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Quasi-Fredholm Linear Relations in Hilbert Spaces

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Abstract. In this paper we obtain some results concerning the ascent and descent of a quasi-Fredholm relation in a Hilbert space and we analyze the behaviour of a polynomial in a quasi-Fredholm relation in a Hilbert space.

1. Introduction

Closed quasi-Fredholm operators in Banach or Hilbert spaces have been studied by different authors, for instance, [1], [4], [9] and [11] among others.

In [9], Labrousse showed that these operators allow an algebraic decomposition, the so-called Kato decomposition. Many years later, in [10] Labrousse et al generalized the above result to the general case of range space relations in Hilbert spaces. On the other hand, Mbekhta [11] gave some results concerning the ascent and descent of a closed quasi-Fredholm operator in a Hilbert space. In [4] Berkani studied, essentially, the behaviour of a polynomial in a closed quasi-Fredholm operator in a Hilbert space. However, the generalization of the results of [4] and [11] mentioned above to the case of range space relations seems still unknown.

On the other hand, in the last years, several authors have paid attention to the research of the theory of linear relations since it has applications in many problems in Physics and other areas of Applied Mathematical. We cite some of them.

Applications of some perturbation results for linear relations to the study of degenerate elliptic-parabolic evolution equations. (See [6] and the references therein).

Applications of the fixed point theory of linear relations in: Game theory and Mathematical Economics; Discontinuous differential equations which occur in the Biological Sciences (for example, population in dynamics and epidemiology); Optimal control and Digital Imaging. (See [8] and the references therein).

Applications of the Fredholm theory of linear relations to the study of many problems of the Operator theory: Theory of pseudoresolvents and theory of linear bundles. (See [3] and the references therein).

In view of the above remarks the attempt to generalize the existing results for operators to the case of linear relations appears as natural and perhaps necessary in view of scientific progress in this field.

The purpose of the present paper is to extend the results of Berkani [4] and Mbekhta [11] for closed operators to the general case of range space relations.

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The paper is organized as follows. Some entirely algebraic properties of linear relations in linear spaces are presented in Section 2, in particular, some general facts about the ascent and descent of a linear relation as well as some results concerning the degree of a polynomial in a linear relation are established. Range space and quasi-Fredholm relations in Hilbert spaces are discussed in Section 3; in particular we recall the interesting characterization of quasi-Fredholm relations by their Kato decomposition due to Labrousse et al. [10]. As an application of this characterization we give a result which relates the quasi-Fredholmness of a range space relation to that of its powers. All results obtained in the above sections 2 and 3 are used in the following sections. Section 4 is devoted to the extension of the results of Mbekhta [11] for operators to the case of range space relations. The analysis is essentially based on the main results of [10]. The fundamental theorem of Section 5 proves that if $p(A) = \sqcap_{i=1}^{n}(A-\lambda_i I)^{m_i}$ (see Definition 2.7 below)is a polynomial in a linear relation A, then $p(A)$ is quasi-Fredholm if and only if $A-\lambda_i I$ is quasi-Fredholm. The proof of this result is based on the results obtained in the previous sections combined with the ingenious techniques due to Berkani [4].

The present paper can be seen as a natural continuation of the paper [10].

2. Algebraic Properties for Linear Relations

In this section we present some entirely algebraic notions and properties of linear relations in linear spaces which are needed in the sequel.

We adhered to the notations and terminology of the monographs [5], [10] and [13]. Let E be a linear space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A linear relation A in E is a subspace of the space $E \times E$, the Cartesian product of E and itself. The subspaces $D(A) := \{x : (x, y) \in A\}, N(A) := \{x : (x, 0) \in A\}, A(0) := \{y : (0, y) \in A\}$ and $R(A) := \{y : (x, y) \in A\}$ are called the domain, the null space, the multivalued part and the range of A, respectively.

A linear relation A is the graph an operator if and only if $A(0) = \{0\}$. The inverse A^{-1} of A is given by $A^{-1} := \{(y, x) : (x, y) \in A\}$. We say that A is injective if $N(A) = \{0\}$ and A is called surjective if $R(A) = E$. The index of A is the quantity $i(A) := dimN(A) - codimR(A)$ provided $dimN(A)$ and $codimR(A)$ are not both infinite where $codimR(A) := dimE/R(A)$.

For linear relations A and B in E and $\lambda \in \mathbb{K}$, the linear relations $A + B, A \oplus B, AB$ and λA are defined by
$A + B := \{(x, y + z) : (x, y) \in A, (x, z) \in B\}$,
$A \oplus B := \{(x + u, y + v) : (x, y) \in A, (u, v) \in B\}$ such that $A \cap B = \{(0, 0)\}$,
$AB := \{(x, z) : (x, y) \in B, (y, z) \in A$ for some $y \in E\}$
while λA stands for $(\lambda I)A$ where I is the identity operator on E.

The product of linear relations is clearly associative. Hence $A^n, n \in \mathbb{Z}$, is defined as usual with $A^0 = I$ and $A^1 = A$.

The following elementary lemma is a preliminary result from which information concerning the commutativity of linear relations will follow.

Lemma 2.1. *Let A and B be linear relations in a linear space E such that $AB = BA$.*

(i) *$(AB)^n = A^nB^n$ and $A^nB^m = B^mA^n$ for all $n, m \in \mathbb{N}$.*
(ii) *$A(N(B)) \subset A(0) + N(B) \subset R(A^n) + N(B)$ for all $n \in \mathbb{N}$.*
(iii) *If A is injective and surjective, then $(A^{-1})^nB \subset B(A^{-1})^n$ for all $n \in \mathbb{N}$.*

Proof. (i) This statement can be easily obtained proceeding by induction.

(ii) Follows immediately from the definitions.

(iii) Since A is injective, we have that $I_{D(A)} = A^{-1}A$ and since A is surjective, it follows that $I \subset AA^{-1}$. So that $A^{-1}B = A^{-1}BI \subset A^{-1}BAA^{-1} = A^{-1}ABA^{-1} \subset BA^{-1}$ and by induction we obtain the validity of (iii). □

In the rest of this section A will be a linear relation in a linear space E.

Lemma 2.2. *Let $\lambda, \mu \in \mathbb{K}$. Then*

(i) *$(A - \lambda I)^n(A - \mu I)^m = (A - \mu I)^m(A - \lambda I)^n$ for all $n, m \in \mathbb{N}$.*

(ii) *If $\lambda \neq \mu$, then $N(A-\lambda I)^n \subset R(A-\mu I)^m$ for all $n, m \in \mathbb{N}$.*

Proof. (i) By [5, Proposition VI.5.1] the linear relations $A-\lambda I$ and $A-\mu I$ commute. The statement now follows from Lemma 2.1 (i).

(ii) See [13, Lemma 7.2]. □

The resolvent set of A is the set $\rho(A) := \{\lambda \in \mathbb{K} : A-\lambda I$ is injective and surjective $\}$ and the spectrum of A is the set $\sigma(A) = \mathbb{K} \setminus \rho(A)$.

Lemma 2.3. *Assume that A has a nonempty resolvent set.*

(i) *$E = D(A^n) + R(A^m)$ and $\{0\} = A^n(0) \cap N(A^m)$ for all $n, m \in \mathbb{N}$.*

(ii) *Let $n, m \in \mathbb{N} \cup \{0\}$. Then $R(A^n)/R(A^{n+m}) \cong D(A^n)/(D(A^n) \cap (N(A^n) + R(A^m))) \cong E/(N(A^n) + R(A^m))$.*

(iii) *Let $\lambda \in \mathbb{K} \setminus \{0\}$ and $n \in \mathbb{N}$. Then $E = R(A-\lambda I) + R(A^n)$.*

Proof. (i) This assertion was proved in [14, Lemma 6.1].

(ii) The first isomorphism was established in [13, Lemma 4.1]. The fact that $D(A^n)/(D(A^n) \cap (N(A^n) + R(A^m))) \cong (D(A^n) + R(A^m))/(N(A^n) + R(A^m))$ together with the part (i) proves the second isomorphism.

(iii) By virtue of (i) it is enough to verify that $D(A^n) \subset R(A-\lambda I) + R(A^n)$. Let $x \in D(A^n)$. It follows from the definition of $D(A^n)$ that there exist $x_o \in \lambda^{-1}(A-\lambda I)x, x_1 \in \lambda^{-2}(A-\lambda I)Ax, \ldots$ and $x_{n-1} \in \lambda^{-n}(A-\lambda I)A^{n-1}x$. Hence

$$x_0 + x_1 + \ldots + x_{n-1} \in \lambda^{-1}Ax - x + \lambda^{-2}A^2x - \lambda^{-1}Ax + \ldots + \lambda^{-(n-1)}A^{n-1}x - \lambda^{-(n-1)}A^{n-1}x + \lambda^{-n}A^nx = -x + \lambda^{-1}A(0) + \lambda^{-2}A^2(0) + \ldots + \lambda^{-(n-1)}A^{n-1}(0) + \lambda^{-n}A^nx$$

which implies that

$x \in -(x_0 + x_1 + \ldots + x_{n-1}) + \lambda^{-1}A(0) + \ldots. + \lambda^{-(n-1)}A^{n-1}(0) + \lambda^{-n}A^nx \subset R(A-\lambda I) + R(A^n)$

since clearly $A^p(0) \subset A^q(0)$ for all $p, q \in \mathbb{N}$. □

The ascent and the descent of A are defined by

$$asc(A) := min\{r \in \mathbb{N} \cup \{0\} : N(A^r) = N(A^{r+1})\},$$

$$des(A) := min\{s \in \mathbb{N} \cup \{0\} : R(A^s) = R(A^{s+1})\},$$

respectively, whenever these minima exist. If no such numbers exist the ascent and descent of A are defined to be ∞.

In [13] the authors introduce and give a systematic treatment of these notions. They show that many of the results of Taylor and Kaashoek for operators remain valid in the context of linear relations only under the additional condition that the linear relation A has a trivial singular chain manifold, that is, if $R_c(A) = \{0\}$ where $R_c(A) := (\cup_{n=1}^{\infty} N(A^n)) \cap (\cup_{n=1}^{\infty} A^n(0))$. Note that by virtue of Lemma 2.3 (i), the condition $\rho(A) \neq \emptyset$ implies that $R_c(A) = \{0\}$.

We shall make extensive use of the following result concerning the ascent and descent of A.

Lemma 2.4. *We have*

(i) *If $N(A) \cap R(A^p) = \{0\}$ for some nonnegative integer p, then $R_c(A) = \{0\}$ and $asc(A) \leq p$. If A has a trivial singular chain manifold and $asc(A) \leq p$ for some $p \in \mathbb{N} \cup \{0\}$, then $N(A^k) \cap R(A^p) = \{0\}$ for $k \in \mathbb{N}$.*

(ii) *Assume that for some nonnegative integer p and $k \in \mathbb{N}$ there exists a subspace M_k such that*

$$M_k \subset N(A^p), D(A^p) = (D(A^p) \cap R(A^k)) + M_k.$$

Then $des(A) \leq p$.

(iii) *Assume that $R_c(A) = \{0\}$. If $asc(A) := p < \infty$ and $des(A) := q < \infty$, then $D(A^q) = (D(A^q) \cap R(A^q)) \oplus N(A^q), p \leq q$ with $p = q$ if $D(A^p) \subset R(A) + D(A^q)$.*

(iv) *If A has a nonempty resolvent set and $N(A^n) \subset R(A)$ for all positive integer n, then $asc(A) = 0$ or ∞ and $des(A) = 0$ or ∞.*

Proof. The statements (i), (ii) and (iii) were proved in [13, Lemmas 5.5 and 5.6, Theorems 5.7 and 5.8].

(iv) We first note that by virtue of [10, Lemma 2.7] the conditions $N(A^n) \subset R(A)$ for all $n \in \mathbb{N}$ and $N(A) \subset R(A^m)$ for all $m \in \mathbb{N}$ are equivalent. Suppose that $asc(A) := p < \infty$. By Lemma 2.3 (i) and part (i) we have that $N(A) \cap R(A^p) = \{0\}$ and hence $N(A) = \{0\}$ equivalently $asc(A) = 0$.

Assume now that $des(A) := q < \infty$, so that $R(A^q) = R(A^{q+1})$. Let $x \in D(A^q)$ and $y \in A^q x \subset R(A^{q+1})$. Then, there exist $z, w \in E$ such that $w \in Az$ and $y \in A^q w$ which imply that $x - w \in N(A^q) \subset R(A)$. Hence $D(A^q) \subset R(A)$ and thus it follows from Lemma 2.3 (i) that $R(A) = E$ equivalently $des(A) = 0$. □.

Following [10, Definition 2.4] we define the degree $\delta(A)$ of A as follows:

$$\delta(A) := min\Delta(A) \text{ if } \Delta(A) \neq \emptyset \text{ and } \delta(A) := \infty \text{ if } \Delta(A) = \emptyset.$$

where $\Delta(A) := min\{m \in \mathbb{N} : N(A) \cap R(A^m) = N(A) \cap R(A^n) \text{ for all } n \geq m\}$.

The following lemma is sometimes useful.

Lemma 2.5. *[10, Lemmas 2.5 and 2.7, Corollary 2.6]*

(i) *Let $d \in \mathbb{N} \cup \{0\}$. Then, $d \in \Delta(A) \Leftrightarrow N(A^m) \subset N(A^d) + R(A^n)$ for all $n, m \in \mathbb{N} \Leftrightarrow N(A) \cap R(A^d) \subset N(A) \cap R(A^m)$ for all $m \in \mathbb{N}$.*

(ii) *$\delta(A) = 0 \Leftrightarrow N(A) \subset R(A^m)$ for all $m \in \mathbb{N}$.*

(iii) *If $\delta(A) < \infty$, then for all $n, m \in \mathbb{N} \cup \{0\}$ we have that*

$$N(A^{\delta(A)}) + R(A^n) = N(A^{m+\delta(A)}) + R(A^n)$$

and

$$N(A^n) \cap R(A^{\delta(A)}) = N(A^n) \cap R(A^{m+\delta(A)}).$$

As a direct consequence we get

Lemma 2.6. *Let $m \in \mathbb{N}$. Then $\delta(A) \leq m\delta(A^m)$.*

Proof. The inequality is trivial if $\delta(A^m) = \infty$. Assume that $0 \leq d := \delta(A^m) < \infty$. Applying Lemma 2.5 (i) we obtain that

$R(A^{md}) \cap N(A) = R(A^{md}) \cap N(A^m) \cap N(A) \subset R(A^{mn}) \cap N(A^m) \cap N(A) \subset R(A^n) \cap N(A)$ for all positive integer n and hence $md \in \Delta(A)$. □

The notion of polynomial $p(A)$ in A is introduced by Sandovici [12, (1.1)] as follows:

Definition 2.7. *Let A be a linear relation in a linear space E, let n and $m_i, 1 \leq i \leq n$ be some positive integers, and let $\lambda_i \in \mathbb{K}, 1 \leq i \leq n$ be some distinct constants. The polynomial $p(A)$ in A is the linear relation*

$$p(A) := \sqcap_{i=1}^{n} (A - \lambda_i I)^{m_i}.$$

The behaviour of the domain, the null space, the multivalued part and the range of $p(A)$ is described in the following useful lemma which is due to Sandovici [12, Theorems 3.2, 3.3, 3.4 and 3.6].

Lemma 2.8. *Let $p(A)$ be as in Definition 2.7. Then*

(i) $D(p(A)) = D(A^{m_1+m_2+\ldots+m_n})$.

(ii) $R(p(A)) = \cap_{i=1}^{n} R((A - \lambda_i I)^{m_i})$.

(iii) $N(p(A)) = \sum_{i=1}^{n} N((A - \lambda_i I)^{m_i})$.

(iv) $p(A)(0) = A^{m_1+m_2+\ldots+m_n}(0)$.

The following result concerning the degree of $p(A)$ will be used to obtain the main theorem of section 5.

Lemma 2.9. *Let $p(A) = \sqcap_{i=1}^{n} (A - \lambda_i I)^{m_i}$ be as in Definition 2.7. Define $d := \delta(p(A))$ and $d_i := \delta((A - \lambda_i I)^{m_i}), 1 \leq i \leq n$. Then $d = max\{d_i : 1 \leq i \leq n\}$.*

Proof. Let us consider various possibilities for d:

Case 1: $d = 0$. By Lemma 2.5 (ii), $N(p(A)^n) \subset R(p(A))$ for all positive integer n and thus we infer from Lemma 2.8 (ii) and (iii) that $N(A - \lambda_i I)^{m_i n} \subset N(p(A)^n) \subset R(p(A)) \subset R(A - \lambda_i I)$ for all $n \in \mathbb{N}$. Again applying Lemma 2.5 (ii) we obtain that $d_i = 0, 1 \leq i \leq n$.

Case 2. $1 \leq d < \infty$. Then, the desired property is obtained by using Lemmas 2.1 and 2.2 and proceeding exactly as in [4, Lemma 2.2]. □

For closed operators in Hilbert spaces the above lemma was proved in [4, Lemma 2.2].

We close this section recalling some basic properties about the notion of linear relation completely reduced which are required for the proofs of the main results of this paper.

Let M be a subspace of E. The restriction A_M is given by $A_M = \{(x, y) \in A : x, y \in M\}$. if M and N are two subspaces of E such that $E = M \oplus N$(that is, $E = M + N$ and $\{0\} = M \cap N$), then we say that A is completely reduced by the pair (M, N) if $A = A_M \oplus A_N$.

Lemma 2.10. *[13, Lemma 8.1, Theorems 8.2 and 8.3]*

(i) *Assume that A is completely reduced by the pair (M, N). Then, for all $n \in \mathbb{N}, A^n$ is completely reduced by the pair (M, N) and $A^n = A^n_M \oplus A^n_N$. Further, $asc(A) < \infty$ if and only if $asc(A_M)$ and $asc(A_N)$ are both finite. Similar property for $des(A)$.*

(ii) *If $E = R(A^p) \oplus N(A^p)$ for some nonnegative integer p, then $asc(A) \leq p, des(A) \leq p$ and A is completely reduced by the pair $(R(A^p), N(A^p))$.*

In the sequel H will be a Hilbert space with the inner product $<, >_H$.

3. Range Space and Quasi-Fredholm Relations

We commence this section recalling some basic properties of range space relations in H. A subspace M of H is said to be a range subspace of H if there exists an inner product $< . >_+$ on M such that $(M, < . >_+)$ is a Hilbert space and $c \parallel u \parallel_H \leq \parallel u \parallel_+, u \in M$ for some $c > 0$. In that case we say that M is a Hilbert space with its own norm.

Following [10, Definition 4.1] we say that a linear relation A in H is closed if it is a closed subspace of $H \times H$ and A is called a range space relation in H if A is a range subspace of $H \times H$. This notion is a useful generalization of the notion of closed linear relation as we can deduce from the following lemma.

Lemma 3.1. *[10, Lemmas 4.2 and 4.10, Propositions 4.7 and 4.8] Let A and B be two range space relations in H.*

(i) *The subspaces $D(A), N(A), A(0)$ and $R(A)$ are range subspaces of H.*

(ii) *If M is a range subspace of H, then A_M is a range space relation in M.*

(iii) *$A + B, A \oplus B$ and AB are range space relations.*

The following result is very useful: it gives circumstances under which one can conclude the closedness of range space relations.

Lemma 3.2. *[10, Corollaries 4.4 and 4.6] Let A be a range space relation in H.*

(i) *If A is an everywhere defined operator, then A is bounded and, hence, closed.*

(ii) *If $N(A)$ and $R(A)$ are closed, then A is closed.*

The adjoint A^* of A is defined by

$$A^* := \{(u, v) \in H \times H :< v, x >_H =< u, y >_H \text{ for all } (x, y) \in A\}$$

so that A^* is a closed linear relation in H.

For a subspace M of H we write $B_M := \{x \in M :\parallel x \parallel_H \leq 1\}$ and $M^\perp := \{x \in H :< x, y >_H = 0 \text{ for all } y \in M\}$.

We note that $N(A^*) = R(A)^\perp$ and if A is closed then $R(A)$ is closed if and only if $R(A^*)$ is closed if and only if $\gamma(A) > 0$ where $\gamma(A) := sup\{\lambda \geq 0 : \lambda B_{R(A)} \subset AB_{D(A)}\}$.

In the second part of this section, our interest concentrates to develop some properties about the notion of quasi-Fredholm relation which will play an important role in the following sections.

Definition 3.3. *[10, Definition 5.1] We say that a linear relation A in H is a quasi-Fredholm relation in H, denoted by $A \in q\phi(H)$, if A is a range space relation and there exists a nonnegative integer d for which*
(i) $d = \delta(A)$.
(ii) $N(A) \cap R(A^d)$ is closed in H.
(iii) $N(A^d) + R(A)$ is closed in H.

In that case, the relation A is called quasi-Fredholm relation of degree d.
As an immediate consequence of Lemma 2.5 (ii) we get

Lemma 3.4. *A range space relation A in H is quasi-Fredholm of degree 0 if and only if the following conditions are satisfied:*
(i) $N(A) \subset R(A^n)$ for all positive integer n.
(ii) $N(A)$ and $R(A)$ are closed subspaces of H.

We now list some properties of quasi-Fredholm relations for future use.

Lemma 3.5. *[10, Theorems 5.2 and 7.3, Proposition 7.4] Let $A \in q\phi(H)$ of degree d. Then*

(i) *A is closed and $N(A^m) + R(A^n) = (N(A^*)^n \cap R((A^*)^m))^\perp$ for all $n, m \in \mathbb{N} \cup \{0\}$ with $n + m \geq d$. Particularly, $R(A^n)$ is closed for all $n \geq d$*
(ii) *$A^* \in q\phi(H)$ of degree d.*

We observe that by virtue of Lemmas 3.4 and 3.5, every quasi-Fredholm relation in H of degree 0 is a regular linear relation in H in the sense of [2, Definition 10].

Quasi-Fredholm relations in H are completely characterized in terms of an algebraic decomposition. Such a decomposition will be essential for the proofs of the main results of the present paper.

Proposition 3.6. *[10, Theorems 5.2 and 6.4] Assume that A is a range space relation in H and let $d \in \mathbb{N} \cup \{0\}$. The following properties are equivalent:*

(i) *$A \in q\phi(H)$ of degree d.*
(ii) *There exist two closed subspaces M and N of H such that*
(1) $H = M \oplus N$ with $R(A^d) \subset M, N \subset N(A^d)$ and, if $d \geq 1$, then N is not contained in $N(A^{d-1})$.
(2) A is completely reduced by the pair (M, N).
(3) A_M is a quasi-Fredholm relation in M of degree 0 and A_N is a bounded operator on N (that is, A_N is everywhere defined and continuous) and A_N^d is the zero operator in N.

The pair (M, N) of invariant subspaces under A which appears in the proposition above is called a Kato decomposition of degree d associated with A.

For a linear relation A in H, the root manifold $R^\infty(A)$ is defined by $R^\infty(A) := \cap_{n=1}^{\infty} R(A^n)$.

Remark 3.7.

Assume that A is quasi-Fredholm of degree d and let (M, N) be a Kato decomposition of A of degree d established in Proposition 3.6. Then

(i) $R^\infty(A) = A(D(A) \cap R^\infty(A)) = R^\infty(A_M)$ is closed.

Indeed, since $A \in q\phi(H)$ of degree d and A_N^d is the zero operator on N, we have that for all $n \geq d$, $R(A_M^n) = R(A^n)$ is closed and, hence $R^\infty(A) = R^\infty(A_M)$ is closed. These properties together with the fact that $A_M \in q\phi(M)$ of degree 0 and [2, Lemma 20] ensure that $A(D(A) \cap R^\infty(A)) = R^\infty(A)$. Therefore (i) holds.

(ii) For all $n \geq d, N(A) \cap R(A^n) = N(A_M) = N(A) \cap R^\infty(A)$.

It is proved in [10, Theorem 5.2 (5.37)].

(iii) For all $n \geq d, N(A^n) + R(A) = R(A_M) \oplus N$.

Follows immediately from [10, Theorem 5.2 (5.34) and Corollary 2.6]. □

We close this section with a result concerning the powers of a quasi-Fredholm relation. In order to prove such a result, we first give the following lemma

Lemma 3.8. *Let A be a range space relation in H with $\rho(A) \neq \emptyset$. Then $N(A)$ is closed.*

Proof. Let $\beta \in \rho(A)$ and we write $S := (A - \beta I)^{-1}$. From [5, (9) and Proposition I.4.2 (e)] it follows that $SA = (A - \beta I)^{-1}((A - \beta I) + \beta I) = I + \beta S$ which implies that SA is an everywhere defined operator. On the other hand, one finds by Lemma 3.1 (iii) that SA is a range space relation. So that, SA is closed by Lemma 3.2 (i) and hence, $N(SA) = N(A)$ is closed. □

Proposition 3.9. *Let A be a range space relation in H with $\rho(A) \neq \emptyset$. The following properties are equivalent:*

(i) $A \in q\phi(H)$.
(ii) $A^n \in q\phi(H)$ *for all nonnegative integer n.*
(iii) $A^m \in q\phi(H)$ *for some nonnegative integer m.*

Proof. (i) ⇒(ii) Let (M, N) be a Kato decomposition of degree d of A established in Proposition 3.6 and let $n \in \mathbb{N}$. Since A_M is closed and it has a nonempty resolvent set, we infer from [7, Lemma 3.1] that A_M^n is closed. This property together with [2, Proposition 11] and Lemma 3.4 ensures that A_M^n is a quasi-Fredholm relation in M of degree 0.

On the other hand, it is clear that A_N^n is a bounded operator on N such that $A_N^n = 0$ if $d \leq n$ and $A_N^{nk} = 0$ if $n \leq d$ for $(k-1)n \leq d \leq nk$. Hence (M, N) is a Kato decomposition associated with A^n which implies by Proposition 3.6 that $A^n \in q\phi(H)$.

(ii) ⇒ (iii) It is obvious.

(iii) ⇒(i) Let m be a positive integer for which A^m is a quasi-Fredholm relation of degree r. From Lemma 2.6 we have that $\delta(A) \leq mr$, so that $mr \in \Delta(A)$. According to Lemma 2.5 (iii) to prove that $A \in q\phi(H)$ it is enough to verify that $N(A) \cap R(A^{mr})$ and $N(A^{mr}) + R(A)$ are both closed subspaces of H. The closedness of $N(A) \cap R(A^{mr})$ follows immediately from Lemmas 3.5 (i) and 3.8.

We now claim that $N(A^{mr}) + R(A)$ is closed. Let $\beta \in \rho(A) \setminus \{0\}$ since if $\beta = 0$ then the desired property is trivially true. We write $S := (A - \beta I)^{-1}$ and $D := S^{m-1}A^{m-1}$. Since $A^{m-1} = ((A - \beta I) + \beta I)^{m-1} = \sum_{k=0}^{m-1} c_k (A - \beta I)^k$ where $c_k, 0 \leq k \leq m-1$ are constants, we infer from [, Proposition I.4.2 (e)] and Lemma 3.2 (i) that

(3.1) $D := S^{m-1}A^{m-1} = \sum_{k=0}^{m-1} c_k C^{m-1-k}$ is a bounded operator where $S := (A - \beta I)^{-1}$.

It follows immediately from the definitions and Lemma 2.1 (iii) that

(3.2) Let $x_n \in D(A), t_n \in Ax_n$ and $y_n \in N(A^{mr})$ such that $t_n + y_n \to z$ for some $z \in H$. Then $Dt_n \in R(A^m)$ and $Dy_n \in N(A^{mr})$.

A combination of (3.1), (3.2) and the fact that $N(A^{mr}) + R(A^m)$ is closed leads to $Dz \in A^m x + y$ for some $x \in D(A^m)$ and $y \in N(A^{mr})$. Let $Dz = u + y$ with $u \in A^m x$. Then there exists $v \in Ax$ such that $u \in A^{m-1}v$ and, therefore $y = Dz - u \in S^{m-1}A^{m-1}z - A^{m-1}v \subset A^{m-1}S^{m-1}z - A^{m-1}v$ (Lemma 2.1 (iii)) $\subset A^{m-1}(S^{m-1}z - v)$ ([5, Proposition I.4.2 (e)]) and since $y \in N(A^{mr})$ we obtain that $0 \in A^{mr+m-1}(S^{m-1}z - v)$ which implies that

(3.3) There exists $v \in R(A)$ such that $S^{m-1}z - v \in N(A^{mr+m})$.

Now, since $N(A^{mr+m}) \subset N(A^{mr}) + R(A)$ by virtue of Lemma 2.5 (i) one deduces from (3.3) that $S^{m-1}z \in N(A^{mr}) + R(A)$ and since S is an invertible operator we conclude that $N(A^{mr}) + R(A)$ is closed. The proof is completed. □

Proposition 3.9 was proved in [4, Proposition 2.4] for closed operators.

4. Ascent and Descent of a Quasi-Fredholm Relation

The first main result of this section represents an extension of [11, Lemma 1.4] to linear relations.

Theorem 4.1. *Let $A \in q\phi(H)$ of degree d. There exists $\eta > 0$ such that if $0 <| \lambda |< \eta$ then*

(i) *$A - \lambda I$ is quasi-Fredholm of degree d.*
(ii) $dimN(A - \lambda I) = dim(N(A) \cap R(A^d))$.
(iii) $codimR(A - \lambda I) = codim(N(A^d) + R(A))$.

Proof. (i) Write $A = A_M \oplus A_N$ with M, N, A_M and A_N as in Proposition 3.6; in particular A_N^d is the zero operator on N which implies by Lemma 2.2 (ii) that $A_N - \alpha I_N$ is an invertible operator for all nonzero scalar α.

On the other hand, since A_M is quasi-Fredholm of degree 0, we infer from Lemma 3.5 (i) combined with [2, Theorem 23] that there exists $\eta > 0$ for which $A_M - \lambda I_M$ is quasi-Fredholm of degree 0 whenever $0 <| \lambda |< \eta$. Therefore, $A - \lambda I = (A - \lambda I)_M \oplus (A - \lambda I)_N$ is a quasi-Fredholm relation in H of degree 0 if $0 <| \lambda |< \eta$.

(ii) Define $A_\infty := A_{R^\infty(A)}$ and $I_\infty := I_{R^\infty(A)}$.

By Lemma 3.5 (i) A is closed and thus it follows from Remark 3.7 (i) that A_∞ is a closed surjective linear relation in $R^\infty(A)$ which implies by [5, Theorem III.7.4 and Corollary V.15.7] that $A_\infty - \lambda I_\infty$ is a closed surjective linear relation in $R^\infty(A)$ with $i(A_\infty) = i(A_\infty - \lambda I_\infty)$ whenever $0 <| \lambda |< \gamma(A_\infty)$. This combined with Lemma 2.2 (ii) and Remark 3.7 (ii) leads to

$dimN(A - \lambda I) = dim(N(A - \lambda I) \cap R^\infty(A)) = i(A_\infty - \lambda I_\infty) = i(A_\infty) = dimN(A_\infty) = dim(N(A) \cap R^\infty(A)) = dim(N(A) \cap R(A^d))$ if $0 <| \lambda |< \gamma(A_\infty)$.

(iii) By Lemma 3.5 (ii) A^* is quasi-Fredholm of degree d and therefore, one finds by part (ii) combined with Lemma 3.5 (i) that

$$codimR(A - \lambda I) = dimN(A^* - \overline{\lambda} I) = dim(N(A^*) \cap R((A^*)^d) = codim(N(A^*) \cap R((A^*)^d))^\perp = codim(N(A^d) + R(A)).$$

The proof is completed. □

Proposition 4.2. *Let A be a range space relation in H with a nonempty resolvent set.*

(i) *If $des(A) = q < \infty$, then there exists $\epsilon > 0$ such that $des(A - \lambda I) = 0$ and $dimN(A - \lambda I) = dim(N(A) \cap R(A^q))$ if $0 <| \lambda |< \epsilon$.*

(ii) *If $asc(A) = p < \infty$ and $R(A^{p+1})$ is closed, then there exists $\mu > 0$ such that $asc(A - \lambda I) = 0$ and $codimR(A - \lambda I) = dim(R(A^p) \cap R(A^{p+1})^\perp)$ if $0 <| \lambda |< \mu$.*

Proof. We first note that by virtue of Lemma 3.1, for all positive integer n, A^n is a range space relation in H and $R(A^n)$ is a range subspace of H. So that, $R(A^n)$ is a Hilbert space with its own norm.

(i) Define $A_q := A_{R(A^q)}$ and $I_q := I_{R(A^q)}$.

Then we infer from Lemma 3.1 (ii) that A_q is a range space relation in $R(A^q)$ and since $des(A) = q$ we obtain that A_q is a closed surjective linear relation in the Hilbert space $R(A^q)$ endowed with its own norm. This fact combined with [5, Theorem III.7.4 (ii) and Corollary V.15.7] ensures that there exists $\epsilon > 0$ for which $A_q - \lambda I_q$ is surjective with $dimN(A_q - \lambda I_q) = dimN(A_q)$ whenever $0 <| \lambda |< \epsilon$. Hence

$dimN(A - \lambda I) = dim(N(A - \lambda I) \cap R(A^q)) = dimN(A_q - \lambda I_q) = dimN(A_q) = dim(N(A) \cap R(A^q))$.

$R(A^q) = R(A^{q+1}) = R(A_q) = R(A_q - \lambda I_q) = R((A - \lambda I)A^q) \subset R(A - \lambda I)$.

Now, applying Lemma 2.3 (iii), we deduce that $R(A - \lambda I) = H$, that is, $des(A - \lambda I) = 0$.

(ii) Since $R(A^{p+1})$ is a closed subspace of H and $R(A^p)$ is a range subspace of H with $R(A^{p+1}) \subset R(A^p)$ we infer easily from the definitions that $R(A^{p+1})$ is a closed subspace of $R(A^p)$ as a range subspace. Hence

(4.1) $R(A^p) = R(A^{p+1}) \oplus (R(A^p) \cap R(A^{p+1})^\perp)$ where $R(A^p)$ is the Hilbert space with its own norm.

Let $\lambda \in \mathbb{K} \setminus \{0\}$. A combination of Lemmas 2.2 (ii), 2.3 (i) and [5, Exercise I.6.5] implies that

$codimR(A - \lambda I) = dim(D((A - \lambda I)^p) + R(A - \lambda I))/R(A - \lambda I) = dim(D(A^p) + R(A - \lambda I))/R(A - \lambda I) = dimD(A^p)/(D(A^p) \cap R(A - \lambda I)) = dimR(A^p)/A^p(D(A^p) \cap R(A - \lambda I))$.

Hence

(4.2) $codimR(A - \lambda I) = dimR(A^p)/R((A - \lambda I)A^p)$ for all nonzero scalar λ.

Define $A_p := A_{R(A^p)}$ and $I_p := I_{R(A^p)}$.

Then, A_p is a range space relation in $R(A^p)$ by Lemma 3.1 (ii), $N(A_p) := N(A) \cap R(A^p) = \{0\}$ by Lemmas 2.3 and 2.4 (i) and $R(A_p) = R(A^{p+1})$ is closed by hypothesis. So that, A_p is closed in $R(A^p)$ by Lemma 3.2 (ii). The use of these properties together with [5, Theorem III.7.4 (i) and Corollary VI.15.7] leads to

(4.3) $A_p - \lambda I_p$ is closed, injective with closed range and $i(A_p) = i(A_p - \lambda I_p)$ if $0 <| \lambda |< \gamma(A_p)$.

Now, it follows from (4.1), (4.2) and (4.3) that $N(A - \lambda I) = \{0\}$ and $codimR(A - \lambda I) = dim(R(A) \cap R(A^{p+1})^\perp)$ if $0 <| \lambda |< \gamma(A_p)$.

Hence (ii) holds. The proof is completed. □

Proposition 4.2 generalizes Lemmas 3.4 and 3.5 in [10].
Now, we are in the position to give the second fundamental result of this section.

Theorem 4.3. *Let A be a quasi-Fredholm relation in H of degree d such that $\rho(A) \neq \emptyset$. The following properties are equivalent:*

(i) 0 *is an element of the boundary of* $\sigma(A)$.
(ii) 0 *is an isolated point of* $\sigma(A)$.
(iii) $asc(A) = des(A) = d$.
(iv) $H = N(A^d) \oplus R(A^d)$ *where both subspaces are closed.*

Proof. Let $\sigma(A)^b$ denote the boundary of $\sigma(A)$.

$(i) \Rightarrow (ii)$ By Theorem 4.1, there is $\eta > 0$ such that $dimN(A - \lambda I)$ and $codimR(A - \lambda I)$ are constants if $0 <| \lambda |< \eta$ and since $0 \in \sigma(A)^b$ we obtain that $\{\lambda \in \mathbb{K} : 0 <| \lambda |< \eta\} \cap \rho(A) \neq \emptyset$. So that, $dimN(A - \lambda I) = codimR(A - \lambda I) = 0$ whenever $0 <| \lambda |< \eta$ which implies that 0 is an isolated point of the spectrum of A.

$(ii) \Rightarrow (iii)$ Since 0 is an isolated point of $\sigma(A)$ we infer from Theorem 4.1 that $N(A) \cap R(A^d) = \{0\}$ and $H = N(A^d) + R(A^d)$. The use of these equalities together with Lemmas 2.3 (i) and 2.4 (i), (ii) and (iii) proves that $asc(A) = des(A) \leq d$. Accordingly, it only remains to verify that $asc(A) = d$. For this, let (M, N) be a Kato decomposition of degree d associated with A established in Proposition 3.6. By Lemma 2.10 (i), $asc(A_M) < \infty$ and thus one finds from Lemma 2.4 (iv) that A_M is injective. The use of this property combined with the facts that $H = M \oplus N$, N contained in $N(A^d)$ and N is not contained in $N(A^{d-1})$ makes us to conclude that $N = N(A^n)$ for all $n \geq d$ and N is not contained in $N(A^{d-1})$. Therefore, $asc(A) = d$, as desired.

$(iii) \Rightarrow (iv)$ The equality $H = N(A^d) \oplus R)A^d)$ follows immediately from Lemmas 2.3 (ii) and 2.4 (i). Furthermore, since A is quasi-Fredholm of degree d, we infer from Lemma 3.5 (i) that $N(A^d)$ and $R(A^d)$ are closed subspaces of H.

$(iv) \Rightarrow (i)$ Proceeding as in the proof of $(ii) \Rightarrow (iii)$ we obtain that $asc(A) = des(A) := p \leq d$. This together with the equality $H = N(A^d) \oplus R(A^d)$ with $p \leq d$ and again Lemma 2.4 (i) proves that $H = N(A^d) \oplus R(A^{p+1})$ and since one has from Proposition 3.9 that $A^n \in q\phi(H)$ for all $n \in \mathbb{N}$, we deduce from Lemma 3.5 (i) and [7, Lemma 3.2] that $R(A^{p+1})$ is closed. After that, using Proposition 4.2 we get that there exists $\epsilon > 0$ such that $A - \lambda I$ is invertible if $0 <| \lambda |< \epsilon$, so that $0 \in \sigma(A)^b$. □

Theorem 4.3 provides an extension of Proposition 3.6 in [11] to linear relations.

Theorem 4.4. *Let A be a range space relation in H having a nonempty resolvent set and assume that* $0 \in \sigma(A)^b$. *The following properties are equivalent:*

(i) $A \in q\phi(H)$ *of degree* d.
(ii) $asc(A) = d$ *and* $R(A^{d+1})$ *is closed.*
(iii) $des(A) = d$.
(iv) $asc(A) = des(A) = d$.
(v) $H = N(A^d) \oplus R(A^d)$.

Proof. $(i) \Rightarrow (ii)$ It is a direct consequence of Theorem 4.3.

$(ii) \Rightarrow (iii)$ Proposition 4.2 (ii) and the fact that $0 \in \sigma(A)^b$ imply that $R(A^d) \cap R(A^{d+1})^\perp = \{0\}$. On the other hand, $R(A^{d+1})$ closed implies that $H = R(A^{d+1}) \oplus R(A^{d+1})^\perp$, so that $R(A^d) = R(A^{d+1}) \oplus (R(A^d) \cap R(A^{d+1})^\perp)$. Hence $R(A^d) = R(A^{d+1})$. So that, applying again Lemma 2.4 we get that $des(A) = d$.

$(iii) \Rightarrow (iv)$ The use of Proposition 4.2 (i) and the condition $0 \in \sigma(A)^b$ proves that $N(A) \cap R(A^d) = \{0\}$. This fact combined with the assumption $des(A) = d$ and Lemma 2.4 ensures that $asc(A) = d$.

$(iv) \Rightarrow (v)$ Follows immediately from Lemmas 2.3 (i) and 2.4 (i) and (iii).

$(v) \Rightarrow (i)$ By Lemma 3.1, $N(A^d)$ and $R(A^d)$ are range subspaces and since $H = N(A^d) \oplus R(A^d)$ by hypothesis one deduces from the known Neubauer's lemma that $N(A^d)$ and $R(A^d)$ are closed subspaces of H.

On the other hand, we infer from Lemma 2.10 (ii) that $asc(A) \leq d$, $des(A) \leq d$ and A is completely reduced by the pair $(R(A^d), N(A^d))$, that is, $A = A_{R(A^d)} \oplus A_{N(A^d)}$. Furthermore, by Lemmas 2.4 and 3.4 the range space relation $A_{R(A^d)}$ is quasi-Fredholm of degree 0. As an immediate consequence of the equality $H = N(A^d) \oplus R(A^d)$ we obtain that $A_{N(A^d)}(0) = \{0\}$ equivalently $A_{N(A^d)}$ is an operator. So that, one has by Lemma 3.2 (i) that $A_{N(A^d)}$ is a bounded operator on $N(A^d)$ and clearly $A^d_{N(A^d)} = 0$. Therefore, $(R(A^d), N(A^d))$ is a Kato decomposition of degree d associated with A. Thus, by Proposition 3.6 we get that $A \in q\phi(H)$ of degree d. □

For closed operators the above Theorem 4.4 was proved by Mbekhta [11, Proposition 3.7].

5. Polynomial in a Quasi-Fredholm Relation

This section is to investigate the behaviour of the a polynomial in a quasi-Fredholm linear relation.

Proposition 5.1. *Let A be a range space relation in H with $\rho(A) \neq \emptyset$. Let $p(A) = \sqcap_{i=1}^{n}(A - \lambda_i I)^{m_i}$ be as in Definition 2.7 with $n \geq 2$ and let $d \in \mathbb{N} \cup \{0\}$. The following properties are equivalent:*

(i) *$N(p(A)^d) + R(p(A))$ is closed.*

(ii) *For all $i, 1 \leq i \leq n, N((A - \lambda_i I)^{m_i d}) + R((A - \lambda_i I)^{m_i})$ is closed.*

Proof. Let $\beta \in \rho(A)$. We write $C := (A - \beta I)^{-1}$ and $B := C^{p-m_i} \sqcap_{j=1, j\neq i}^{n} (A - \lambda_j)^{m_j}$ where $p := \sum_{i=1}^{n} m_i$

$(i) \Rightarrow (ii)$ Reasoning as in the proof of (3.1) and (3.2) in Proposition 3.9 we obtain that B is a bounded operator in H and that if $x_n \in D((A - \lambda_i I)^{m_i}), t_n \in (A - \lambda_i I)^{m_i} x_n$ and $y_n \in N((A - \lambda_i I)^{m_i d})$ with $t_n + y_n \to z$ for some $z \in H$, then $Bt_n \in R(p(A))$ and $By_n \in N(p(A)^d)$. The rest of the proof is along the lines of the proof of Theorem 2.5 (1) ⇒ (2) in [4], with the appropriate modifications.

$(ii) \Rightarrow (i)$ Let $x_n \in D(p(A)^d), t_n \in p(A)^d x_n$ and $y_n \in N(p(A)^d)$ such that $t_n + y_n \to z$ for some $z \in H$. Define $D := C^{(p-m_i)d} \sqcap_{j=i, j\neq i}^{n} (A - \lambda_j I)^{m_j d}$.

The same techniques used previously show that D is a bounded operator satisfying $Dt_n \in R((A - \lambda_i I)^{m_i})$ and $Dy_n \in N((A - \lambda_i I)^{m_i d})$. Now, the rest of the proof proceeds as in the proof of the implication (2) ⇒ (1) in Theorem 2.5 in [4] with only minor changes. □

Proposition 5.1 represents an improvement of [4, Theorem 2.5] to linear relations.

Proposition 5.2. *Let A be a range space relation in H such that $\rho(A) \neq \emptyset$. Let $p(A) = \sqcap_{i=1}^{n}(A - \lambda_i I)^{m_i}$ be as in Definition 2.7 with $n \geq 2$ and let d be a nonnegative integer. The following properties are equivalent:*

(i) *$N(p(A)) \cap R(p(A)^d)$ is closed.*

(ii) *For all $i, 1 \leq i \leq n, N((A - \lambda_i I)^{m_i}) \cap R((A - \lambda_i I)^{m_i d})$ is closed.*

Proof. The proof may be sketched in a similar way as Proposition 5.1. □

Proposition 5.2 provides an extension of Theorem 2.6 in [4].

We are ready to give the main result of this section.

Theorem 5.3. *Let A be a range space relation in H such that $\rho(A) \neq \emptyset$ and let $p(A) = \sqcap_{i=1}^{n}(A - \lambda_i I)^{m_i}$ be as in Definition 2.7. Then $p(A) \in q\phi(H)$ if and only if for all $i, 1 \leq i \leq n, A - \lambda_i I \in q\phi(H)$.*

Proof. The case $n = 1$ is covered by Proposition 3.9. Accordingly, assume $n \geq 2$.

Suppose that $p(A)$ is a quasi-Fredholm relation in H of degree d and let $\beta \in \rho(A)$. Let us consider two possibilities for d.

Case 1: $d = 0$. By Lemmas 2.6 and 2.9 we have that $\delta(A - \lambda_i I) = 0$. On the other hand, using Lemma 2.8 one deduces that $N(A - \lambda_i I) \subset R((A - \lambda_i I)^n)$ for all $n \in \mathbb{N}$ and by Lemma 3.8, $N(A - \lambda_i I)$ is closed. So that, in order to apply Lemma 3.4 we only need to prove that $R(A - \lambda_i I)$ is closed. For this, we first note that one finds by Proposition 5.1 that $R((A - \lambda_i I)^{m_i})$ is closed.

Let $x_n \in D(A - \lambda_i I)$ and $t_n \in (A - \lambda_i I)x_n$ such that $t_n \to z$ for some $z \in H$. Define $U := C^{m_i - 1}(A - \lambda_i I)^{m_i - 1}$ where $C := (A - \beta I)^{-1}$. Then U is a bounded operator in H and $Ut_n \in R((A - \lambda_i I)^{m_i})$. Consequently, $Uz \in (A - \lambda_i I)^{m_i})w$ for some $w \in D((A - \lambda_i I)^{m_i})$ which implies that
$0 \in (A - \lambda_i I)^{m_i - 1}(C^{m_i - 1}z - (A - \lambda_i I)w)$.

So, there exists $y \in (A - \lambda_i I)w$ such that

$$C^{m_i - 1}z - y \in N((A - \lambda_i I)^{m_i - 1}) \subset N(p(A)) \subset R(p(A)) \subset R(A - \lambda_i I)$$

and hence $C^{m_i - 1}z \in R(A - \lambda_i I)$ and since C is invertible we conclude that $R(A - \lambda_i I)$ is closed, as desired.

Assume now that for all $i, 1 \leq i \leq n, A - \lambda_i I \in q\phi(H)$. If $A - \lambda_i I$ is quasi-Fredholm of degree 0, then it follows from [2, Theorem 21] that $p(A)$ is quasi-Fredholm of degree 0. Suppose that $A - \lambda_i I$ is quasi-Fredholm with $\delta(A - \lambda_i I) \geq 1$. Then, applying Lemmas 2.6 and 2.8 together with Proposition 3.9 we deduce that $(A - \lambda_i I)^{m_i} \in q\phi(H)$ and $d := \delta(p(A)) = max d_i$. This implies by the use of Lemma 2.5 (iii) combined with Propositions 5.1 and 5.2 that $p(A)$ is a quasi-Fredholm relation in H. □

Theorem 5.3 provides an extension of Theorem 3.1 in [4] to range space relations.

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