \overline{x}, \text{ and } t_n > 0 \text{ for all } n = 1,2,...\right\}.$$

Definition 3.2. *Let $\overline{x} \in D$. The linearizing cone at $\overline{x}$ for problem (SMFP) is the set defined by*

$$C(\overline{x}) \equiv \Big\{h \in R^n : \left\langle \nabla G_j(\overline{x},t), h\right\rangle \leqq 0 \text{ for all } t \in \widehat{T}_j(\overline{x}),\ j = 1,...,q,$$
$$\left\langle \nabla H_k(\overline{x},s), h\right\rangle = 0 \text{ for all } s \in S_k,\ k = 1,...,r\Big\}.$$

Definition 3.3. *The problem (SMFP) satisfies the generalized Abadie constraint qualification at a given point $\overline{x} \in D$ if the following relation*

$$C(\overline{x}) \subseteq T(D;\overline{x})$$

holds.

Now, we give a useful lemma which we use to prove the main results in the paper.

Lemma 3.4. [54] *For each $x \in X$,*

$$\varphi(x) \equiv \max_{1\leqq i\leqq p} \frac{f_i(x)}{g_i(x)} = \max_{\lambda\in\Lambda} \frac{\sum_{i=1}^{p} \lambda_i f_i(x)}{\sum_{i=1}^{p} \lambda_i g_i(x)},$$

where $\Lambda = \left\{\lambda = (\lambda_1,...,\lambda_p) \in R^p : \lambda_i \geq 0 \wedge \sum_{i=1}^{p} \lambda_i = 1\right\}$.

For the considered semi-infinite minimax fractional programming problem (SMFP), we now give the parametric necessary optimality conditions established by Zalmai and Zhang [54].

Theorem 3.5. *Let $\overline{x} \in D$ be an optimal point in the considered semi-infinite minimax fractional programming problem (SMFP) with the corresponding optimal value equal to $\overline{v} = \max_{1\leqq i\leqq p} \frac{f_i(\overline{x})}{g_i(\overline{x})}$, the functions $t \to G_j(x,t)$, $t \in T_j$ and $s \to H_k(x,s)$, $s \in S_k$ be continuously differentiable at $\overline{x}$, the generalized Abadie constraint qualification (Definition 3.3) be satisfied at $\overline{x}$ and the set cone $\left\{\nabla G_j(\overline{x},t) : t \in \widehat{T}_{j_m}(\overline{x}),\ j = 1,...,q\right\} + span\left\{\nabla H_k(\overline{x},s) : s \in S_k,\ k = 1,...,r\right\}$ is closed. Then, there exist $\overline{\lambda} \in \Lambda = \left\{\lambda = (\lambda_1,...,\lambda_p) \in R^p : \lambda \geq 0,\ \sum_{i=1}^{p} \lambda_i = 1\right\}$ and integers $\overline{w}_0$ and $\overline{w}$, with $0 \leqq \overline{w}_0 \leqq \overline{w} \leqq n+1$, such that there exist $\overline{w}_0$ indices j_m, with $1 \leqq j_m \leqq q$, together with $\overline{w}_0$ points $t^m \in \widehat{T}_{j_m}(\overline{x})$, $m = 1,...,\overline{w}_0$, $\overline{w} - \overline{w}_0$ indices k_m, with $1 \leqq k_m \leqq r$, together with $\overline{w} - \overline{w}_0$ points $s^m \in S_{k_m}$, $m = 1,...,\overline{w} - \overline{w}_0$ and $\overline{w}$ real numbers $\overline{\mu}_m$ with $\overline{\mu}_m > 0$, $m = 1,...,\overline{w}_0$ such that the following conditions are satisfied:*

$$\sum_{i=1}^{p} \overline{\lambda}_i \left[\nabla f_i(\overline{x}) - \overline{v}\nabla g_i(\overline{x})\right] + \sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x},t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m \nabla H_{k_m}(\overline{x},s^m) = 0, \tag{2}$$

$$\overline{\lambda}_i \left[f_i(\overline{x}) - \overline{v} g_i(\overline{x})\right] = 0, \quad i = 1,...,p. \tag{3}$$

For the considered semi-infinite minimax fractional programming problem (SMFP), we define the vector-valued Lagrange function L, where each its component is defined by

$$L_i\left(z,\lambda,\mu,v,w,w_0,\overline{t},\overline{s}\right) = \lambda_i \left[f_i(z) - v g_i(z)\right] + \frac{1}{p}\left[\sum_{m=1}^{w_0} \mu_m G_{j_m}(z,t^m) + \sum_{m=w_0+1}^{w} \mu_m H_{k_m}(z,s^m)\right]. \tag{4}$$

Now, we give the definition of a vector saddle point of the vector-valued Lagrange function L in the considered semi-infinite minimax fractional programming problem (SMFP).

Definition 3.6. *A point* $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right)$ *is said to be a saddle point of the vector-valued Lagrange function L defined for the considered semi-infinite minimax fractional programming problem (SMFP) if,*

i) $L\left(\overline{x}, \overline{\lambda}, \mu, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right) \leqq L\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right) \quad \forall \mu \in R^{\overline{w}}$,

ii) $L\left(x, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right) \nleq L\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right) \quad \forall x \in D$.

Theorem 3.7. *Let* $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right)$ *be a saddle point of the Lagrange function in the considered semi-infinite minimax fractional programming problem (SMFP). Then* $\overline{x}$ *is optimal in problem (SMFP).*

Proof. Since $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right)$ is a saddle point of the vector-valued Lagrange function in the considered semi-infinite minimax fractional programming problem (SMFP), by Definition 3.6, conditions i) and ii) are satisfied. Thus, by condition i) and the definition of the vector-valued Lagrange function L, the following inequalities

$$\begin{aligned} &\overline{\lambda}_i \left[f_i(\overline{x}) - \overline{v} g_i(\overline{x})\right] + \tfrac{1}{p}\left[\textstyle\sum_{m=1}^{\overline{w}_0} \mu_m G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \mu_m H_{k_m}(\overline{x}, s^m)\right] \\ &\quad \leqq \overline{\lambda}_i \left[f_i(\overline{x}) - \overline{v} g_i(\overline{x})\right] + \tfrac{1}{p}\left[\textstyle\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m)\right], \ i \in I \end{aligned} \tag{5}$$

hold for all $\mu \in R^{\overline{w}}$. Thus, (5) gives

$$\sum_{m=1}^{\overline{w}_0} \mu_m G_{j_m}(\overline{x}, t^m) \leqq \sum_{m=1}^{\overline{w}_0} \overline{\mu}_m G_{j_m}(\overline{x}, t^m). \tag{6}$$

Hence, for $\mu_m = 0$, $m = 1, ..., \overline{w}_0$, (6) yields

$$\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m G_{j_m}(\overline{x}, t^m) \geqq 0. \tag{7}$$

Using $\overline{x} \in D$ together with $\overline{\mu}_m > 0$, $m = 1, ..., \overline{w}_0$, we obtain

$$\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m G_{j_m}(\overline{x}, t^m) \leqq 0. \tag{8}$$

Thus, by (7) and (8), it follows that

$$\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m G_{j_m}(\overline{x}, t^m) = 0. \tag{9}$$

We proceed by contradiction. Suppose, contrary to the result, that $\overline{x} \in D$ is not optimal in the considered semi-infinite minimax fractional programming problem. Then, there exists $\widetilde{x} \in D$ such that

$$\varphi(\widetilde{x}) < \varphi(\overline{x}) = \overline{v}. \tag{10}$$

Since $\varphi(\widetilde{x}) \equiv \max_{1 \leqq i \leqq p} \frac{f_i(\widetilde{x})}{g_i(\widetilde{x})}$, the inequality (10) yields

$$f_i(\widetilde{x}) - \overline{v} g_i(\widetilde{x}) < 0, \ i = 1, ..., p. \tag{11}$$

Thus,

$$f_i(\widetilde{x}) - \overline{v} g_i(\widetilde{x}) < f_i(\overline{x}) - \overline{v} g_i(\overline{x}), \ i = 1, ..., p. \tag{12}$$

Multiplying the inequality above by the associated Lagrange multiplier $\overline{\lambda}_i$, $i = 1, ..., p$, we obtain

$$\overline{\lambda}_i \left[f_i(\widetilde{x}) - \overline{v} g_i(\widetilde{x})\right] \leqq \overline{\lambda}_i \left[f_i(\overline{x}) - \overline{v} g_i(\overline{x})\right], \ i = 1, ..., p, \tag{13}$$

$$\overline{\lambda}_i \left[f_i\left(\widetilde{x}\right) - \overline{v}g_i\left(\widetilde{x}\right)\right] < \overline{\lambda}_i \left[f_i\left(\overline{x}\right) - \overline{v}g_i\left(\overline{x}\right)\right] \text{ for at least one } i \in I. \tag{14}$$

Using $\overline{x} \in D$ and $\widetilde{x} \in D$ together with (9), we get, respectively,

$$\begin{aligned}&\overline{\lambda}_i \left[f_i\left(\widetilde{x}\right) - \overline{v}g_i\left(\widetilde{x}\right)\right] + \tfrac{1}{p}\left[\textstyle\sum_{m=1}^{\overline{w}_0} \mu_m G_{j_m}(\widetilde{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \mu_m H_{k_m}(\widetilde{x}, s^m)\right] \\ &\leqq \overline{\lambda}_i \left[f_i\left(\overline{x}\right) - \overline{v}g_i\left(\overline{x}\right)\right] + \tfrac{1}{p}\left[\textstyle\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m)\right], \ i \in I,\end{aligned} \tag{15}$$

$$\begin{aligned}&\overline{\lambda}_i \left[f_i\left(\widetilde{x}\right) - \overline{v}g_i\left(\widetilde{x}\right)\right] + \tfrac{1}{p}\left[\textstyle\sum_{m=1}^{\overline{w}_0} \mu_m G_{j_m}(\widetilde{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \mu_m H_{k_m}(\widetilde{x}, s^m)\right] \\ &< \overline{\lambda}_i \left[f_i\left(\overline{x}\right) - \overline{v}g_i\left(\overline{x}\right)\right] + \tfrac{1}{p}\left[\textstyle\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m)\right] \text{ for at least one } i \in I.\end{aligned} \tag{16}$$

By the definition of the vector-valued Lagrange function, (15) and (16) imply, respectively,

$$L_i\left(\widetilde{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right) \leqq L_i\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right), \ i \in I, \tag{17}$$

$$L_i\left(\widetilde{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right) < L_i\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right), \text{ for at least one } i \in I, \tag{18}$$

contradicting condition ii) in the definition of a saddle point of the Lagrange function L defined for problem (SMFP) (see Definition 3.6). This completes the proof of this theorem. □

Theorem 3.8. *Let $\overline{x} \in D$ be an optimal point in the considered semi-infinite minimax fractional programming problem (SMFP) with the corresponding optimal value equal to $\overline{v} = \max_{1 \leqq i \leqq p} \frac{f_i(\overline{x})}{g_i(\overline{x})}$. Further, assume that there exist $\overline{\lambda} \in \Lambda$ and integers $\overline{w}_0$ and $\overline{w}$, with $0 \leqq \overline{w}_0 \leqq \overline{w} \leqq n+1$, such that there exist $\overline{w}_0$ indices j_m, with $1 \leqq j_m \leqq q$, together with $\overline{w}_0$ points $t^m \in \widehat{T}_{j_m}(\overline{x})$, $m = 1, ..., \overline{w}_0$, $\overline{w} - \overline{w}_0$ indices k_m, with $\overline{w} - \overline{w}_0$ points $s^m \in S_{k_m}$, $m = 1, ..., \overline{w} - \overline{w}_0$, and $\overline{w}$ real numbers $\overline{\mu}_m$ with $\overline{\mu}_m > 0$ for $m = 1, ..., \overline{w}_0$, with the property that the necessary optimality conditions (2)-(3) are fulfilled at $\overline{x}$. Assume, furthermore, that any one of the following seven sets of hypotheses is fulfilled:*

A) a) $f_i(\cdot)$, $i = 1, ..., p$, is a $\left(\Phi, \rho_{f_i}\right)$-invex function at $\overline{x}$ on D,
b) $-\overline{v}g_i(\cdot)$, $i = 1, ..., p$, is a $\left(\Phi, \rho_{g_i}\right)$-invex function at $\overline{x}$ on D,
c) $G_{j_m}(\cdot, t^m)$, $t^m \in \widehat{T}_{j_m}(\overline{x})$, $m = 1, ..., \overline{w}_0$, is a $\left(\Phi, \rho_{G_{j_m}}\right)$-invex function at $\overline{x}$ on D,
d) $H_{k_m}(\cdot, s^m)$, $s^m \in S^+_{k_m}(\overline{x}) \equiv \left\{s^m \in S_{k_m} : \overline{\mu}_m > 0\right\}$, $m \in \{\overline{w}_0 + 1, ..., \overline{w}\}$, is a $\left(\Phi, \rho^+_{H_{k_m}}\right)$-invex function at $\overline{x}$ on D,
e) $-H_{k_m}(\cdot, s^m)$, $s^m \in S^-_{k_m}(\overline{x}) \equiv \left\{s^m \in S_{k_m} : \overline{\mu}_m < 0\right\}$, $m \in \{\overline{w}_0 + 1, ..., \overline{w}\}$, is a $\left(\Phi, \rho^-_{H_{k_m}}\right)$-invex function at $\overline{x}$ on D,
f) $\sum_{i=1}^{p} \overline{\lambda}_i \left(\rho_{f_i} + \overline{v}\rho_{g_i}\right) + \sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \rho_{G_{j_m}} + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \overline{\mu}_m \rho^+_{H_{k_m}} - \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \overline{\mu}_t \rho^-_{H_{k_m}} \geqq 0$,

B) a) $f_i(\cdot) - \overline{v}g_i(\cdot)$, $i = 1, ..., p$, is a (Φ, ρ_i)-invex function at $\overline{x}$ on D,
b) $G_{j_m}(\cdot, t^m)$, $t^m \in \widehat{T}_{j_m}(\overline{x})$, $m = 1, ..., \overline{w}_0$, is a $\left(\Phi, \rho_{G_{j_m}}\right)$-invex function at $\overline{x}$ on D,
c) $H_{k_m}(\cdot, s^m)$, $s^m \in S^+_{k_m}(\overline{x}) \equiv \left\{s^m \in S_{k_m} : \overline{\mu}_m > 0\right\}$, $m \in \{\overline{w}_0 + 1, ..., \overline{w}\}$, is a $\left(\Phi, \rho^+_{H_{k_m}}\right)$-invex function at $\overline{x}$ on D,
d) $-H_{k_m}(\cdot, s^m)$, $s^m \in S^-_{k_m}(\overline{x}) \equiv \left\{s^m \in S_{k_m} : \overline{\mu}_m < 0\right\}$, $m \in \{\overline{w}_0 + 1, ..., \overline{w}\}$, is a $\left(\Phi, \rho^-_{H_{k_m}}\right)$-invex function at $\overline{x}$ on D,
e) $\sum_{i=1}^{p} \overline{\lambda}_i \rho_i + \sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \rho_{G_{j_m}} + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \overline{\mu}_m \rho^+_{H_{k_m}} - \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \overline{\mu}_t \rho^-_{H_{k_m}} \geqq 0$,

C) a) $f_i(\cdot) - \overline{v}g_i(\cdot)$, $i = 1, ..., p$, is a (Φ, ρ_i)-invex function at $\overline{x}$ on D,
b) $\overline{\mu}_m G_{j_m}(\cdot, t^m)$, $t^m \in \widehat{T}_{j_m}(\overline{x})$, $m = 1, ..., \overline{w}_0$, is a $\left(\Phi, \rho_{G_{j_m}}\right)$-invex function at $\overline{x}$ on D,
c) $\overline{\mu}_m H_{k_m}(\cdot, s^m)$, $s^m \in S_{k_m}$, $m = \overline{w}_0 + 1, ..., \overline{w}$, is a $\left(\Phi, \rho_{H_{k_m}}\right)$-invex function at $\overline{x}$ on D,
d) $\sum_{i=1}^{p} \overline{\lambda}_i \rho_i + \sum_{m=1}^{\overline{w}_0} \rho_{G_{j_m}} + \sum_{m=\overline{w}_0+1}^{\overline{w}} \rho_{H_{k_m}} \geqq 0$,

D) a) $f_i(\cdot) - \overline{v}g_i(\cdot)$, $i = 1, ..., p$, is a (Φ, ρ_i)-invex function at $\overline{x}$ on D,
b) $\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m G_{j_m}(\cdot, t^m)$, $t^m \in \widehat{T}_{j_m}(\overline{x})$, is a (Φ, ρ_G)-invex function at $\overline{x}$ on D,
c) $\sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\cdot, s^m)$, $s^m \in S_{k_m}$, is a (Φ, ρ_H)-invex function at $\overline{x}$ on D,
d) $\sum_{i=1}^{p} \overline{\lambda}_i \rho_i + \rho_G + \rho_H \geqq 0$,

E) a) $f_i(\cdot) - \overline{v}g_i(\cdot)$, $i = 1, ..., p$, *is a* (Φ, ρ_i)*-invex function at* $\overline{x}$ *on* D,

b) $\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m G_{j_m}(\cdot, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\cdot, s^m)$, $t^m \in \widehat{T}_{j_m}(\overline{x})$, $s^m \in S_{k_m}$, *is a* (Φ, ρ_{HG})*-invex function at* $\overline{x}$ *on* D,

c) $\sum_{i=1}^{p} \overline{\lambda}_i \rho_i + \rho_{GH} \geqq 0$,

F) each component of the vector-valued Lagrange function $L\left(\cdot, \overline{\lambda}, p\overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right)$, *that is, each function* $z \to L_i\left(z, \overline{\lambda}, p\overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right)$, $i = 1, ...p$, *is* (Φ, ρ_i)*-invex at* $\overline{x}$ *on* D, *where* $\overline{t} \equiv \left(t^1, ..., t^{\overline{w}_0}\right)$, $\overline{s} \equiv \left(s^{\overline{w}_0+1}, ..., s^{\overline{w}}\right)$ *and* $\sum_{i=1}^{p} \rho_i \geqq 0$,

G) each function $\psi_i\left(\cdot, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right) = f_i(\cdot) - \overline{v}g_i(\cdot) + \sum_{m=1}^{\overline{w}_0} \overline{\mu}_m G_{j_m}(\cdot, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\cdot, s^m)$, $i = 1, ..., p$, *is* (Φ, ρ_i)*-invex at* $\overline{x}$ *on* D, *where* $\overline{t} \equiv \left(t^1, ..., t^{\overline{w}_0}\right)$, $\overline{s} \equiv \left(s^{\overline{w}_0+1}, ..., s^{\overline{w}}\right)$ *and* $\sum_{i=1}^{p} \overline{\lambda}_i \rho_i \geqq 0$.

Then $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right)$ *is a saddle point of the vector-valued Lagrange function in the considered semi-infinite minimax fractional programming problem (SMFP).*

Proof. First, we prove inequality i) in Definition 3.6. By assumption, $\overline{x} \in D$ is an optimal point in the considered semi-infinite minimax fractional programming problem (SMFP) with the corresponding optimal value equal to $\overline{v} = \max_{1 \leqq i \leqq p} \frac{f_i(\overline{x})}{g_i(\overline{x})}$ and here exist $\overline{\lambda} \in \Lambda$ and integers $\overline{w}_0$ and $\overline{w}$, with $0 \leqq \overline{w}_0 \leqq \overline{w} \leqq n+1$, such that there exist $\overline{w}_0$ indices j_m, with $1 \leqq j_m \leqq q$, together with $\overline{w}_0$ points $t^m \in \widehat{T}_{j_m}(\overline{x})$, $m = 1, ..., \overline{w}_0$, $\overline{w} - \overline{w}_0$ indices k_m, with $\overline{w} - \overline{w}_0$ points $s^m \in S_{k_m}$, $m = 1, ..., \overline{w} - \overline{w}_0$, and $\overline{w}$ real numbers $\overline{\mu}_m$ with $\overline{\mu}_m > 0$ for $m = 1, ..., \overline{w}_0$, with the property that the necessary optimality conditions (2)-(3) are fulfilled at this point. Hence, from the feasibility of $\overline{x}$, it follows that the relation

$$\sum_{m=\overline{w}_0+1}^{\overline{w}} \mu_m H_{k_m}(\overline{x}, s^m) = \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m) \tag{19}$$

holds for all $\mu = \left(\mu_{\overline{w}_0+1}, ..., \mu_{\overline{w}}\right) \in R_+^{\overline{w}-\overline{w}_0}$. Since $\overline{w}_0$ points t^m belong to $\widehat{T}_{j_m}(\overline{x})$, therefore, using the feasibility of $\overline{x}$ in problem (SMFP) together with $\overline{\mu}_m > 0$ for $m = 1, ..., \overline{w}_0$, we obtain that the inequality

$$\sum_{m=1}^{\overline{w}_0} \mu_m G_{j_m}(\overline{x}, t^m) \leqq \sum_{m=1}^{\overline{w}_0} \overline{\mu}_m G_{j_m}(\overline{x}, t^m) \tag{20}$$

holds for all $\mu = \left(\mu_1, ..., \mu_{\overline{w}_0}\right) \in R_+^{\overline{w}_0}$. Adding both sides of (19) and (20), it follows that the following inequality

$$\sum_{m=1}^{\overline{w}_0} \mu_m G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \mu_m H_{k_m}(\overline{x}, s^m) \leqq \sum_{m=1}^{\overline{w}_0} \overline{\mu}_m G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m). \tag{21}$$

holds for all $\mu = (\mu_1, ..., \mu_{\overline{w}}) \in R_+^{\overline{w}}$. Thus, for each $i = 1, ...p$, we have

$$\overline{\lambda}_i \left[f_i(\overline{x}) - \overline{v}g_i(\overline{x})\right] + \frac{1}{p}\left[\sum_{m=1}^{\overline{w}_0} \mu_m G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \mu_m H_{k_m}(\overline{x}, s^m)\right]$$
$$\leqq \overline{\lambda}_i \left[f_i(\overline{x}) - \overline{v}g_i(\overline{x})\right] + \frac{1}{p}\left[\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m)\right].$$

Hence, by the definition of the vector-valued Lagrange function, the following inequalities

$$L_i\left(\overline{x}, \overline{\lambda}, \mu, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right) \leqq L_i\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right), i \in I \tag{22}$$

hold for any $\mu = (\mu_1, ..., \mu_{\overline{w}}) \in R_+^{\overline{w}}$. This means that inequality i) in the definition of a saddle point of the vector-valued Lagrange function is satisfied.

Now, we prove the second inequality in Definition 3.6.

We proceed by contradiction. Suppose, contrary to the result, that there exists $\widetilde{x} \in D$ such that $L\left(\widetilde{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right) \leq L\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right)$. Then, by the definition of the vector-valued Lagrange function, it follows that

$$L_i\left(\widetilde{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right) \leqq L_i\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right), \, i \in I, \tag{23}$$

$$L_{i^*}\left(\widetilde{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right) < L_{i^*}\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right) \text{ for some } i^* \in I. \tag{24}$$

Taking into account (4) in (23) and (24), we have

$$\begin{aligned} &\overline{\lambda}_i\left[f_i\left(\widetilde{x}\right) - \overline{v}g_i\left(\widetilde{x}\right)\right] + \tfrac{1}{p}\left[\textstyle\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m G_{j_m}(\widetilde{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\widetilde{x}, s^m)\right] \\ &\leqq \overline{\lambda}_i\left[f_i\left(\overline{x}\right) - \overline{v}g_i\left(\overline{x}\right)\right] + \tfrac{1}{p}\left[\textstyle\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m)\right], \, i \in I, \end{aligned} \tag{25}$$

$$\begin{aligned} &\overline{\lambda}_{i^*}\left[f_{i^*}\left(\widetilde{x}\right) - \overline{v}g_{i^*}\left(\widetilde{x}\right)\right] + \tfrac{1}{p}\left[\textstyle\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m G_{j_m}(\widetilde{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\widetilde{x}, s^m)\right] \\ &< \overline{\lambda}_{i^*}\left[f_{i^*}\left(\overline{x}\right) - \overline{v}g_{i^*}\left(\overline{x}\right)\right] + \tfrac{1}{p}\left[\textstyle\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m)\right] \quad \text{for some } i^* \in I. \end{aligned} \tag{26}$$

We prove the second inequality in Definition 3.6 under hypothesis A).

By assumption, f_i, $i = 1, ..., p$, is a $\left(\Phi, \rho_{f_i}\right)$-invex function at $\overline{x}$ on D and $-\overline{v}g_i$, $i = 1, ..., p$, is a $\left(\Phi, \rho_{g_i}\right)$-invex function at $\overline{x}$ on D, $G_{j_m}(\cdot, t^m)$, $t^m \in \widehat{T}_{j_m}\left(\overline{x}\right)$, $m = 1, ..., \overline{w}_0$, is a $\left(\Phi, \rho_{G_{j_m}}\right)$-invex function at $\overline{x}$ on D, $H_{k_m}(\cdot, s^m)$, $s^m \in S^+_{k_m}\left(\overline{x}\right) \equiv \left\{s^m \in S_{k_m} : \overline{\mu}_m > 0\right\}$, $m \in \{\overline{w}_0 + 1, ..., \overline{w}\}$, is a $\left(\Phi, \rho^+_{H_{k_m}}\right)$-invex function at $\overline{x}$ on D, $-H_{k_m}(\cdot, s^m)$, $s^m \in S^-_{k_m}\left(\overline{x}\right) \equiv \left\{s^m \in S_{k_m} : \overline{\mu}_m < 0\right\}$, $m \in \{\overline{w}_0 + 1, ..., \overline{w}\}$, is a $\left(\Phi, \rho^-_{H_{k_m}}\right)$-invex function at $\overline{x}$ on D. Hence, by Definition 2.2, the following inequalities

$$f_i(x) - f_i(\overline{x}) \geqq \Phi\left(x, \overline{x}, \left(\nabla f_i\left(\overline{x}\right), \rho_{f_i}\right)\right), \, i = 1, ..., p, \tag{27}$$

$$-\overline{v}g_i(x) + \overline{v}g_i(\overline{x}) \geqq \Phi\left(x, \overline{x}, \left(-\overline{v}\nabla g_i\left(\overline{x}\right), \rho_{g_i}\right)\right), \, i = 1, ..., p, \tag{28}$$

$$G_{j_m}(x, t^m) - G_{j_m}(\overline{x}, t^m) \geqq \Phi\left(x, \overline{x}, \left(\nabla G_{j_m}(\overline{x}, t^m), \rho_{G_{j_m}}\right)\right), \, t^m \in \widehat{T}_{j_m}\left(\overline{x}\right), \; m = 1, ..., \overline{w}_0, \tag{29}$$

$$H_{k_m}(x, s^m) - H_{k_m}(\overline{x}, s^m) \geqq \Phi\left(x, \overline{x}, \left(\nabla H_{k_m}(\overline{x}, s^m), \rho^+_{H_{k_m}}\right)\right), \, s^m \in S^+_{k_m}\left(\overline{x}\right), \; m \in \{\overline{w}_0 + 1, ..., \overline{w}\}, \tag{30}$$

$$-H_{k_m}(x, s^m) + H_{k_m}(\overline{x}, s^m) \geqq \Phi\left(x, \overline{x}, \left(-\nabla H_{k_m}(\overline{x}, s^m), \rho^-_{H_{k_m}}\right)\right), \, s^m \in S^-_{k_m}\left(\overline{x}\right), \; m \in \{\overline{w}_0 + 1, ..., \overline{w}\}. \tag{31}$$

hold for all $x \in D$. Therefore, they are also satisfied for $\widetilde{x} \in D$. Multiplying each inequality (27)-(31) by the associated Lagrange multiplier, we obtain, respectively

$$\overline{\lambda}_i f_i(\widetilde{x}) - \overline{\lambda}_i f_i(\overline{x}) \geqq \overline{\lambda}_i \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla f_i\left(\overline{x}\right), \rho_{f_i}\right)\right), \, i = 1, ..., p, \tag{32}$$

$$-\overline{\lambda}_i \overline{v} g_i(\widetilde{x}) + \overline{\lambda}_i \overline{v} g_i(\overline{x}) \geqq \overline{\lambda}_i \Phi\left(\widetilde{x}, \overline{x}, \left(-\overline{v}\nabla g_i\left(\overline{x}\right), \rho_{g_i}\right)\right), \, i = 1, ..., p, \tag{33}$$

$$\overline{\mu}_m G_{j_m}(\widetilde{x}, t^m) - \overline{\mu}_m G_{j_m}(\overline{x}, t^m) \geqq \overline{\mu}_m \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla G_{j_m}(\overline{x}, t^m), \rho_{G_{j_m}}\right)\right), \, t^m \in \widehat{T}_{j_m}\left(\overline{x}\right), \; m = 1, ..., \overline{w}_0, \tag{34}$$

$$\overline{\mu}_m H_{k_m}(\widetilde{x}, s^m) - \overline{\mu}_m H_{k_m}(\overline{x}, s^m) \geqq \overline{\mu}_m \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla H_{k_m}(\overline{x}, s^m), \rho^+_{H_{k_m}}\right)\right), \, s^m \in S^+_{k_m}\left(\overline{x}\right), \; m \in \{\overline{w}_0 + 1, ..., \overline{w}\}, \tag{35}$$

$$\overline{\mu}_m H_{k_m}(\widetilde{x}, s^m) - \overline{\mu}_m H_{k_m}(\overline{x}, s^m) \geqq -\overline{\mu}_m \Phi\left(\widetilde{x}, \overline{x}, \left(-\nabla H_{k_m}(\overline{x}, s^m), \rho^-_{H_{k_m}}\right)\right), \, s^m \in S^-_{k_m}\left(\overline{x}\right), \; m \in \{\overline{w}_0 + 1, ..., \overline{w}\}. \tag{36}$$

Adding both sides of the inequalities (34)-(36) and the adding both sides of the obtained inequalities and both sides of (32)-(33), we get

$$\overline{\lambda}_i \left[f_i(\widetilde{x}) - \overline{v} g_i(\widetilde{x}) \right] + \frac{1}{p} \left[\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m G_{j_m}(\widetilde{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\widetilde{x}, s^m) \right]$$
$$- \left(\overline{\lambda}_i \left[f_i(\overline{x}) - \overline{v} g_i(\overline{x}) \right] + \frac{1}{p} \left[\sum_{m=1}^{w_0} \overline{\mu}_m G_{j_m}(\overline{x}, t^m) + \sum_{m=w_0+1}^{w} \overline{\mu}_m H_{k_m}(\overline{x}, s^m) \right] \right)$$
$$\geqq \overline{\lambda}_i \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla f_i(\overline{x}), \rho_{f_i}\right)\right) + \overline{\lambda}_i \Phi\left(\widetilde{x}, \overline{x}, \left(-\overline{v}\nabla g_i(\overline{x}), \rho_{g_i}\right)\right) + \frac{1}{p} \sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla G_{j_m}(\overline{x}, t^m), \rho_{G_{j_m}}\right)\right)$$
$$+ \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla H_{k_m}(\overline{x}, s^m), \rho^+_{H_{k_m}}\right)\right) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \left(-\overline{\mu}_m\right) \Phi\left(\widetilde{x}, \overline{x}, \left(-\nabla H_{k_m}(\overline{x}, s^m), \rho^-_{H_{k_m}}\right)\right), \ i \in I. \quad (37)$$

Combining (25), (26) and (37), and then adding both sides of the obtained inequalities, we get, respectively,

$$\overline{\lambda}_i \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla f_i(\overline{x}), \rho_{f_i}\right)\right) + \overline{\lambda}_i \Phi\left(\widetilde{x}, \overline{x}, \left(-\overline{v}\nabla g_i(\overline{x}), \rho_{g_i}\right)\right) + \frac{1}{p} \left[\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla G_{j_m}(\overline{x}, t^m), \rho_{G_{j_m}}\right)\right) \right.$$
$$\left. + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla H_{k_m}(\overline{x}, s^m), \rho^+_{H_{k_m}}\right)\right) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \left(-\overline{\mu}_m\right) \Phi\left(\widetilde{x}, \overline{x}, \left(-\nabla H_{k_m}(\overline{x}, s^m), \rho^-_{H_{k_m}}\right)\right) \right] \leqq 0, \ i \in I, \quad (38)$$

$$\overline{\lambda}_{i^*} \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla f_{i^*}(\overline{x}), \rho_{f_{i^*}}\right)\right) + \overline{\lambda}_{i^*} \Phi\left(\widetilde{x}, \overline{x}, \left(-\overline{v}\nabla g_{i^*}(\overline{x}), \rho_{g_{i^*}}\right)\right) + \frac{1}{p} \left[\sum_{m=1}^{w_0} \overline{\mu}_m \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla G_{j_m}(\overline{x}, t^m), \rho_{G_{j_m}}\right)\right) \right.$$
$$+ \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla H_{k_m}(\overline{x}, s^m), \rho^+_{H_{k_m}}\right)\right)$$
$$\left. + \sum_{m=\overline{w}_0+1}^{\overline{w}} \left(-\overline{\mu}_m\right) \Phi\left(\widetilde{x}, \overline{x}, \left(-\nabla H_{k_m}(\overline{x}, s^m), \rho^-_{H_{k_m}}\right)\right) \right] < 0 \text{ for at least one } i^* \in I. \quad (39)$$

Adding both sides of the above inequalities, we obtain

$$\sum_{i=1}^{p} \overline{\lambda}_i \left(\Phi\left(\widetilde{x}, \overline{x}, \left(\nabla f_i(\overline{x}), \rho_{f_i}\right)\right) + \Phi\left(\widetilde{x}, \overline{x}, \left(-\overline{v}\nabla g_i(\overline{x}), \rho_{g_i}\right)\right) \right) + \sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla G_{j_m}(\overline{x}, t^m), \rho_{G_{j_m}}\right)\right)$$
$$+ \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla H_{k_m}(\overline{x}, s^m), \rho^+_{H_{k_m}}\right)\right) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \left(-\overline{\mu}_m\right) \Phi\left(\widetilde{x}, \overline{x}, \left(-\nabla H_{k_m}(\overline{x}, s^m), \rho^-_{H_{k_m}}\right)\right) < 0. \quad (40)$$

Let us introduce the following notations:

$$\widetilde{\lambda}_i = \frac{\overline{\lambda}_i}{A}, \ i = 1, ..., p, \quad (41)$$

$$\widetilde{\mu}_m = \frac{\overline{\mu}_m}{A}, \ m = 1, ..., \overline{w}_0, \quad (42)$$

$$\widetilde{\mu}^+_m = \frac{\overline{\mu}_m}{A}, \ m \in \{\overline{w}_0 + 1, ..., \overline{w}\} \text{ if } \overline{\mu}_m > 0, \quad (43)$$

$$\widetilde{\mu}_m^- = \frac{-\overline{\mu}_m}{A},\ m \in \{\overline{w}_0 + 1, ..., \overline{w}\} \text{ if } \overline{\mu}_m < 0, \tag{44}$$

where

$$A = \sum_{i=1}^{p} \overline{\lambda}_i + \sum_{m=1}^{\overline{w}_0} \overline{\mu}_m + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\ :\ \overline{\mu}_m > 0\}} \overline{\mu}_m + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\ :\ \overline{\mu}_m < 0\}} \left(-\overline{\mu}_m\right). \tag{45}$$

By (41)-(44), it follows that $0 \leqq \widetilde{\lambda}_i \leqq 1$, $i = 1, ..., p$, $\widetilde{\lambda}_i > 0$ for at least one $i \in I$, $0 \leqq \widetilde{\mu}_m \leqq 1$, $m = 1, ..., \overline{w}_0$, $0 < \widetilde{\mu}_m^+ \leqq 1$, $m \in \{\overline{w}_0 + 1, ..., \overline{w}\}$, $0 < \widetilde{\mu}_m^- \leqq 1$, $m \in \{\overline{w}_0 + 1, ..., \overline{w}\}$, and also

$$\sum_{i=1}^{p} \widetilde{\lambda}_i + \sum_{m=1}^{\overline{w}_0} \widetilde{\mu}_m + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \widetilde{\mu}_m^+ + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \widetilde{\mu}_m^- = 1. \tag{46}$$

Taking into account (41)-(44) in (40), we get

$$\sum_{i=1}^{p} \widetilde{\lambda}_i \left(\Phi\left(\widetilde{x}, \overline{x}, \left(\nabla f_i(\overline{x}), \rho_{f_i}\right)\right) + \Phi\left(\widetilde{x}, \overline{x}, \left(-\overline{v}\nabla g_i(\overline{x}), \rho_{g_i}\right)\right)\right) + \sum_{m=1}^{\overline{w}_0} \widetilde{\mu}_m \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla G_{j_m}(\overline{x}, t^m), \rho_{G_{j_m}}\right)\right)$$
$$+ \sum_{m=\overline{w}_0+1}^{\overline{w}} \widetilde{\mu}_m^+ \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla H_{k_m}(\overline{x}, s^m), \rho_{H_{k_m}}^+\right)\right) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \widetilde{\mu}_m^- \Phi\left(\widetilde{x}, \overline{x}, \left(-\nabla H_{k_m}(\overline{x}, s^m), \rho_{H_{k_m}}^-\right)\right) < 0. \tag{47}$$

By Definition 2.2, it follows that $\Phi(\widetilde{x}, \overline{x}, (\cdot, \cdot))$ is a convex function on R^{n+1}. Hence, by (46), the definition of a convex function yields

$$\sum_{i=1}^{p} \widetilde{\lambda}_i \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla f_i(\overline{x}), \rho_{f_i}\right)\right) + \sum_{i=1}^{p} \widetilde{\lambda}_i \Phi\left(\widetilde{x}, \overline{x}, \left(-\overline{v}\nabla g_i(\overline{x}), \rho_{g_i}\right)\right) + \sum_{m=1}^{\overline{w}_0} \widetilde{\mu}_m \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla G_{j_m}(\overline{x}, t^m), \rho_{G_{j_m}}\right)\right)$$
$$+ \sum_{m=\overline{w}_0+1}^{\overline{w}} \widetilde{\mu}_m^+ \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla H_{k_m}(\overline{x}, s^m), \rho_{H_{k_m}}^+\right)\right) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \widetilde{\mu}_m^- \Phi\left(\widetilde{x}, \overline{x}, \left(-\nabla H_{k_m}(\overline{x}, s^m), \rho_{H_{k_m}}^-\right)\right)$$
$$\geqq \Phi\left(\widetilde{x}, \overline{x}, \left(\sum_{i=1}^{p} \widetilde{\lambda}_i \left(\nabla f_i(\overline{x}), \rho_{f_i}\right) + \sum_{i=1}^{p} \widetilde{\lambda}_i \overline{v}\left(-\nabla g_i(\overline{x}), \rho_{g_i}\right) + \sum_{m=1}^{\overline{w}_0} \widetilde{\mu}_m \left(\nabla G_{j_m}(\overline{x}, t^m), \rho_{G_{j_m}}\right)\right.\right.$$
$$\left.\left. + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \widetilde{\mu}_m^+ \left(\nabla H_{k_m}(\overline{x}, s^m), \rho_{H_{k_m}}^+\right) + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \widetilde{\mu}_m^- \left(-\nabla H_{k_m}(\overline{x}, s^m), \rho_{H_{k_m}}^-\right)\right)\right). \tag{48}$$

Combining (47) and (48), we have

$$\Phi\left(\widetilde{x}, \overline{x}, \left(\sum_{i=1}^{p} \widetilde{\lambda}_i \left(\nabla f_i(\overline{x}), \rho_{f_i}\right) + \sum_{i=1}^{p} \widetilde{\lambda}_i \left(-\overline{v}\nabla g_i(\overline{x}), \rho_{g_i}\right) + \sum_{m=1}^{\overline{w}_0} \widetilde{\mu}_m \left(\nabla G_{j_m}(\overline{x}, t^m), \rho_{G_{j_m}}\right)\right.\right.$$
$$\left.\left. + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \widetilde{\mu}_m^+ \left(\nabla H_{k_m}(\overline{x}, s^m), \rho_{H_{k_m}}^+\right) + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \widetilde{\mu}_m^- \left(-\nabla H_{k_m}(\overline{x}, s^m), \rho_{H_{k_m}}^-\right)\right)\right) < 0. \tag{49}$$

Taking into account (41)-(44) in (49), we get

$$\Phi\left(\widetilde{x}, \overline{x}, \frac{1}{A}\left(\sum_{i=1}^{p} \overline{\lambda}_i \left[\nabla f_i(\overline{x}) - \overline{v}\nabla g_i(\overline{x})\right] + \sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m \nabla H_{k_m}(\overline{x}, s^m),\right.\right.$$
$$\left.\left. \sum_{i=1}^{p} \overline{\lambda}_i \left[\rho_{f_i} + \overline{v}\rho_{g_i}\right] + \sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \rho_{G_{j_m}} + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \overline{\mu}_m \rho_{H_{k_m}}^+ - \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \overline{\mu}_m \rho_{H_{k_m}}^-\right)\right) < 0. \tag{50}$$

Hence, the necessary optimality condition (2) implies

$$\Phi\left(\widetilde{x},\overline{x},\frac{1}{A}\left(0\,,\,\sum_{i=1}^{p}\overline{\lambda}_i\rho_{f_i}+\overline{v}\sum_{i=1}^{p}\overline{\lambda}_i\rho_{g_i}+\sum_{m=1}^{\overline{w}_0}\overline{\mu}_m\rho_{G_{j_m}}+\sum_{m\in\{\overline{w}_0+1,\dots,\overline{w}\}}\overline{\mu}_m\rho^{+}_{H_{k_m}}-\sum_{m\in\{\overline{w}_0+1,\dots,\overline{w}\}}\overline{\mu}_m\rho^{-}_{H_{k_m}}\right)\right)<0. \quad (51)$$

By Definition 2.2, it follows that $\Phi\left(\widetilde{x},\overline{x},(0,a)\right)\geqq 0$ for each $a\in R_+$. Therefore, hypothesis f)) implies that the inequality

$$\Phi\left(x,\overline{x},\frac{1}{A}\left(0,\sum_{i=1}^{p}\overline{\lambda}_i\rho_{f_i}+\sum_{i=1}^{p}\overline{\lambda}_i\overline{v}\rho_{g_i}+\sum_{m=1}^{\overline{w}_0}\overline{\mu}_m\rho_{G_{j_m}}+\sum_{m\in\{\overline{w}_0+1,\dots,\overline{w}\}}\overline{\mu}_m\rho^{+}_{H_{k_m}}-\sum_{m\in\{\overline{w}_0+1,\dots,\overline{w}\}}\overline{\mu}_t\rho^{-}_{H_{k_m}}\right)\right)\geqq 0 \quad (52)$$

holds, contradicts (52). This completes the proof of the second inequality in Definition 3.6 under hypothesis A).

Proofs of the second inequality in Definition 3.6 under hypotheses B)-D) are similar to the proof under hypotheses A) or E) and, therefore, they have been omitted in the paper.

We now prove the second inequality in Definition 3.6 under hypothesis E). We proceed by contradiction. Thus, there exists $\widetilde{x}\in D$ such that (25) and (26) are satisfied. Therefore, adding both sides of (25) and (26), we have

$$\begin{aligned}&\sum_{i=1}^{p}\overline{\lambda}_i\left[f_i\left(\widetilde{x}\right)-\overline{v}g_i\left(\widetilde{x}\right)\right]+\sum_{m=1}^{\overline{w}_0}\overline{\mu}_mG_{j_m}(\widetilde{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\mu}_mH_{k_m}(\widetilde{x},s^m)\\ &\quad<\sum_{i=1}^{p}\overline{\lambda}_i\left[f_i\left(\overline{x}\right)-\overline{v}g_i\left(\overline{x}\right)\right]+\sum_{m=1}^{\overline{w}_0}\overline{\mu}_mG_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\mu}_mH_{k_m}(\overline{x},s^m).\end{aligned} \quad (53)$$

By assumptions E) a) and E) b), Definition 2.2 gives, respectively,

$$f_i\left(\widetilde{x}\right)-\overline{v}g_i\left(\widetilde{x}\right)-\left(f_i\left(\overline{x}\right)-\overline{v}g_i\left(\overline{x}\right)\right)\geqq\Phi\left(\widetilde{x},\overline{x},\left(\nabla f_i\left(\overline{x}\right)-\overline{v}\nabla g_i\left(\overline{x}\right),\rho_i\right)\right), \quad (54)$$

$$\begin{aligned}&\sum_{m=1}^{\overline{w}_0}\overline{\mu}_mG_{j_m}(\widetilde{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\mu}_mH_{k_m}(\widetilde{x},s^m)-\left(\sum_{m=1}^{\overline{w}_0}\overline{\mu}_mG_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\mu}_mH_{k_m}(\overline{x},s^m)\right)\\ &\quad\geqq\Phi\left(\widetilde{x},\overline{x},\left(\sum_{m=1}^{\overline{w}_0}\overline{\mu}_m\nabla G_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\mu}_m\nabla H_{k_m}(\overline{x},s^m),\rho_{GH}\right)\right).\end{aligned} \quad (55)$$

Multiplying (53) by $\overline{\lambda}_i$, $i=1,...,p$, and the adding both sides of the obtained inequalities, we get

$$\sum_{i=1}^{p}\overline{\lambda}_i\left(f_i\left(\widetilde{x}\right)-\overline{v}g_i\left(\widetilde{x}\right)\right)-\sum_{i=1}^{p}\overline{\lambda}_i\left(f_i\left(\overline{x}\right)-\overline{v}g_i\left(\overline{x}\right)\right)\geqq\sum_{i=1}^{p}\overline{\lambda}_i\Phi\left(\widetilde{x},\overline{x},\left(\nabla f_i\left(\overline{x}\right)-\overline{v}\nabla g_i\left(\overline{x}\right),\rho_i\right)\right). \quad (56)$$

Combining (55) and (56), we have

$$\begin{aligned}&\sum_{i=1}^{p}\overline{\lambda}_i\left(f_i\left(\widetilde{x}\right)-\overline{v}g_i\left(\widetilde{x}\right)\right)-\sum_{i=1}^{p}\overline{\lambda}_i\left(f_i\left(\overline{x}\right)-\overline{v}g_i\left(\overline{x}\right)\right)+\sum_{m=1}^{\overline{w}_0}\overline{\mu}_mG_{j_m}(\widetilde{x},t^m)\\ &\quad+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\mu}_mH_{k_m}(\widetilde{x},s^m)-\left(\sum_{m=1}^{\overline{w}_0}\overline{\mu}_mG_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\mu}_mH_{k_m}(\overline{x},s^m)\right)\\ &\geqq\sum_{i=1}^{p}\overline{\lambda}_i\Phi\left(\widetilde{x},\overline{x},\left(\nabla f_i\left(\overline{x}\right)-\overline{v}\nabla g_i\left(\overline{x}\right),\rho_i\right)\right)+\Phi\left(\widetilde{x},\overline{x},\left(\sum_{m=1}^{\overline{w}_0}\overline{\mu}_m\nabla G_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\mu}_m\nabla H_{k_m}(\overline{x},s^m),\rho_{GH}\right)\right).\end{aligned} \quad (57)$$

Hence, (53) and (57) yield

$$\sum_{i=1}^{p} \overline{\lambda}_i \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla f_i\left(\overline{x}\right) - \overline{v}\nabla g_i\left(\overline{x}\right), \rho_i\right)\right) + \Phi\left(\widetilde{x}, \overline{x}, \left(\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m \nabla H_{k_m}(\overline{x}, s^m), \rho_{GH}\right)\right) < 0. \tag{58}$$

Let us introduce the following notations:

$$\widetilde{\lambda}_i = \frac{\overline{\lambda}_i}{1 + \sum_{i=1}^{p} \overline{\lambda}_i}, \ i = 1, ..., p, \tag{59}$$

$$\widetilde{\lambda}_{p+1} = \frac{1}{1 + \sum_{i=1}^{p} \overline{\lambda}_i}. \tag{60}$$

By (57) and (58), it follows that $0 \leqq \widetilde{\lambda}_i \leqq 1$, $i = 1, ..., p$, $\widetilde{\lambda}_i > 0$ for at least one $i \in I$, $0 < \widetilde{\lambda}_{p+1} < 1$, and also

$$\sum_{i=1}^{p+1} \widetilde{\lambda}_i = 1. \tag{61}$$

Using (59) and (60) together with (58), we get

$$\sum_{i=1}^{p} \widetilde{\lambda}_i \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla f_i\left(\overline{x}\right) - \overline{v}\nabla g_i\left(\overline{x}\right), \rho_i\right)\right) + \widetilde{\lambda}_{p+1} \Phi\left(\widetilde{x}, \overline{x}, \left(\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m \nabla H_{k_m}(\overline{x}, s^m), \rho_{GH}\right)\right) < 0. \tag{62}$$

By Definition 2.2, it follows that $\Phi\left(\widetilde{x}, \overline{x}, (\cdot, \cdot)\right)$ is a convex function on R^{n+1}. Hence, by (61), the definition of a convex function implies

$$\Phi\left(\widetilde{x}, \overline{x}, \left(\sum_{i=1}^{p} \widetilde{\lambda}_i \left(\nabla f_i\left(\overline{x}\right) - \overline{v}\nabla g_i\left(\overline{x}\right), \rho_i\right) + \ \widetilde{\lambda}_{p+1} \left(\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m \nabla H_{k_m}(\overline{x}, s^m), \rho_{GH}\right)\right)\right) < 0. \tag{63}$$

Thus,

$$\Phi\left(\widetilde{x}, \overline{x}, \left(\sum_{i=1}^{p} \widetilde{\lambda}_i \left(\nabla f_i\left(\overline{x}\right) - \overline{v}\nabla g_i\left(\overline{x}\right), \rho_i\right) + \widetilde{\lambda}_{p+1} \left(\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m \nabla H_{k_m}(\overline{x}, s^m)\right), \ \sum_{i=1}^{p} \widetilde{\lambda}_i \rho_i + \widetilde{\lambda}_{p+1} \rho_{GH}\right)\right) < 0. \tag{64}$$

Using (59) and (60) together with (64), we get

$$\Phi\left(\widetilde{x}, \overline{x}, \frac{1}{1 + \sum_{i=1}^{p} \overline{\lambda}_i} \left(\sum_{i=1}^{p} \overline{\lambda}_i \nabla f_i\left(\overline{x}\right) - \overline{v}\nabla g_i\left(\overline{x}\right) + \sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m \nabla H_{k_m}(\overline{x}, s^m), \sum_{i=1}^{p} \overline{\lambda}_i \rho_i + \rho_{GH}\right)\right) < 0. \tag{65}$$

Hence, the necessary optimality condition (2) implies

$$\Phi\left(\widetilde{x}, \overline{x}, \frac{1}{1 + \sum_{i=1}^{p} \overline{\lambda}_i} \left(0, \ \sum_{i=1}^{p} \overline{\lambda}_i \rho_i + \rho_{GH}\right)\right) < 0. \tag{66}$$

By Definition 2.2, it follows that $\Phi\left(\widetilde{x}, \overline{x}, (0, a)\right) \geqq 0$ for each $a \in R_+$. Therefore, hypothesis E) c) implies that the following inequality

$$\Phi\left(\widetilde{x}, \overline{x}, \frac{1}{1 + \sum_{i=1}^{p} \overline{\lambda}_i} \left(0, \ \sum_{i=1}^{p} \overline{\lambda}_i \rho_i + \rho_{GH}\right)\right) \geqq 0$$

holds, contradicting (66). This completes the proof of the second inequality in Definition 3.6 under hypothesis E).

We now prove the second inequality in Definition 3.6 under hypothesis F). We proceed by contradiction. Suppose, contrary to the result, that there exists $\widetilde{x} \in D$ such that $L\left(\widetilde{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right) \leq L\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right)$. Then, by the definition of the vector-valued Lagrange function, it follows that the inequalities (25) and (26) are satisfied. By assumption, the vector-valued Lagrange function $L\left(\cdot, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right)$ is (Φ, ρ)-invex at $\overline{x}$ on D. Thus, by Definition 2.1, the following inequalities

$$L_i\left(x, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right) - L_i\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right) \geqq \Phi\left(x, \overline{x}, \left(\nabla L_i\left(\overline{x}, \overline{\lambda}, p\overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right)\ , \rho_i\right)\right), i = 1, ..., p$$

hold. Therefore, they are also satisfied for $x = \widetilde{x} \in D$. Taking into account the definition of the Lagrange function (see (4)), we obtain

$$\begin{aligned}
&\overline{\lambda}_i \left[f_i\left(\widetilde{x}\right) - \overline{v} g_i\left(\widetilde{x}\right)\right] + \frac{1}{p}\left[\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m G_{j_m}(\widetilde{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\widetilde{x}, s^m)\right) \\
&- \left(\overline{\lambda}_i \left[f_i\left(\overline{x}\right) - \overline{v} g_i\left(\overline{x}\right)\right] + \frac{1}{p}\left[\sum_{m=1}^{\overline{w}_0} p\overline{\mu}_m G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m)\right]\right) \\
&\geqq \Phi\left(\widetilde{x}, \overline{x}, \left(\overline{\lambda}_i \left[\nabla f_i\left(\overline{x}\right) - \overline{v} \nabla g_i\left(\overline{x}\right)\right] + \frac{1}{p}\left[\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m)\right]\ , \rho_i\right)\right), i = 1, ..., p. \quad (67)
\end{aligned}$$

Combining (25), (26) and (67), we obtain, respectively,

$$\Phi\left(\widetilde{x}, \overline{x}, \left(\overline{\lambda}_i \left[\nabla f_i\left(\overline{x}\right) - \overline{v} \nabla g_i\left(\overline{x}\right)\right] + \frac{1}{p}\left[\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x}, t^m) \quad + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m)\right]\ , \rho_i\right)\right) \leqq 0, i = 1, ..., p, \quad (68)$$

$$\Phi\left(\widetilde{x}, \overline{x}, \left(\overline{\lambda}_{i^*} \left[\nabla f_{i^*}\left(\overline{x}\right) - \overline{v} \nabla g_{i^*}\left(\overline{x}\right)\right] + \frac{1}{p}\left[\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m)\right]\ , \rho_i\right)\right) < 0 \text{ for some } i^* \in I. \quad (69)$$

Multiplying each inequality above by $\frac{1}{p}$ and then adding both sides of the obtained inequality, we get

$$\sum_{i=1}^{p} \frac{1}{p} \Phi\left(\widetilde{x}, \overline{x}, \left(\overline{\lambda}_i \left[\nabla f_i\left(\overline{x}\right) - \overline{v} \nabla g_i\left(\overline{x}\right)\right] + \frac{1}{p}\left[\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m)\right]\ , \rho_i\right)\right) < 0.$$

By Definition 2.1, it follows that $\Phi\left(x, \overline{x}, (\cdot, \cdot)\right)$ is a convex function on R^{n+1}. Then, by the definition of a convex function, it follows that

$$\begin{aligned}
&\sum_{i=1}^{p} \frac{1}{p} \Phi\left(\widetilde{x}, \overline{x}, \left(\overline{\lambda}_i \left[\nabla f_i\left(\overline{x}\right) - \overline{v} \nabla g_i\left(\overline{x}\right)\right] + \frac{1}{p}\left[\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m)\right]\ , \rho_i\right)\right) \\
&\geqq \Phi\left(\widetilde{x}, \overline{x}, \sum_{i=1}^{p} \frac{1}{p}\left(\overline{\lambda}_i \left[\nabla f_i\left(\overline{x}\right) - \overline{v} \nabla g_i\left(\overline{x}\right)\right] + \frac{1}{p}\left[\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m)\right]\ , \rho_i\right)\right). \quad (70)
\end{aligned}$$

Combining (69) and (70), we have

$$\Phi\left(\widetilde{x}, \overline{x}, \sum_{i=1}^{p} \frac{1}{p}\left(\overline{\lambda}_i \left[\nabla f_i\left(\overline{x}\right) - \overline{v} \nabla g_i\left(\overline{x}\right)\right] + \frac{1}{p}\left[\sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m)\right]\ , \rho_i\right)\right) < 0.$$

Thus, the inequality above yields

$$\Phi\left(\widetilde{x},\overline{x},\frac{1}{p}\left(\sum_{i=1}^{p}\overline{\lambda}_i\left[\nabla f_i(\overline{x})-\overline{v}\nabla g_i(\overline{x})\right]+\sum_{m=1}^{\overline{w}_0}\overline{\mu}_m\nabla G_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\mu}_m H_{k_m}(\overline{x},s^m)\,,\,\rho_i\right)\right)<0.$$

By the necessary optimality condition (2), it follows that

$$\Phi\left(\widetilde{x},\overline{x},\frac{1}{p}\left(0\,,\,\sum_{i=1}^{p}\rho_i\right)\right)<0. \tag{71}$$

By Definition 2.1, it follows that $\Phi(\widetilde{x},\overline{x},(0,a))\geqq 0$ for each $a\in R_+$. Since $\sum_{i=1}^{p}\rho_i\geq 0$, the inequality

$$\Phi\left(\widetilde{x},\overline{x},\frac{1}{p}\left(0\,,\,\sum_{i=1}^{p}\rho_i\right)\right)\geqq 0$$

holds, contradicting (71). Hence, the second inequality in Definition 3.6 is satisfied and, therefore, the proof under hypothesis F) is completed.

Now, we prove the second inequality in Definition 3.6 under hypotheses G).

We proceed by contradiction. Suppose, contrary to the result, that there exists $\widetilde{x}\in D$ such that $L\left(\widetilde{x},\overline{\lambda},\overline{\mu},\overline{v},\overline{w},\overline{w}_0,\overline{t},\overline{s}\right)\leq L\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{v},\overline{w},\overline{w}_0,\overline{t},\overline{s}\right)$. Then, by the definition of the vector-valued Lagrange function, it follows that (25) and (26) are satisfied. Adding both sides of the inequalities (25) and (26), we get

$$\begin{aligned}\sum_{i=1}^{p}\overline{\lambda}_i\left[f_i(\widetilde{x})-\overline{v}g_i(\widetilde{x})\right]+\sum_{m=1}^{w_0}\overline{\mu}_m G_{j_m}(\widetilde{x},t^m)+\sum_{m=w_0+1}^{w}\overline{\mu}_m H_{k_m}(\widetilde{x},s^m)\\ <\sum_{i=1}^{p}\overline{\lambda}_i\left[f_i(\overline{x})-\overline{v}g_i(\overline{x})\right]+\sum_{m=1}^{w_0}\overline{\mu}_m G_{j_m}(\overline{x},t^m)+\sum_{m=w_0+1}^{w}\overline{\mu}_m H_{k_m}(\overline{x},s^m).\end{aligned} \tag{72}$$

By assumption, $\psi\left(\cdot,\overline{\mu},\overline{v},\overline{w},\overline{w}_0,\overline{t},\overline{s}\right)$ is a (Φ,ρ)-invex function at $\overline{x}$ on D. Thus, by Definition 2.1, the following inequalities

$$\psi_i\left(\widetilde{x},\overline{\mu},\overline{v},\overline{w},\overline{w}_0,\overline{t},\overline{s}\right)-\psi_i\left(\overline{x},\overline{\mu},\overline{v},\overline{w},\overline{w}_0,\overline{t},\overline{s}\right)\geqq\Phi\left(\widetilde{x},\overline{x},\left(\nabla\psi_i\left(\overline{x},\overline{\mu},\overline{v},\overline{w},\overline{w}_0,\overline{t},\overline{s}\right),\rho_i\right)\right),\ i=1,...,p \tag{73}$$

hold. Multiplying each above inequality by $\overline{\lambda}_i$, $i=1,...,p$, and using the definition of ψ, we get

$$\begin{aligned}&\overline{\lambda}_i\left[f_i(\widetilde{x})-\overline{v}g_i(\widetilde{x})+\sum_{m=1}^{w_0}\overline{\mu}_m G_{j_m}(\widetilde{x},t^m)+\sum_{m=w_0+1}^{w}\overline{\mu}_m H_{k_m}(\widetilde{x},s^m)\right]\\ &\quad-\overline{\lambda}_i\left[f_i(\overline{x})-\overline{v}g_i(\overline{x})+\sum_{m=1}^{w_0}\overline{\mu}_m G_{j_m}(\overline{x},t^m)+\sum_{m=w_0+1}^{w}\overline{\mu}_m H_{k_m}(\overline{x},s^m)\right]\\ &\quad\geqq\overline{\lambda}_i\Phi\left(\widetilde{x},\overline{x},\left(\nabla f_i(\overline{x})-\overline{v}\nabla g_i(\overline{x})+\sum_{m=1}^{\overline{w}_0}\overline{\mu}_m\nabla G_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\mu}_m H_{k_m}(\overline{x},s^m),\rho_i\right)\right),\ i=1,...,p.\end{aligned} \tag{74}$$

Adding both sides of the above inequalities and taking into account $\sum_{i=1}^{p} \overline{\lambda}_i = 1$, we obtain

$$\sum_{i=1}^{p} \overline{\lambda}_i \left[f_i(\widetilde{x}) - \overline{v} g_i(\widetilde{x}) \right] + \sum_{m=1}^{w_0} \overline{\mu}_m G_{j_m}(\widetilde{x}, t^m) + \sum_{m=w_0+1}^{w} \overline{\mu}_m H_{k_m}(\widetilde{x}, s^m)$$
$$- \sum_{i=1}^{p} \overline{\lambda}_i \left[f_i(\overline{x}) - \overline{v} g_i(\overline{x}) \right] + \sum_{m=1}^{w_0} \overline{\mu}_m G_{j_m}(\overline{x}, t^m) + \sum_{m=w_0+1}^{w} \overline{\mu}_m H_{k_m}(\overline{x}, s^m)$$
$$\geqq \sum_{i=1}^{p} \overline{\lambda}_i \Phi\left(\widetilde{x}, \overline{x}, \left[\nabla f_i(\overline{x}) - \overline{v} \nabla g_i(\overline{x}) + \sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m), \rho_i \right] \right). \tag{75}$$

Thus, (72) and (75) yield

$$\sum_{i=1}^{p} \overline{\lambda}_i \Phi\left(\widetilde{x}, \overline{x}, \left[\nabla f_i(\overline{x}) - \overline{v} \nabla g_i(\overline{x}) + \sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m), \rho_i \right] \right) < 0. \tag{76}$$

By Definition 2.1, it follows that $\Phi(\widetilde{x}, \overline{x}, (\cdot, \cdot))$ is a convex function on R^{n+1}. Since $\overline{\lambda}_i \geq 0$, $i = 1, ..., p$, $\sum_{i=1}^{p} \overline{\lambda}_i = 1$, by the definition of a convex function, it follows that

$$\sum_{i=1}^{p} \overline{\lambda}_i \Phi\left(\widetilde{x}, \overline{x}, \left[\nabla f_i(\overline{x}) - \overline{v} \nabla g_i(\overline{x}) + \sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m), \rho_i \right] \right)$$
$$\geqq \Phi\left(\widetilde{x}, \overline{x}, \sum_{i=1}^{p} \overline{\lambda}_i \left[\nabla f_i(\overline{x}) - \overline{v} \nabla g_i(\overline{x}) + \sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m) \right], \sum_{i=1}^{p} \overline{\lambda}_i \rho_i \right). \tag{77}$$

Combining (76) and (77), we get

$$\Phi\left(\widetilde{x}, \overline{x}, \sum_{i=1}^{p} \overline{\lambda}_i \left[\nabla f_i(\overline{x}) - \overline{v} \nabla g_i(\overline{x}) + \sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m) \right], \sum_{i=1}^{p} \overline{\lambda}_i \rho_i \right) < 0.$$

Since $\overline{\lambda}_i \geq 0$, $i = 1, ..., p$, $\sum_{i=1}^{p} \overline{\lambda}_i = 1$, the above inequality gives

$$\Phi\left(\widetilde{x}, \overline{x}, \sum_{i=1}^{p} \overline{\lambda}_i \left[\nabla f_i(\overline{x}) - \overline{v} \nabla g_i(\overline{x}) \right] + \sum_{m=1}^{\overline{w}_0} \overline{\mu}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\mu}_m H_{k_m}(\overline{x}, s^m), \sum_{i=1}^{p} \overline{\lambda}_i \rho_i \right) < 0.$$

The last part of this proof is the same as in the proof under hypothesis G).

We have established under each of the assumptions A)-G) that inequality ii) in Definition 3.6 is satisfied. This means that $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right)$ is a saddle point of the vector-valued Lagrange function in the considered semi-infinite minimax fractional programming problem (SMFP). □

Now, we give an example of a nonconvex semi-infinite minimax fractional programming problem (SMFP) involving (Φ, ρ)-invex functions. It turns out that, in order to prove optimality of a feasible point $\overline{x}$ at which the necessary optimality (2)-(3) are satisfied, the concept of (Φ, ρ)-invexity may be applied.

Example 3.9. *Consider the following semi-infinite minimax fractional programming problem:*

$$\begin{array}{c} \min\limits_{x \in R^2} \max\limits_{1 \leqq i \leqq p} \left(\frac{x_1^2 + x_2^2 + i}{2i + \arctan(x_1 x_2)} \right) \\ G_1(x, t) = -\frac{t}{1+t} - x_1 x_2 \leqq 0, \;\; t \in T_1 = [0, 1], \end{array} \qquad (P1)$$

where p is a finite positive integer number. Note that the set of all feasible solutions $D = \left\{(x_1, x_2) \in R^2 : -\frac{t}{1+t} - x_1 x_2 \leqq 0,\ t \in T_1 = [0, 1]\right\}$. Further, note that $\overline{x} = (0, 0)$ is a feasible point in the considered semi-infinite minimax fractional programming problem, for which there exist $\overline{\lambda} \in \Lambda$, $\overline{\lambda}_i = \frac{1}{p}$, $i = 1, ..., p$, and an integer $\overline{w}_0 = 1$ such that there exist one index $j_1 = 1$ together with a point $t^1 \in \widehat{T}_{j_1}(\overline{x}) = \{0\}$ and $\overline{\mu}_1 = \frac{1}{2}$ such that the necessary optimality conditions (2)-(3) are satisfied. It is not difficult to see that $\overline{v} = \max_{1 \leqq i \leqq p} \frac{f_i(\overline{x})}{g_i(\overline{x})} = \frac{1}{2}$. Let a functional $\Phi : X \times X \times R^{n+1} \to R$ and scalars ρ_{f_i}, ρ_{g_i}, $i = 1, ..., p$, ρ_{G_1} be defined as follows:

$$\Phi(x, \overline{x}, (\varsigma, \rho)) = \frac{1}{2}(\varsigma_1 + \varsigma_2)\left(x_1^2 - \overline{x}_1^2 + x_2^2 - \overline{x}_2^2\right) + 2\left(2^{\rho} - 1\right)\left|(x_1 - \overline{x}_1)(x_2 - \overline{x}_2)\right|,$$

and

$$\rho_{f_i} = 1,\ \rho_{g_i} = -1,\ \rho_{G_1} = -1.$$

It is not difficult to prove that all objective functions and the constraint function are (Φ, ρ)-invex at $\overline{x}$ on D with respect to the functional Φ and scalars ρ_{f_i}, ρ_{g_i}, $i = 1, ..., p$, ρ_{G_1} given above. Since all hypotheses of Theorem 3.8 are satisfied, therefore, $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}_0, \overline{t}\right) = \left((0, 0), \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}_0, 0\right)$ is a saddle point in the considered semi-infinite minimax fractional programming problem (P1). Further, since $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}_0, \overline{t}\right) = \left((0, 0), \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}_0, 0\right)$ is a saddle point in the considered semi-infinite minimax fractional programming problem (P1), optimality of $\overline{x} = (0, 0)$ in problem (P1) follows directly from Theorem 3.7.

Remark 3.10. *Note that we are not in a position to prove the equivalence between an optimal solution $\overline{x}$ and a saddle point $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}, \overline{w}_0, \overline{t}\right)$ of the Lagrange function in the semi-infinite minimax fractional programming problem (P1) considered in Example 3.9 under invexity and many generalized convexity notions defined earlier in the literature (for instance, r-invexity [6], ρ-convexity [50], F-convexity [27], (F, ρ)-convexity [42], (b, η)-invexity [10], ρ-invexity [18], univexity [12], (p, r)-invexity, [5], B-(p, r)-invexity [4], [7] G-invexity [8]). This is a consequence of the fact that a stationary point of each objective function g_i, $i = 1, ..., p$, and a stationary point of the constraint function $G_1(\cdot, t)$ are not their global minimizers (see Ben-Israel and Mond [14]). Then, each objective function g_i, $i = 1, ..., p$, and the constraint function are neither invex [26] nor generalized convex (that is, r-invex [6], (b, η)-invex [10], univex [12], B-(p, r)-invex [4], [7], G-invex [8]) with respect to any function $\eta : D \times D \to R^2$. As it follows even from this example of a nonconvex semi-infinite minimax fractional programming problem, the saddle point criteria established under (Φ, ρ)-invexity are useful for a larger class of such optimization problems than similar criteria established under other generalized invexity notions, even those mentioned above.*

4. Conclusions

In the paper, a new class of nonconvex semi-infinite minimax fractional programming problems with both inequality and equality constraints has been considered. Parametric saddle point criteria have been established for this class of nonconvex semi-infinite minimax fractional programming problems in which the involved functions are (Φ, ρ)-invex. Subsequently, we have illustrated the results established in the paper by an example of a nonconvex semi-infinite minimax fractional programming problem involving (Φ, ρ)-invex functions with respect to the same functional Φ and with respect to, not necessarily the same ρ. It turns out that, in order to prove the equivalence between an optimal solution and a saddle point of the Lagrange function in the considered nonconvex semi-infinite minimax fractional programming problem, a fairly large number of conditions established in the literature under other generalized convexity notions existing in the literature is not applicable. This is consequence of the fact that the concept of (Φ, ρ)-invexity generalizes many generalized convexity notions previously defined in the literature (see Remark 2.3). In this way, the saddle point criteria presented here have been proved for a larger class of nonconvex semi-infinite minimax fractional programming problems than those ones previously established in the literature.

Further, all parametric saddle point criteria established in the paper can easily be modified and used for each one of the following nonlinear mathematical programming problems, which are special cases of the considered optimization problem (SMFP):

$$\underset{x\in D}{\text{minimize}}\ f_1(x),$$

$$\underset{x\in D}{\text{minimize}}\ \frac{f_1(x)}{g_1(x)},$$

$$\underset{x\in D}{\text{minimize}}\ \max_{1\leqq i\leqq p} f_i(x),$$

$$\underset{x\in D}{\text{minimize}}\ \max_{1\leqq i\leqq p} \frac{f_i(x)}{g_i(x)}.$$

In this way, the concept of (Φ, ρ)-invexity can be used to prove several saddle point criteria for various classes of nonconvex semi-infinite minimax fractional programming problems. Moreover, as it follows from the above, this concept of generalized convexity is a useful tool in proving saddle point conditions also for such a nonconvex optimization problem for which other generalized convexity notions may avoid in such a case.

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