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Optimization of a Nonlinear Hermitian Matrix Expression with Application

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Abstract. Extremal ranks and inertias of a nonlinear Hermitian matrix expression are established over a complex field $\mathbb{C}$. In addition, the constraint extremal ranks and inertias of the nonlinear matrix function are also observed. Our research extends already work done in the literature.

1. Introduction

In this article, the notation $\mathbb{C}$ interprets the complex number field. I pertains an identity matrix having acceptable dimension. For a matrix A, the notations $r(A)$, $R(A)$ and A^* refer for the rank, the column space and the conjugate transpose of A, respectively. The Moore-Penrose inverse of A is designated by $A^\dagger$ and is defined to be the solution of the following four matrix equations

$$AA^\dagger A = A,\ A^\dagger AA^\dagger = A^\dagger,\ (AA^\dagger)^* = AA^\dagger,\ (A^\dagger A)^* = A^\dagger A.$$

$L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$ are pair of projectors made by A, respectively, and these are idempotent and Hermitian by the definition of the Moore-Penrose inverse. Inertia of a Hermitian matrix A is the set consisting of the positive, negative and zero eigen values of A counting with multiplicities, respectively. It is understood that

$$r(A) = i_+(A) + i_-(A),$$

and $i_+(A)$ and $i_-(A)$ are known as the positive and negative signature of A.

Linear matrix functions have backbone standing in matrix theory and its applications [4, 5, 7, 9–11, 13, 16, 19, 21, 25–29, 32–41, 43–53, 56, 58, 60]. For example, the usage of

$$BX + (BX)^* = A \tag{1}$$

in model reduction, stability analysis and optimal control can be found in [26, 29]. Liao and Bai analyzed

$$CYC^* + DZD^* + A \tag{2}$$

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in [20]. Zhang et al. evaluated the extremal ranks and inertias of

$$\begin{aligned} A &= BXB^* + CYC^*, \\ DX &= E, GY = H, X = X^*, Y = Y^*, \end{aligned} \tag{3}$$

in [61]. Farid et al. [13] found the Hermitian solution of

$$\begin{aligned} &A_{22}Y = C_{22}\ ,\ YB_{22} = D_{22}, \\ &A_{33}Z = C_{33}\ ,\ ZB_{33} = D_{33}, \\ &C_{44}YC_{44}^* + D_{44}ZD_{44}^* = A_{44}, \end{aligned} \tag{4}$$

in the setting of Hilbert C^*-module. The η-Hermitian solution to (4) was calculated by He and Wang in [17]. The range of inertias of

$$g_2(X, Y) = A - BX - (BX)^* - CYD - (CYD)^* \tag{5}$$

were examined in [42]. The outstanding applications of extremal ranks and inertias of Hermitian matrix function can be found in [4, 5].

The nonlinear matrix functions also have vital importance in different fields like quadratic programming and control theory [1, 3, 12, 18, 24, 30]. Wang and Zhang et al. designed the extremal ranks and inertias of

$$\begin{aligned} \phi(X) &= XX^* - P_{11}, \\ A_{11}X &= C_{11},\ XB_{11} = C_{22}, P_{11} = P_{11}^*, \end{aligned} \tag{6}$$

in [54]. Yao computed the extremal ranks and inertias of

$$\begin{aligned} \varphi &= Q_{11} - XP_{11}X^*, \\ A_{11}X &= C_{11},\ XB_{11} = C_{22}, Q_{11} = Q_{11}^*, \end{aligned} \tag{7}$$

in [59]. The extremal ranks of

$$f(X, Y) = Q - XPY - Y^*WX^* \tag{8}$$

were inspected by Xiong and Qin [57] in 2014. Tian [31] derived the extremal ranks and inertias of

$$g(X, Y) = Q - XPY - (XPY)^*. \tag{9}$$

Some latest related research on matrix equations including coupled Sylvester matrix equation can be consulted in [8, 14, 15, 55].

Note that (1) and (9) are particular cases of

$$\mu(X, Y, Z) = A - BX - (BX)^* - YCZ - (YCZ)^*. \tag{10}$$

Motivated by the work stated above, the extremal ranks and inertias of (10) will be set up in this article. As an application, the extremal ranks and inertias of (10) are also considered when X, Y and Z are solution of the following consistent matrix equations

$$DX = E,\ \ FY = G,$$

and

$$ZH = J.$$

Since the rank and inertia of a matrix are always a nonnegative integer and hence they are discontinuous in nature. So, we can not use the differential and Lagrangian method to optimize them as in the case of continuous optimization. Hence, we will apply some linearization method introduce in Section 3 to compute the extremal ranks and inertias of the above mentioned problems.

Our article is composed as follows. Some Lemmata are given in Section 2. Key results and some consequences are established in Section 3. Constraint extremal ranks and inertias of (10) are constituted in Section 4. Conclusion is presented in Section 5.

2. Preliminaries

We start with some known results which have crucial importance in the construction of the main results of this article.

Lemma 2.1. *[23]. Let $D_{11} \in \mathbb{C}^{s\times t}$, $E_{11} \in \mathbb{C}^{s\times k}$ and $F_{11} \in \mathbb{C}^{l\times t}$ be known. Then*

(1) $r(D_{11}) + r(R_{D_{11}}E_{11}) = r(E_{11}) + r(R_{E_{11}}D_{11}) = r\begin{bmatrix} D_{11} & E_{11} \end{bmatrix}$.

(2) $r(D_{11}) + r(FL_{D_{11}}) = r(F_{11}) + r(D_{11}L_{F_{11}}) = r\begin{bmatrix} D_{11} \\ F_{11} \end{bmatrix}$.

(3) $r(E_{11}) + r(F_{11}) + r(R_{E_{11}}D_{11}L_{F_{11}}) = r\begin{bmatrix} D_{11} & E_{11} \\ F_{11} & 0 \end{bmatrix}$.

Lemma 2.2. *[2]. Let A_1 and C_1 be given matrices with allowable dimensions. Then $A_1X = C_1$ is consistent if and only if*

$$C_1 = A_1A_1^{\dagger}C_1.$$

In this condition, its general solution is

$$X = A_1^{\dagger}C_1 + L_{A_1}U,$$

where U is a free matrix with suitable dimension.

Lemma 2.3. *[2]. Let B_1 and D_1 be known matrices with suitable dimensions. Then $YB_1 = D_1$ is solvable if and only if*

$$D_1 = D_1B_1^{\dagger}B_1.$$

Under this condition, its general solution is

$$Y = D_1B_1^{\dagger} + WR_{B_1},$$

where W is free matrix with feasible dimension.

Lemma 2.4. *[6]. Let $A_{11} \in \mathbb{C}_h^{m\times m}$ and $B_{11} \in \mathbb{C}^{m\times p}$ be known. Then*

$$\max_{X\in C^{p\times m}} r[A_{11} - B_{11}X - (B_{11}X)^*] = \min\left\{m, r\begin{bmatrix} A_{11} & B_{11} \\ B_{11}^* & 0 \end{bmatrix}\right\}, \tag{11}$$

$$\min_{X\in C^{p\times m}} r[A_{11} - B_{11}X - (BX_{11})^*] = r\begin{bmatrix} A_{11} & B_{11} \\ B_{11}^* & 0 \end{bmatrix} - 2r(B_{11}), \tag{12}$$

$$\max_{X\in C^{p\times m}} i_{\pm}[A_{11} - B_{11}X - (B_{11}X)^*] = i_{\pm}\begin{bmatrix} A_{11} & B_{11} \\ B_{11}^* & 0 \end{bmatrix}, \tag{13}$$

$$\min_{X\in C^{p\times m}} i_{\pm}[A_{11} - B_{11}X - (B_{11}X)^*] = r\begin{bmatrix} A_{11} & B_{11} \\ B_{11}^* & 0 \end{bmatrix} - r(B_{11}). \tag{14}$$

Lemma 2.5. *[22]. $A_{11} \in \mathbb{C}_h^{m\times m}$, $B_{11} \in \mathbb{C}^{m\times p}$, and $C_{11} \in \mathbb{C}^{q\times m}$ be given, and assign*

$$K_1 = \begin{bmatrix} A_{11} & B_{11} \\ B_{11}^* & 0 \end{bmatrix}, K_2 = \begin{bmatrix} A_{11} & C_{11}^* \\ C_{11} & 0 \end{bmatrix},$$
$$K_3 = \begin{bmatrix} A_{11} & B_{11} & C_{11}^* \\ B_{11}^* & 0 & 0 \end{bmatrix}, K_4 = \begin{bmatrix} A_{11} & B_{11} & C_{11}^* \\ C_{11} & 0 & 0 \end{bmatrix}.$$

Then

$$\max_{Y\in C^{p\times q}} r_{\pm}[A_{11} - B_{11}YC_{11} - (B_{11}YC_{11})^*] = \min\{r\begin{bmatrix} A_{11} & B_{11} & C_{11}^* \end{bmatrix}, r(K_1), r(K_2)\}, \tag{15}$$

$$\min_{Y\in C^{p\times q}} r_{\pm}[A_{11} - B_{11}YC_{11} - (B_{11}YC_{11})^*] = 2r\begin{bmatrix} A_{11} & B_{11} & C_{11}^* \end{bmatrix} + \max\{w_+ + w_-, g_+ + g_-, w_+ + g_-, w_- + g_+\}, \tag{16}$$

$$\max_{Y\in C^{p\times q}} i_{\pm}[A_{11} - B_{11}YC_{11} - (B_{11}YC_{11})^*] = \min\{i_{\pm}(K_1), i_{\pm}(K_2)\}, \tag{17}$$

$$\min_{Y\in C^{p\times q}} i_{\pm}[A_{11} - B_{11}YC_{11} - (B_{11}YC_{11})^*] = r\begin{bmatrix} A_{11} & B_{11} & C_{11}^* \end{bmatrix} + \max\{i_{\pm}(K_1) - r(K_3), i_{\pm}(K_2) - r(K_4)\}, \tag{18}$$

where $w_{\pm} = i_{\pm}(K_1) - r(K_3), g_{\pm} = i_{\pm}(K_2) - r(K_4)\}$.

3. Main Result

The fundamental theorem of this paper is established in this section.

Theorem 3.1. *Let* $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times p}$ *and* $C \in \mathbb{C}^{s\times t}$ *be given matrices over* $\mathbb{C}$ *of executable shapes. Then*

$$\max_{X,Y} r[A - BX - (BX)^* - YCZ - (YCZ)^*] = \min\left\{n, n + r(B), 2r(C) + r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}\right\}, \tag{19}$$

$$\min_{X,Y} r[A - BX - (BX)^* - YCZ - (YCZ)^*] = \max\left\{r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - 2r(C), 0, i_+\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - r(C), i_-\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - r(C)\right\}, \tag{20}$$

$$\max_{X,Y} i_{\pm}[A - BX - (BX)^* - YCZ - (YCZ)^*] = \min\left\{i_{\pm}\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} + r(C), n\right\}, \tag{21}$$

$$\min_{X,Y} i_{\pm}[A - BX - (BX)^* - YCZ - (YCZ)^*] = \max\left\{i_{\pm}\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - r(C), 0\right\}. \tag{22}$$

Proof. *Using formula (11) in the Lemma 2.4 to the equation (10), we have*

$$\max_{X} r[\mu(X,Y)] = \min\left\{n, r\begin{bmatrix} A - YCZ - (YCZ)^* & B \\ B^* & 0 \end{bmatrix}\right\}. \tag{23}$$

Since

$$r\begin{bmatrix} A - YCZ - (YCZ)^* & B \\ B^* & 0 \end{bmatrix} = r\left[A_{11} - B_{11}Y_{11}C_{11} - (B_{11}Y_{11}C_{11})^*\right] - 2r(C), \tag{24}$$

where

$$A_{11} = \begin{bmatrix} A & B & 0 & 0 \\ B^* & 0 & 0 & 0 \\ 0 & 0 & 0 & C \\ 0 & 0 & C^* & 0 \end{bmatrix}, \; B_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & C \\ C^* & 0 \end{bmatrix},$$

$$Y_{11} = \begin{bmatrix} Y^* \\ Z \end{bmatrix}, \quad C_{11} = \begin{bmatrix} -I_n & 0 & 0 & 0 \end{bmatrix}.$$

Now using formula (15) in the Lemma 2.5 to (24), we have

$$\max_{Y} r\begin{bmatrix} A-YCZ-(YCZ)^{*} & B \\ B^{*} & 0 \end{bmatrix} = \min\left\{r\begin{bmatrix} A_{11} & B_{11} & C_{11}^{*} \end{bmatrix}, r(P_{11}), r(P_{22})\right\} - 2r(C), \tag{25}$$

where

$$r\begin{bmatrix} A_{11} & B_{11} & C_{11}^{*} \end{bmatrix} = r\begin{bmatrix} A & B & 0 & 0 & 0 & 0 & -I_n \\ B^{*} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & 0 & C & 0 \\ 0 & 0 & C^{*} & 0 & C^{*} & 0 & 0 \end{bmatrix}$$
$$= n + r(B) + 2r(C), \tag{26}$$

$$r(P_{11}) = r\begin{bmatrix} A & B & 0 & 0 & 0 & 0 \\ B^{*} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & 0 & C \\ 0 & 0 & C^{*} & 0 & C^{*} & 0 \\ 0 & 0 & 0 & C & 0 & 0 \\ 0 & 0 & C^{*} & 0 & 0 & 0 \end{bmatrix}$$
$$= r\begin{bmatrix} A & B \\ B^{*} & 0 \end{bmatrix} + 4r(C), \tag{27}$$

$$r(P_{22}) = r\begin{bmatrix} A & B & 0 & 0 & -I_n \\ B^{*} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & 0 \\ 0 & 0 & C^{*} & 0 & 0 \\ -I_n & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= 2n + 2r(C). \tag{28}$$

By putting (26)-(28) in (25) and then (25) in (23) produces (19).
By using formula (12) in the Lemma 2.4 to (10), we have

$$\min_{X} r[\mu(X,Y)] = r\begin{bmatrix} A-YCZ-(YCZ)^{*} & B \\ B^{*} & 0 \end{bmatrix} - 2r(B). \tag{29}$$

By using formula (16) in the Lemma 2.4 to the term of (29), we have

$$\min_{Y} r\begin{bmatrix} A-YCZ-(YCZ)^{*} & B \\ B^{*} & 0 \end{bmatrix}$$
$$= 2r\begin{bmatrix} A_{11} & B_{11} & C_{11}^{*} \end{bmatrix} + \max\left\{w_{+}+w_{-}, g_{+}+g_{-}, w_{+}+g_{-}, w_{-}+g_{+}\right\} - 2r(C)$$
$$= 2n + 2r(B) + 2r(C) + \max\left\{w_{+}+w_{-}, g_{+}+g_{-}, w_{+}+g_{-}, w_{-}+g_{+}\right\}. \tag{30}$$

where

$$\begin{aligned}
w_{\pm} &= i_{\pm}(P_{11}) - r(P_{33}) \\
&= i_{\pm}\begin{bmatrix} A & B & 0 & 0 & 0 & 0 \\ B^* & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & 0 & C \\ 0 & 0 & C^* & 0 & C^* & 0 \\ 0 & 0 & 0 & C & 0 & 0 \\ 0 & 0 & C^* & 0 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B & 0 & 0 & 0 & 0 & -I_n \\ B^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & 0 & C & 0 \\ 0 & 0 & C^* & 0 & C^* & 0 & 0 \\ 0 & 0 & 0 & C & 0 & 0 & 0 \\ 0 & 0 & C^* & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= i_{\pm}\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - n - 2r(C), \qquad (31) \\
g_{\pm} &= i_{\pm}(P_{22}) - r(P_{44}) \\
&= i_{\pm}\begin{bmatrix} A & B & 0 & 0 & -I_n \\ B^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & 0 \\ 0 & 0 & C^* & 0 & 0 \\ -I_n & 0 & 0 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B & 0 & 0 & 0 & 0 & -I_n \\ B^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & 0 & C & 0 \\ 0 & 0 & C^* & 0 & C^* & 0 & 0 \\ -I_n & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= -n - r(C). \qquad (32)
\end{aligned}$$

With the assistance of (29)-(32), we gain (20).
By using formula (13) in the Lemma 2.4 to (10), we have

$$\max_X i_{\pm}[\mu(X,Y)] = i_{\pm}\begin{bmatrix} A - YCZ - (YCZ)^* & B \\ B^* & 0 \end{bmatrix}. \qquad (33)$$

By using formula (17) in the Lemma 2.5 to the right hand side of (33) and simplifying the block matrices in it by the same fashion as we did earlier, we obtain

$$\begin{aligned}
\max_Y i_{\pm}\begin{bmatrix} A - YCZ - (YCZ)^* & B \\ B^* & 0 \end{bmatrix} &= \min_{Y_{11}}\left\{A_{11} - B_{11}Y_{11}C_{11} - (B_{11}Y_{11}C_{11})^*\right\} - r(C) \\
&= \min\left\{i_{\pm}\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} + r(C), n\right\}. \qquad (34)
\end{aligned}$$

From (33) and (34), we receive (21).
By utilizing formula (14) in the Lemma 2.4, we have

$$\min_X i_{\pm}[\mu(X,Y)] = i_{\pm}\begin{bmatrix} A - YCZ - (YCZ)^* & B \\ B^* & 0 \end{bmatrix} - r(B). \qquad (35)$$

By applying formula (18) in the Lemma 2.5 to a term on the right hand side of (35), we have

$$\begin{aligned}
\min_Y i_{\pm}\begin{bmatrix} A - YCZ - (YCZ)^* & B \\ B^* & 0 \end{bmatrix} &= \min_{Y_{11}} i_{\pm}[A_{11} - B_{11}Y_{11}C_{11} - (B_{11}Y_{11}C_{11})^*] - r(C) \\
&= r\begin{bmatrix} A_{11} & B_{11} & C_{11}^* \end{bmatrix} + \max\{w_{\pm}, g_{\pm}\} - r(C) \\
&= n + r(B) + 2r(C) + \max\left\{i_{\pm}\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - n - 2r(C), -n - r(C)\right\} - r(C) \\
&= \max\left\{i_{\pm}\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - r(C) + r(B), r(B)\right\}. \qquad (36)
\end{aligned}$$

By (35) and (36), we gain (22). Hence the theorem is accomplished. □

If B is zero in (10), then I get the following outcome.

Corollary 3.2. *Let A and C be given matrices over $\mathbb{C}$ of conformable shapes. Then*

$$\max_{X,Y} r[A - YCZ - (YCZ)^*] = \min\left\{n, 2r(C) + r(A)\right\},$$
$$\min_{X,Y} r[A - YCZ - (YCZ)^*] = \max\left\{r(A) - 2r(C), 0, i_+(A) - r(C), i_-(A) - r(C)\right\},$$
$$\max_{X,Y} i_\pm[A - YCZ - (YCZ)^*] = \min\left\{i_\pm(A) + r(C), n\right\},$$
$$\min_{X,Y} i_\pm[A - YCZ - (YCZ)^*] = \max\left\{i_\pm(A) - r(C), 0\right\}.$$

Comment 3.3. *Corollary 3.2 is the Theorem of [31].*

4. Constraint Extremal Ranks and Inertias of (10)

As an application of the Theorem 3.1, the extremal ranks and inertias of

$$\begin{aligned} &DX = E,\ \ FY = G,\ \ ZH = J, \\ &\mu(X, Y, Z) = A - BX - (BX)^* - YCZ - (YCZ)^* \end{aligned} \tag{37}$$

are investigated in this section.
By Lemma 2.2 and Lemma 2.3, the general solution to $DX = E,\ FY = G$ and $ZH = J$ are given by

$$X = D^\dagger E + L_D X_1 = X_0 + L_D X_1, \tag{38}$$
$$Y = F^\dagger G + L_F Y_1 = Y_0 + L_F Y_1, \tag{39}$$

and

$$Z = JH^\dagger + Z_1 R_H = Z_0 + Z_1 R_H. \tag{40}$$

Using (38)-(40) in (10), we have

$$\begin{aligned} \mu(X, Y, Z) =& A - BX_0 - (BX_0)^* - BL_D X_1 - (BL_D X_1)^* \\ & -(Y_0 + L_F Y_1)C(Z_0 + Z_1 R_H) - (Z_0^* + R_H Z_1^*)C^*(Y_0^* + Y_1^* L_F). \end{aligned} \tag{41}$$

Theorem 4.1. *Let A, B, C, D, E, F, G, H and J are known matrices of adjustable shapes over $\mathbb{C}$. Assign*

$$N_1 = \begin{bmatrix} B^* & 0 & D^* & 0 & 0 & 0 \\ E & D & 0 & 0 & 0 & 0 \\ H^*A & H^*B & H^*E^* & J^*C^* & 0 & -H^* \\ 0 & 0 & 0 & 0 & GC & F \end{bmatrix}, \ \zeta_1 = \begin{bmatrix} A & B & E^* \\ B^* & 0 & D^* \\ E & D & 0 \end{bmatrix},$$

$$\zeta_2 = \begin{bmatrix} 0 & D^* & 0 & B^*H & 0 \\ D & 0 & 0 & EH & 0 \\ 0 & 0 & 0 & -H & F^* \\ H^*B & H^*E^* & -H^* & -H^*AH & -J^*C^*G^* \\ 0 & 0 & F & -GCJ & 0 \end{bmatrix}, \ \zeta_3 = \begin{bmatrix} B^* & 0 & D^* & 0 \\ E & D & 0 & 0 \\ H^*A & H^*B & H^*E^* & -H^* \\ 0 & 0 & 0 & F \end{bmatrix},$$

$$\zeta_4 = \begin{bmatrix} 0 & D^* & 0 & 0 & 0 & B^*H & 0 \\ D & 0 & 0 & 0 & 0 & EH & 0 \\ 0 & 0 & 0 & 0 & 0 & -H & F^* \\ H^*B & H^*E^* & J^*C^* & 0 & -H^* & -H^*AH & 0 \\ 0 & 0 & 0 & GC & F & 0 & 0 \end{bmatrix},$$

$$w_\pm = i_\pm(\zeta_1) - r(\zeta_3) - n - 2r(C) + r(F) + r(H),$$
$$g_\pm = i_\pm(\zeta_2) - r(\zeta_4) - n - r(C) + r(F) + r(H).$$

Then

(1) *The maximal rank of (37) is*

$$\max_{\substack{DX=E\\FY=G,ZH=J}} r[\mu(X,Y,Z)]$$
$$= \min\left\{n, r(N_1) - r(H) - r(F) - 2r(D), r(\zeta_1) + 2r(C) - 2r(D), r(\zeta_2) + 2n - 2r(F) - 2r(H) - 2r(D)\right\}. \quad (42)$$

(2) *The minimal rank of (37) is*

$$\min_{\substack{DX=E\\FY=G,ZH=J}} r[\mu(X,Y,Z)] =$$
$$2r(N_1) + 2r(C) - 2r(H) - 2r(F) + \max\left\{w_+ + w_-, g_+ + g_-, w_+ + g_-, w_- + g_+\right\} - 2r\begin{bmatrix} B \\ D \end{bmatrix}. \quad (43)$$

(3) *The maximal positive and negative signature of (37) is*

$$\max_{\substack{DX=E\\FY=G,ZH=J}} i_\pm[\mu(X,Y,Z)] = \min\left\{i_\pm(\zeta_1) + r(C) - r(D), i_\pm(\zeta_2) + n - r(H) - r(F) - r(D)\right\}. \quad (44)$$

(4) *The minimal positive and negative signature of (37) is*

$$\min_{\substack{DX=E\\FY=G,ZH=J}} i_\pm[\mu(X,Y,Z)] = r(N_1) + r(C) - r(H) - r(F) + \max\{w_\pm, g_\pm\} - r\begin{bmatrix} B \\ D \end{bmatrix}. \quad (45)$$

Proof. *Applying formula (11) of the Lemma 2.4 to (41), we have*

$$\begin{aligned}
\max_{X_1} r[\mu(X,Y,Z)] &= \min\left\{n, r\begin{bmatrix} \begin{smallmatrix} A-BX_0-(BX_0)^*-(Y_0+L_FY_1)C(Z_0+Z_1R_H) \\ -(Z_0^*+R_HZ_1^*)C^*(Y_0^*+Y_1^*L_F) \end{smallmatrix} & BL_D \\ L_DB^* & 0 \end{bmatrix}\right\} \\
&= \min\left\{n, r\begin{bmatrix} \begin{smallmatrix} A-BX_0-(BX_0)^*-(Y_0+L_FY_1)C(Z_0+Z_1R_H) \\ -(Z_0^*+R_HZ_1^*)C^*(Y_0^*+Y_1^*L_F) \end{smallmatrix} & B & 0 \\ B^* & 0 & D^* \\ 0 & D & 0 \end{bmatrix} - 2r(D)\right\} \\
&= \min\left\{n, r\begin{bmatrix} \begin{smallmatrix} A-BX_0-(BX_0)^*-(Y_0+L_FY_1)C(Z_0+Z_1R_H) \\ -(Z_0^*+R_HZ_1^*)C^*(Y_0^*+Y_1^*L_F) \end{smallmatrix} & B & E^* \\ B^* & 0 & D^* \\ E & D & 0 \end{bmatrix} - 2r(D)\right\}. \quad (46)
\end{aligned}$$

Now we consider the block matrix in (46) as follows:

$$\begin{aligned}
& r\begin{bmatrix} \begin{smallmatrix} A-BX_0-(BX_0)^*-(Y_0+L_FY_1)C(Z_0+Z_1R_H) \\ -(Z_0^*+R_HZ_1^*)C^*(Y_0^*+Y_1^*L_F) \end{smallmatrix} & B & E^* \\ B^* & 0 & D^* \\ E & D & 0 \end{bmatrix} \\
&= r\begin{bmatrix} A & B & E^* & (Z_0^* + R_HZ_1^*)C^* & (Y_0 + L_FY_1)C \\ B^* & 0 & D^* & 0 & 0 \\ E & D & 0 & 0 & 0 \\ C(Z_0 + Z_1R_H) & 0 & 0 & 0 & C \\ C^*(Y_0^* + Y_1^*L_F) & 0 & 0 & C^* & 0 \end{bmatrix} - 2r(C) \\
&= r(A_{11} - B_{11}X_{11}C_{11} - (B_{11}X_{11}C_{11})^*) - 2r(C), \quad (47)
\end{aligned}$$

where

$$A_{11}=\begin{bmatrix} A & B & E^* & Z_0^*C^* & Y_0C \\ B^* & 0 & D^* & 0 & 0 \\ E & D & 0 & 0 & 0 \\ CZ_0 & 0 & 0 & 0 & C \\ C^*Y_0^* & 0 & 0 & C^* & 0 \end{bmatrix},\ B_{11}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & C \\ C^* & 0 \end{bmatrix},$$
$$C_{11}=\begin{bmatrix} -R_H & 0 & 0 & 0 & 0 \\ -L_F & 0 & 0 & 0 & 0 \end{bmatrix},\ X_{11}=\begin{bmatrix} 0 & Y_1^* \\ Z_1 & 0 \end{bmatrix}. \tag{48}$$

Now utilizing formula (15) of the Lemma 2.5 to the matrix pencil in (47), we have

$$\max_{X_{11}} r(A_{11}-B_{11}X_{11}C_{11}-(B_{11}X_{11}C_{11})^*)$$
$$=\min\left\{r\begin{bmatrix} A_{11} & B_{11} & C_{11}^* \end{bmatrix}, r\begin{bmatrix} A_{11} & B_{11} \\ B_{11}^* & 0 \end{bmatrix}, r\begin{bmatrix} A_{11} & C_{11}^* \\ C_{11} & 0 \end{bmatrix}\right\}. \tag{49}$$

Now we calculate the block matrices in (49) by using the Lemma 2.1 as follows:

$$r\begin{bmatrix} A_{11} & B_{11} & C_{11}^* \end{bmatrix}=r\begin{bmatrix} A & B & E^* & Z_0^*C^* & Y_0C & 0 & 0 & -R_H & -L_F \\ B^* & 0 & D^* & 0 & 0 & 0 & 0 & 0 & 0 \\ E & D & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ CZ_0 & 0 & 0 & 0 & C & 0 & C & 0 & 0 \\ C^*Y_0^* & 0 & 0 & C^* & 0 & C^* & 0 & 0 & 0 \end{bmatrix}$$
$$=r\begin{bmatrix} A & B & E^* & Z_0^*C^* & Y_0C & 0 & 0 & -I_n & -I_n \\ B^* & 0 & D^* & 0 & 0 & 0 & 0 & 0 & 0 \\ E & D & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ CZ_0 & 0 & 0 & 0 & C & 0 & C & 0 & 0 \\ C^*Y_0^* & 0 & 0 & C^* & 0 & C^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & H^* & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F \end{bmatrix}-r(F)-r(H)$$
$$=r(N_1)+2r(C)-r(H)-r(F). \tag{50}$$

Similarly

$$r\begin{bmatrix} A_{11} & B_{11} \\ B_{11}^* & 0 \end{bmatrix}=r\begin{bmatrix} A & B & E^* & Z_0^*C^* & Y_0C & 0 & 0 \\ B^* & 0 & D^* & 0 & 0 & 0 & 0 \\ E & D & 0 & 0 & 0 & 0 & 0 \\ CZ_0 & 0 & 0 & 0 & C & 0 & C \\ C^*Y_0^* & 0 & 0 & C^* & 0 & C^* & 0 \\ 0 & 0 & 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & C^* & 0 & 0 & 0 \end{bmatrix}$$
$$=r(\zeta_1)+4r(C), \tag{51}$$
$$r\begin{bmatrix} A_{11} & C_{11}^* \\ C_{11} & 0 \end{bmatrix}=r\begin{bmatrix} A & B & E^* & Z_0^*C^* & Y_0C & -R_H & -L_F \\ B^* & 0 & D^* & 0 & 0 & 0 & 0 \\ E & D & 0 & 0 & 0 & 0 & 0 \\ CZ_0 & 0 & 0 & 0 & C & 0 & 0 \\ C^*Y_0^* & 0 & 0 & C^* & 0 & 0 & 0 \\ -R_H & 0 & 0 & 0 & 0 & 0 & 0 \\ -L_F & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$=r(\zeta_2)+2r(C)+2n-2r(H)-2r(F). \tag{52}$$

Combining (46)-(52), we gain (42).
Now by virtue of the formula (12) of the Lemma (2.4) to (41), we have

$$\min_{DX=E} r[\mu(X,Y,Z)] = r\begin{bmatrix} \substack{A-BX_0-(BX_0)^*-(Y_0+L_FY_1)C(Z_0+Z_1R_H)\\ -(Z_0^*+R_HZ_1^*)C^*(Y_0^*+Y_1^*L_F)} & BL_D \\ L_DB^* & 0 \end{bmatrix} - 2r(BL_D)$$

$$= r\begin{bmatrix} \substack{A-BX_0-(BX_0)^*-(Y_0+L_FY_1)C(Z_0+Z_1R_H)\\ -(Z_0^*+R_HZ_1^*)C^*(Y_0^*+Y_1^*L_F)} & B & 0 \\ B^* & 0 & D^* \\ 0 & D & 0 \end{bmatrix} - 2r\begin{bmatrix} B \\ D \end{bmatrix}. \tag{53}$$

Since

$$r\begin{bmatrix} \substack{A-BX_0-(BX_0)^*-(Y_0+L_FY_1)C(Z_0+Z_1R_H)\\ -(Z_0^*+R_HZ_1^*)C^*(Y_0^*+Y_1^*L_F)} & B & E^* \\ B^* & 0 & D^* \\ E & D & 0 \end{bmatrix}$$
$$= r(A_{11} - B_{11}X_{11}C_{11} - (B_{11}X_{11}C_{11})^*) - 2r(C), \tag{54}$$

where $A_{11},\ B_{11},\ C_{11}$ and X_{11} are the same as declared in (48).
By utilizing the formula (16) to the matrix expression of (54), we have

$$\min_{X_{11}} r(A_{11} - B_{11}X_{11}C_{11} - (B_{11}X_{11}C_{11})^*) = 2r\begin{bmatrix} A_{11} & B_{11} & C_{11}^* \end{bmatrix} + \max\Big\{k_+ + k_-, l_+ + l_-, k_+ + l_-, k_- + l_+\Big\}. \tag{55}$$

Now we compute the block matrices in (55) as follows:

$$i_\pm\begin{bmatrix} A_{11} & B_{11} \\ B_{11}^* & 0 \end{bmatrix} = i_\pm\begin{bmatrix} A & B & E^* & Z_0^*C^* & Y_0C & 0 & 0 \\ B^* & 0 & D^* & 0 & 0 & 0 & 0 \\ E & D & 0 & 0 & 0 & 0 & 0 \\ CZ_0 & 0 & 0 & 0 & C & 0 & C \\ C^*Y_0^* & 0 & 0 & C^* & 0 & C^* & 0 \\ 0 & 0 & 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & C^* & 0 & 0 & 0 \end{bmatrix}$$
$$= i_\pm(\zeta_1) + 2r(C), \tag{56}$$

$$i_\pm\begin{bmatrix} A_{11} & C_{11}^* \\ C_{11} & 0 \end{bmatrix} = i_\pm\begin{bmatrix} A & B & E^* & Z_0^*C^* & Y_0C & -R_H & -L_F \\ B^* & 0 & D^* & 0 & 0 & 0 & 0 \\ E & D & 0 & 0 & 0 & 0 & 0 \\ CZ_0 & 0 & 0 & 0 & C & 0 & 0 \\ C^*Y_0^* & 0 & 0 & C^* & 0 & 0 & 0 \\ -R_H & 0 & 0 & 0 & 0 & 0 & 0 \\ -L_F & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= i_\pm\begin{bmatrix} A & B & E^* & Z_0^*C^* & Y_0C & -I_n & -I_n & 0 & 0 \\ B^* & 0 & D^* & 0 & 0 & 0 & 0 & 0 & 0 \\ E & D & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ CZ_0 & 0 & 0 & 0 & C & 0 & 0 & 0 & 0 \\ C^*Y_0^* & 0 & 0 & C^* & 0 & 0 & 0 & 0 & 0 \\ -I_n & 0 & 0 & 0 & 0 & 0 & 0 & H & 0 \\ -I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F^* \\ 0 & 0 & 0 & 0 & 0 & H^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & F & 0 & 0 \end{bmatrix}$$
$$- r(H) - r(F)$$
$$= i_\pm(\zeta_2) + r(C) + n - r(H) - r(F). \tag{57}$$

Similarly,

$$r\begin{bmatrix} A_{11} & B_{11} & C_{11}^{*} \\ B_{11}^{*} & 0 & 0 \end{bmatrix} = r\begin{bmatrix} A & B & E^{*} & Z_0^{*}C^{*} & Y_0C & 0 & 0 & -R_H & -L_F \\ B^{*} & 0 & D^{*} & 0 & 0 & 0 & 0 & 0 & 0 \\ E & D & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ CZ_0 & 0 & 0 & 0 & C & 0 & C & 0 & 0 \\ C^{*}Y_0^{*} & 0 & 0 & C^{*} & 0 & C^{*} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C^{*} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= r(\zeta_3) + n + 4r(C) - r(F) - r(H), \tag{58}$$

$$r\begin{bmatrix} A_{11} & B_{11} & C_{11}^{*} \\ C_{11} & 0 & 0 \end{bmatrix} = r\begin{bmatrix} A & B & E^{*} & Z_0^{*}C^{*} & Y_0C & 0 & 0 & -R_H & -L_F \\ B^{*} & 0 & D^{*} & 0 & 0 & 0 & 0 & 0 & 0 \\ E & D & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ CZ_0 & 0 & 0 & 0 & C & 0 & C & 0 & 0 \\ C^{*}Y_0^{*} & 0 & 0 & C^{*} & 0 & C^{*} & 0 & 0 & 0 \\ -R_H & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -L_F & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= r(\zeta_4) + 2n + 2r(C) - 2r(F) - 2r(H), \tag{59}$$

From (56)-(59), we get

$$k_{\pm} = i_{\pm}(\zeta_1) - r(\zeta_3) - n - 2r(C) + r(F) + r(H), \tag{60}$$
$$l_{\pm} = i_{\pm}(\zeta_2) - r(\zeta_4) - n - r(C) + r(H) + r(F). \tag{61}$$

From(50), (53)-(55) and (60)-(61), we get (43).
Now by aid of formula (13) of the Lemma 2.4 to (41), we have

$$\max_{DX=E} i_{\pm}[\mu(X,Y,Z)] = i_{\pm}\begin{bmatrix} \begin{matrix} A-BX_0-(BX_0)^{*}-(Y_0+L_FY_1)C(Z_0+Z_1R_H) \\ -(Z_0^{*}+R_HZ_1^{*})C^{*}(Y_0^{*}+Y_1^{*}L_F) \end{matrix} & BL_D \\ L_DB^{*} & 0 \end{bmatrix}$$
$$= i_{\pm}\begin{bmatrix} \begin{matrix} A-BX_0-(BX_0)^{*}-(Y_0+L_FY_1)C(Z_0+Z_1R_H) \\ -(Z_0^{*}+R_HZ_1^{*})C^{*}(Y_0^{*}+Y_1^{*}L_F) \end{matrix} & B & 0 \\ B^{*} & 0 & D^{*} \\ 0 & D & 0 \end{bmatrix} - r(D), \tag{62}$$

Since

$$i_{\pm}\begin{bmatrix} \begin{matrix} A-BX_0-(BX_0)^{*}-(Y_0+L_FY_1)C(Z_0+Z_1R_H) \\ -(Z_0^{*}+R_HZ_1^{*})C^{*}(Y_0^{*}+Y_1^{*}L_F) \end{matrix} & B & E^{*} \\ B^{*} & 0 & D^{*} \\ E & D & 0 \end{bmatrix}$$
$$= i_{\pm}(A_{11} - B_{11}X_{11}C_{11} - (B_{11}X_{11}C_{11})^{*}) - r(C), \tag{63}$$

where A_{11}, B_{11}, C_{11} and X_{11} are the same as declared in (48).
By applying formula (17) of the Lemma 2.5 to the matrix pencil in (63), we have

$$\max_{X_{11}} i_{\pm}(A_{11} - B_{11}X_{11}C_{11} - (B_{11}X_{11}C_{11})^{*}) = \left\{ i_{\pm}\begin{bmatrix} A_{11} & B_{11} \\ B_{11}^{*} & 0 \end{bmatrix}, i_{\pm}\begin{bmatrix} A_{11} & C_{11}^{*} \\ C_{11} & 0 \end{bmatrix} \right\}. \tag{64}$$

From (56)-(57) and (62)-(64), we get (44).

By using formula (14) of the Lemma 2.5 to the matrix pencil in (41), we have

$$\min_{DX=E} i_{\pm}[\mu(X,Y,Z)] = i_{\pm}\begin{bmatrix} \begin{matrix} A-BX_0-(BX_0)^*-(Y_0+L_FY_1)C(Z_0+Z_1R_H) \\ -(Z_0^*+R_HZ_1^*)C^*(Y_0^*+Y_1^*L_F) \end{matrix} & BL_D \\ L_DB^* & 0 \end{bmatrix} - r(BL_D)$$
$$= i_{\pm}\begin{bmatrix} \begin{matrix} A-BX_0-(BX_0)^*-(Y_0+L_FY_1)C(Z_0+Z_1R_H) \\ -(Z_0^*+R_HZ_1^*)C^*(Y_0^*+Y_1^*L_F) \end{matrix} & B & 0 \\ B^* & 0 & D^* \\ 0 & D & 0 \end{bmatrix} - r\begin{bmatrix} B \\ D \end{bmatrix}. \quad (65)$$

Sine

$$i_{\pm}\begin{bmatrix} \begin{matrix} A-BX_0-(BX_0)^*-(Y_0+L_FY_1)C(Z_0+Z_1R_H) \\ -(Z_0^*+R_HZ_1^*)C^*(Y_0^*+Y_1^*L_F) \end{matrix} & B & E^* \\ B^* & 0 & D^* \\ E & D & 0 \end{bmatrix}$$
$$= i_{\pm}(A_{11} - B_{11}X_{11}C_{11} - (B_{11}X_{11}C_{11})^*) - r(C). \quad (66)$$

where A_{11}, B_{11}, C_{11} and X_{11} are the same as declared in (48).
By utilizing formula (18) of the Lemma 2.5 to the matrix pencil in (65), we have

$$\min_{X_{11}} i_{\pm}(A_{11} - B_{11}X_{11}C_{11} - (B_{11}X_{11}C_{11})^*) = r\begin{bmatrix} A_{11} & B_{11} & C_{11}^* \end{bmatrix} + \max\{k_{\pm}, l_{\pm}\}. \quad (67)$$

By combining (50), (60)-(61) and (65)-(67), we receive (45). Hence the theorem is finished. □

A following corollary can be derived with the assistance of the Theorem 4.1.
Let

$$Q_1 = \{X \mid DX = E\},$$
$$Q_2 = \{Y \mid FY = G\},$$
$$Q_3 = \{Z \mid ZH = J\}.$$

Corollary 4.2. *Let A, B, C, D, E, F, G, H and J are known matrices of adjustable shapes over $\mathbb{C}$ as mentioned in Theorem 4.1. Assume that Q_1, Q_2 and Q_3 are non empty sets. Then*

(1) There exist $X \in Q_1$, $Y \in Q_2$ and $Z \in Q_3$ such that $A - BX - (BX)^* - YCZ - (YCZ)^* > 0$ if and only if

$$i_+(\zeta_1) + r(C) - r(D) \geq n,$$

$$i_+(\zeta_2) - r(H) - r(F) - r(D) \geq 0.$$

(2) There exist $X \in Q_1$, $Y \in Q_2$ and $Z \in Q_3$ such that $A - BX - (BX)^* - YCZ - (YCZ)^* < 0$ if and only if

$$i_-(\zeta_1) + r(C) - r(D) \geq n,$$

$$i_-(\zeta_2) - r(H) - r(F) - r(D) \geq 0.$$

(3) There exist $X \in Q_1$, $Y \in Q_2$ and $Z \in Q_3$ such that $A - BX - (BX)^* - YCZ - (YCZ)^* \geq 0$ if and only if

$$r(N_1) + r(C) - r(H) - r(F) + w_- - r\begin{pmatrix} B \\ D \end{pmatrix} = 0,$$

$$r(N_1) + r(C) - r(H) - r(F) + g_- - r\begin{pmatrix} B \\ D \end{pmatrix} = 0.$$

(4) *There exist* $X \in Q_1$, $Y \in Q_2$ *and* $Z \in Q_3$ *such that* $A - BX - (BX)^* - YCZ - (YCZ)^* \leq 0$ *if and only if*

$$r(N_1) + r(C) - r(H) - r(F) + w_+ - r\begin{pmatrix} B \\ D \end{pmatrix} = 0,$$

$$r(N_1) + r(C) - r(H) - r(F) + g_+ - r\begin{pmatrix} B \\ D \end{pmatrix} = 0.$$

(5) $A - BX - (BX)^* - YCZ - (YCZ)^* > 0 \ \forall \ X \in Q_1$, $Y \in Q_2$ *and* $Z \in Q_3$ *if and only if*

$$r(N_1) + r(C) - r(H) - r(F) + w_+ - r\begin{pmatrix} B \\ D \end{pmatrix} = n,$$

or

$$r(N_1) + r(C) - r(H) - r(F) + g_+ - r\begin{pmatrix} B \\ D \end{pmatrix} = n.$$

(6) $A - BX - (BX)^* - YCZ - (YCZ)^* < 0 \ \forall \ X \in Q_1$, $Y \in Q_2$ *and* $Z \in Q_3$ *if and only if*

$$r(N_1) + r(C) - r(H) - r(F) + w_- - r\begin{pmatrix} B \\ D \end{pmatrix} = n,$$

or

$$r(N_1) + r(C) - r(H) - r(F) + g_- - r\begin{pmatrix} B \\ D \end{pmatrix} = n.$$

(7) $A - BX - (BX)^* - YCZ - (YCZ)^* \geq 0 \ \forall \ X \in Q_1$, $Y \in Q_2$ *and* $Z \in Q_3$ *if and only if*

$$i_-(\zeta_1) + r(C) - r(D) = 0,$$

or

$$i_-(\zeta_2) + n - r(H) - r(F) - r(D) = 0.$$

(8) $A - BX - (BX)^* - YCZ - (YCZ)^* \leq 0 \ \forall \ X \in Q_1$, $Y \in Q_2$ *and* $Z \in Q_3$ *if and only if*

$$i_+(\zeta_1) + r(C) - r(D) = 0,$$

or

$$i_+(\zeta_2) + n - r(H) - r(F) - r(D) = 0.$$

(9) *There exist* $X \in Q_1$, $Y \in Q_2$ *and* $Z \in Q_3$ *such that* $A - BX - (BX)^* - YCZ - (YCZ)^*$ *is nonsingular if and if*

$$r(N_1) - r(H) - r(F) - 2r(D) \geq n,$$

$$r(\zeta_1) + 2r(C) - 2r(D) \geq n,$$

$$r(\zeta_2) + n - 2r(F) - 2r(H) - 2r(D) \geq 0.$$

5. Conclusion

The extremal ranks and inertias of a nonlinear Hermitian matrix function involving three variables are constituted in this paper with the assistance of linearization. As an application, the constraint extremal ranks and inertias of (10) are also established in this paper. Our investigation contains the primary research of [31]. Some remarkable consequences are also derived from the main theorems of this paper.

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