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On q-Analogues of the Natural Transform of Certain q-Bessel Functions and Some Application

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Abstract. Theory and applications of q-integral transforms have been evolving rapidly over the recent years. Consequently, several q-analogues of certain classical integrals have been investigated by many authors in various citations. In this paper, we give the q-analogues of the Natural transform and we apply further the resulting analogues to three families of q-Bessel functions. Results of this paper are new and complement previously known results in this theory. Moreover, we give some examples to show effectiveness of the proposed results in case of q-Sumudu and q-Laplace transforms.

1. Introduction

Jackson in [11] presented a precise definition of the so-called q-Jackson integral and developed a q-calculus in a systematic way. Consequently, q-calculus has gained a noticeable importance and popularity due to mainly its demonstrated applications in many seemingly diverse fields of science and engineering. Some remarkable integral transforms have different q-analogues in the theory of q-calculus. Among those q-integrals that we recall here are : the q-Laplace integral transform [2, 3, 4, 18], the q-Sumudu integral transform [5, 6] , the Weyl fractional q-integral operator [7] , the q-Wavelet integral transform [8] , the q-Mellin integral transform [9] , and some others.
Over the set A of functions, where

$$A = \left\{ f(t) \,|\exists M, \tau_1 \text{ and /or } \tau_2 > 0 \text{ such that } \left| f(t) \right| < Me^{\frac{|t|}{\tau_j}}, t \in (-1)^j \times [0, \infty) \right\},$$

$j = 1, 2$, the Natural transform of a function f of exponential order is proclaimed as [13]

$$N(f(t))(u; v) = \int_0^\infty f(ut) \exp(-vt)\, dt, \ \ \mathrm{Re}\, v > 0, \ u \in (-\tau_1, \tau_2),$$

where u and v denote the Natural transform variables.
Over the same set A, the Natural transform strictly converges to the Sumudu transform [24]

$$S(f(t))(u) = \begin{cases} \int_0^\infty f(ut) \exp(-t)\, dt & \text{for } 0 \le u < \tau_2 \\ \int_0^\infty f(ut) \exp(-t)\, dt & \text{for } -\tau_1 < u \le 0 \end{cases}$$

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when $v \equiv 1$ and, in that like manner, it strictly converges to the Laplace transform

$$L(f(t))(v) = \int_0^\infty f(t)\exp(-vt)\,dt,$$

when $u \equiv 1$ and $\operatorname{Re} v > 0$.
The natural dualities, Natural-Laplace and Natural-Sumudu transforms dualities, are given in [19, (5) – (8)] as

$$N(f(t))(u;v) = \frac{1}{u}\int_0^\infty f(t)\exp\left(-\frac{vt}{u}\right)dt \ \left(\text{i.e. } N(f(t))(u;v) = \frac{1}{u}L(f(t))\left(\frac{v}{u}\right)\right)$$

and

$$N(f(t))(u;v) = \frac{1}{v}\int_0^\infty f\left(\frac{ut}{v}\right)\exp(-t)\,dt \ \left(\text{i.e. } N(f(t))(u;v) = \frac{1}{v}S(f(t))\left(\frac{u}{v}\right)\right),$$

respectively.

Linearity of the Natural transform was announced by the linearity of Sumudu and Laplace transforms. Whereas, utilization of the Natural transform to fluid flow problems was exposed in [13]. Some further application of the transform to Maxwell and Bessel functions was also discussed in [15, 16].
For more investigation of the Natural transform we refer to the citations [13 – 17] and [19].

We organize this paper as follows. In Section 2, we present some notations and terminologies from the q-calculus. In Section 3, we derive the first type q-analogue of the Natural transform and apply to certain family of q-Bessel functions. In Section 4, we give the definition of the second type q-analogue of the transform and find out its value of certain class of q-Bessel functions that are treated in Section 3. Finally, we develop some special corollaries of the previous theorems for the transforms q-Natural, q-Sumudu and q-Laplace transform as well.

2. Definitions and Preliminaries

Bessel functions mainly describe a series of solutions to the second order differential equation

$$x^2y'' + xy' + \left(x^2 - \mu^2\right)y = 0$$

that arise in many diverse situations. Bessel functions studied by Euler, Lagrange, and Bernoulli's were first used by F. W. Bessel to describe three body motion appearing in series expansion on planetary perturbation .
The mainly best known q-analogues of the remarkable Bessel function

$$J_\mu(x) = \sum_{k=0}^{\infty}\frac{(-1)^k (x/2)^{\mu+2k}}{k!\Gamma(\mu+k+1)} \tag{1}$$

were first introduced by Jackson [1] and studied later by Hahn [18] and Ismail [20],

$$J_\mu^{(1)}(z;q) = \left(\frac{z}{2}\right)^\mu \sum_{n=0}^{\infty}\frac{\left(\frac{-z^2}{4}\right)^n}{(q,q)_{\mu+n}(q;q)_n},\ |z| < 2 \tag{2}$$

$$J_\mu^{(2)}(z;q) = \left(\frac{z}{2}\right)^\mu \sum_{n=0}^{\infty}\frac{q^{n(n+\mu)}\left(\frac{-z^2}{4}\right)^n}{(q,q)_{\mu+n}(q;q)_n},\ z \in \mathbb{C}. \tag{3}$$

In terms of q-hypergeometric functions they were fairly expressed as follows

$$J_{\mu}^{(1)}(z;q)=\frac{\left(q^{\mu+1};q\right)_{\infty}}{(q;q)_{\infty}}\left(\frac{z}{2}\right)^{\mu}{}_{2}\phi_{1}\left[\begin{matrix}0\\ q^{\mu+1}\end{matrix};q,\frac{-z^{2}}{4}\right] \tag{4}$$

and

$$J_{\mu}^{(2)}(z;q)=\frac{\left(q^{\mu+1};q\right)_{\infty}}{(q;q)_{\infty}}\left(\frac{z}{2}\right)^{\mu}{}_{0}\phi_{1}\left[\begin{matrix}-\\ q^{\mu+1}\end{matrix};q,\frac{q^{\mu+1}z^{2}}{4}\right], \tag{5}$$

respectively.
The q-Bessel functions $J_{\mu}^{(1)}$ and $J_{\mu}^{(2)}$ are related by [25]

$$J_{\mu}^{(2)}(z;q)=\left(\frac{-z^{2}}{4};q\right)J_{\mu}^{(1)}(z;q),\ |z|<2.$$

The third q-Bessel function (Hahn-Exton q-Bessel function) was introduced by Hahn [22] (In a special case) and by Exton [23] (In full case) as

$$J_{\mu}^{(3)}(z;q)=z^{\mu}\sum_{n=0}^{\infty}\frac{(-1)^{n}q^{\frac{n(n-1)}{2}}\left(qz^{2}\right)^{n}}{(q,q)_{\mu+n}(q;q)_{n}},\ z\in\mathbb{C}. \tag{6}$$

This in terms of q-hypergeometric functions is demonstrated as follows

$$J_{\mu}^{(3)}(z;q)=\frac{\left(q^{\mu+1};q\right)_{\infty}}{(q;q)_{\infty}}z^{\mu}{}_{1}\phi_{1}\left[\begin{matrix}0\\ q^{\mu+1}\end{matrix};q,qz^{2}\right].$$

Throughout this paper, by fixing $a\in\mathbb{C}$, the q-shifted factorials are defined in literature as

$$(a;q)_{0}=1;\ (a;q)_{n}=\prod_{k=0}^{n-1}\left(1-aq^{k}\right),\ n=1,2,...;\ (a;q)_{\infty}=\lim_{n\to\infty}(a;q)_{n}. \tag{7}$$

We also denote by

$$\left.\begin{aligned}&[x]_{q}=\frac{1-q^{x}}{1-q},\ x\in\mathbb{C};\\ &\left([n]_{q}\right)!=\frac{(q;q)_{n}}{(1-q)^{n}},\ n\in\mathbb{N};\\ &(a;q)_{x}=\frac{(a;q)_{\infty}}{(aq^{x};q)_{\infty}},\ x\in\mathbb{R}\end{aligned}\right\}. \tag{8}$$

The second type q-analogue of the exponential function was introduced in literature as

$$e_{q}(x)=\sum_{0}^{\infty}\frac{x^{n}}{(q;q)_{n}}=\frac{1}{(x;q)_{\infty}},\ |x|<1. \tag{9}$$

Whereas, the first type q-analogue of the exponential function was introduced as

$$E_{q}(x)=\sum_{0}^{\infty}\frac{(-1)^{n}q^{n\frac{(n-1)}{2}}x^{n}}{(q;q)_{n}}=(x,q)_{\infty},\ x\in\mathbb{C}. \tag{10}$$

The q-Jackson integrals from 0 to x and from 0 to ∞ were displayed by [1] as

$$\int_0^x f(t)\, d_q t = x(1-q)\sum_{k=0}^{\infty} q^k f\left(xq^k\right) \tag{11}$$

$$\int_0^{\infty/A} f(t)\, d_q t = (1-q)\sum_{k\in\mathbb{Z}} \frac{q^k}{A} f\left(\frac{q^k}{A}\right). \tag{12}$$

We also have

$$(q^{x+m};q)_\infty = \frac{(q^x;q)_\infty}{(q^x;q)_m},\ m\in\mathbb{N}. \tag{13}$$

Hahn in [21] defines the q-analogues of the Laplace transform of first type (respectively of second type) as

$$L_q(f(t))(v) = \frac{1}{1-q}\int_0^{\frac{1}{v}} f(t)\, E_q(qvt)\, d_q t \tag{14}$$

and

$${}_qL(f(t))(u) = \frac{1}{1-q}\int_0^{\infty} f(t)\, e_q(-vt)\, d_q t. \tag{15}$$

The q-analogue of the Sumudu transform of $f(t)$ of first type was defined on the set A_1 by [5] and [6] as

$$S_q(f(t))(u) = \frac{1}{(1-q)u}\int_0^{u} f(t)\, E_q\left(q\frac{t}{u}\right) d_q t, \tag{16}$$

where A_1 is defined by

$$A_1 = \left\{ f(t) \middle| \exists M, \tau_1, \tau_2 > 0, \left|f(t)\right| < ME_q\left(\frac{|t|}{\tau_j}\right), t\in(-1)^j\times[0,\infty) \right\}. \tag{17}$$

On the other hand, the q-analogue of the Sumudu transform of $f(t)$ of second type was defined on the set A_2 by [5, 6] as

$${}_qS(f(t))(u) = \frac{1}{(1-q)}\int_0^{\infty} f(t)\, e_q\left(-\frac{t}{u}\right) d_q t, \tag{18}$$

where A_2 is defined as

$$A_2 = \left\{ f(t) \middle| \exists M, \tau_1, \tau_2 > 0, \left|f(t)\right| < Me_q\left(\frac{|t|}{\tau_j}\right),\ t\in(-1)^j\times[0,\infty) \right\}. \tag{19}$$

In this note, we introduce two q-analogues of the Natural transform in the following manner:

Definition 1. Let A_1 and A_2 have their usual meaning above. Then, we have the following.
(*i*) We define the q-analogue of the Natural transform of first kind over the set A_1 as

$$N_q(f(t))(u;v) = \frac{1}{(1-q)u}\int_0^{\frac{u}{v}} f(t)\, E_q\left(q\frac{v}{u}t\right) d_q t. \tag{20}$$

(*ii*) We define the q-analogue of the Natural transform of the second kind over the set A_2 as

$${}_qN(f(t))(u;v) = \frac{1}{(1-q)}\int_0^{\infty} f(t)\, e_q\left(-\frac{t}{u}t\right) d_q t. \tag{21}$$

Subsequently, readers can easily verify the respective dualities of q-Natural-Laplace and q-Natural-Sumudu transforms of first and second types. We also can derive that

$$N_q(f(t))(1;v) = L_q f(t)(v) \text{ and } {}_qN(f(t))(1;v) = {}_qL(f(t))(v) \tag{22}$$

and

$$N_q(f(t))(u;1) = S_q(f(t))(u) \text{ and } {}_qN(f(t))(u;1) = {}_qS(f(t))(u). \tag{23}$$

The integral representations of the q-gamma function are defined by [25, p. 3]

$$\left.\begin{array}{c} \Gamma_q(\alpha) = \int_0^{1/(1-q)} x^{\alpha-1} E_q(q(1-q)x)\, d_qx, (\alpha > 0) \\ \text{and} \\ {}_q\Gamma(\alpha) = K(A;\alpha) \int_0^{\infty/A(1-q)} x^{\alpha-1} e_q(-(1-q)x)\, d_qx \end{array}\right\}, \tag{24}$$

where, $\alpha > 0$, $K(A;\alpha)$ is the remarkable function

$$K(A;\alpha) = A^{\alpha-1} \frac{(-q/\alpha;q)_\infty (-\alpha;q)_\infty}{(-q^t/\alpha;q)_\infty (-\alpha q^{1-t};q)_\infty}. \tag{25}$$

It is of importance to us to recall the following properties of $\Gamma_q(\alpha)$ and ${}_q\Gamma(\alpha)$ functions :

$$\Gamma_q(\alpha) = \frac{(q;q)_\infty}{(1-q)^{\alpha-1}} \sum_{k=0}^{\infty} \frac{q^{k\alpha}}{(q;q)_k} = \frac{(q;q)_\infty}{(q^\alpha - q)_\infty} (1-q)^{1-x}, \tag{26}$$

$x \neq 0, -1, -2, ...,$ and

$${}_q\Gamma(\alpha) = \frac{K(A;\alpha)_\infty}{(1-q)^{\alpha-1}\left(-\frac{1}{A};q\right)_\infty} \sum_{k\in\mathbb{Z}} \left(\frac{q^k}{A}\right)\left(-\frac{1}{A};q\right)_k. \tag{27}$$

3. N_q Transforms of q-Bessel Functions

By following techniques of [25] we focus our attention in this part of this paper to the first type q-Natural transform.
By aid of Equation 20 and Equation 11, the N_q transform can be written in terms of a series expansion as

$$N_q(f(t))(u;v) = \frac{1}{v} \sum_{k=0}^{\infty} q^k f\left(\frac{u}{v} q^k\right) E_q\left(q^{k+1}\right).$$

Hence, the parity of Equation 10 and Equation 8 puts the transform N_q into a generic form as

$$N_q(f(t))(u;v) = \frac{(q;q)_\infty}{v} \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k} f\left(q^k \frac{u}{v}\right). \tag{28}$$

Hereafter, in this section, we estimate some values of the N_q transform of certain class of q-Bessel functions.

Theorem 2. Let $J_{2\mu_1}^{(1)}\left(2\sqrt{a_1 t};q\right), ..., J_{2\mu_n}^{(1)}\left(2\sqrt{a_n t};q\right)$ be a set of first kind q-Bessel functions, $f(t) = t^{\Delta-1} \prod_{j=1}^{n} J_{2\mu_j}^{(1)}\left(2\sqrt{a_j t};q\right)$ and $B_\Delta = \frac{(1-q)^\Delta}{(q;q)_\infty} \frac{u^{\Delta-1}}{v^\Delta}$. Then, the q-Natural transform N_q of $f(t)$ is given as

$$N_q(f(t))(u;v) = B_\Delta \prod_{j=1}^{n} \sum_{m_j=0}^{\infty} (1-q)^{2\mu_j+m_j-1} \frac{\left(a_j^{\mu_j+m_j}\right)\left(q^{2\mu_j+m_j+1};q\right)_\infty}{(q;q)_{m_j}} \left(\frac{u}{v}\right)^{\mu_j+m_j} \Gamma_q\left(2\mu_j + m_j - 1\right).$$

Proof On aid of Equation 28 and Equation 2 we write

$$\begin{aligned} N_q(f(t))(u;v) &= \frac{(q;q)_\infty}{v}\sum_{k=0}^{\infty}\frac{q^k}{(q;q)_k}\left(q^k\frac{u}{v}\right)^{\Delta-1}\prod_{j=1}^{n}J_{2\mu_j}^{(1)}\left(2\sqrt{a_n q^k\frac{u}{v}};q\right) \\ &= \frac{(q;q)_\infty}{v}\sum_{k=0}^{\infty}\frac{q^k}{(q;q)_k}\left(q^k\frac{u}{v}\right)^{\Delta-1}\prod_{j=1}^{n}\left(a_j q^k\frac{u}{v}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{\left(a_j q^k\frac{u}{v}\right)^{m_j}}{(q;q)_{2\mu_j+m_j}(q;q)_{m_j}}. \end{aligned}$$

Hence,

$$N_q(f(t))(u;v) = \frac{u^{\Delta-1}(q;q)_\infty}{v^\Delta}\prod_{j=1}^{n}\left(a_j\frac{u}{v}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{\left(a_j\frac{u}{v}\right)^{m_j}}{(q;q)_{2\mu_j+m_j}(q;q)_{m_j}}\times\sum_{k=0}^{\infty}\frac{q^{k(\Delta+\mu_j+m_j)}}{(q;q)_k}. \tag{29}$$

By using Equation 26, Equation 29 gives

$$N_q(f(t))(u;v) = A\prod_{j=1}^{n}\left(a_j\frac{u}{v}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{\left(a_j\frac{u}{v}\right)^{m_j}}{(q;q)_{2\mu_j+m_j}(q;q)_{m_j}}(1-q)^{\mu_j+m_j}\Gamma_q\left(\Delta+\mu_j+m_j\right),$$

where

$$A = (1-q)^{\Delta-1}\frac{u^{\Delta-1}}{v^\Delta}.$$

Hence, on taking into account Equation 8, we get

$$N_q(f(t))(u;v) = B_\Delta\prod_{j=1}^{n}\left(a_j\frac{u}{v}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{\left(a_j\frac{u}{v}\right)^{m_j}(q^{2\mu_j+m_j+1};q)_\infty}{(q;q)_{m_j}}(1-q)^{\mu_j+m_j}\Gamma_q\left(\Delta+\mu_j+m_j\right),$$

where $B_\Delta = \frac{A}{(q;q)_\infty}$.
Or, equivalently,

$$N_q(f(t))(u;v) = B_\Delta\prod_{j=1}^{n}\sum_{m_j=0}^{\infty}(1-q)^{2\mu_j+m_j-1}\frac{\left(a_j^{\mu_j+m_j}\right)(q^{2\mu_j+m_j+1};q)_\infty}{(q;q)_{m_j}}\left(\frac{u}{v}\right)^{\mu_j+m_j}\Gamma_q\left(\mu_j+m_j-1\right),$$

where $B_\Delta = \frac{(1-q)^\Delta}{(q;q)_\infty}\frac{u^{\Delta-1}}{v^\Delta}$.
This completes the proof of the theorem.

We apply now the N_q transform to a family of $J_\mu^{(2)}$ Bessel functions.

Theorem 3. Let $J_{2\mu_1}^{(2)}\left(2\sqrt{a_1 t};q\right),...,J_{2\mu_n}^{(2)}\left(2\sqrt{a_n t};q\right)$ be a set of second order q-Bessel functions, $f(t) = t^{\Delta-1}\prod_{j=1}^{n}J_{2\mu_j}^{(2)}\left(2\sqrt{a_n t};q\right)$ and $B_{\hat\Delta} = \frac{u^{\Delta-1}}{v^\Delta}\frac{(1-q)^{\Delta-1}}{(q;q)_\infty}$. Then, the N_q transform of $f(t)$ is given as

$$N_q(f(t))(u;v) = B_{\hat\Delta}\prod_{j=1}^{n}\sum_{m_j=0}^{\infty}\frac{q^{m_j\left(m_j+2\mu_j\right)}\left(a_j\right)^{\mu_j+m_j}(q^{2\mu_j+m_j+1};q)_\infty}{(q;q)_{m_j}}(1-q)^{\mu_j+m_j}\Gamma_q\left(\Delta+\mu_j+m_j\right)\frac{u}{v}.$$

Proof On a ccount of Equation 28, we obtain that

$$N_q\left(f(t)\right)(u;v)=\frac{(q;q)_\infty}{v}\sum_{k=0}^{\infty}\frac{q^k}{(q;q)_k}f\left(q^k\frac{u}{v}\right)=\frac{(q;q)_\infty}{v}\sum_{k=0}^{\infty}\frac{q^k}{(q;q)_k}\left(\frac{u}{v}q^k\right)^{\Delta-1}\prod_{j=1}^{n}J^{(2)}_{2\mu_j}\left(2\sqrt{a_j\frac{u}{v}q^k};q\right). \tag{30}$$

By invoking Equation 3 in Equation 30, we find that

$$N_q\left(f(t)\right)(u;v)=(q;q)_\infty\frac{u^{\Delta-1}}{v^\Delta}\sum_{k=0}^{\infty}\frac{q^{k\Delta}}{(q;q)_k}\prod_{j=1}^{n}\left(a_j\frac{u}{v}\right)^{\mu_j}q^{k\mu_j}\sum_{m_j=0}^{\infty}\frac{q^{\left(m_j+2\mu_j\right)m_j}\left(a_j\frac{u}{v}q^k\right)^{m_j}}{(q;q)_{2\mu_j+m_j}(q;q)_{m_j}}. \tag{31}$$

By employing Equation 8, Equation 31 can be put into the form

$$N_q\left(f(t)\right)(u;v)=\frac{u^{\Delta-1}}{v^\Delta}\prod_{j=1}^{n}\sum_{m_j=0}^{\infty}\frac{q^{m_j\left(m_j+2\mu_j\right)}\left(a_j\frac{u}{v}\right)^{\mu_j+m_j}(q^{2\mu_j+m_j+1};q)_\infty}{(q;q)_{m_j}}\sum_{k=0}^{\infty}\frac{q^{k\left(\Delta+\mu_j+m_j\right)}}{(q;q)_k}. \tag{32}$$

Finally, on aid of Equation 26, Equation 32 fairly gives

$$N_q\left(f(t)\right)(u;v) = B_{\hat{\Delta}}\prod_{j=1}^{n}\sum_{m_j=0}^{\infty}\frac{q^{m_j\left(m_j+2\mu_j\right)}\left(a_j\right)^{\mu_j+m_j}(q^{2\mu_j+m_j+1};q)_\infty}{(q;q)_{m_j}}(1-q)^{\mu_j+m_j}\Gamma_q\left(\Delta+\mu_j+m_j\right)\frac{u}{v},$$

where

$$B_{\hat{\Delta}}=\frac{u^{\Delta-1}}{v^\Delta}\frac{(1-q)^{\Delta-1}}{(q;q)_\infty}.$$

This completes the proof of the theorem.

As final in this section, we apply N_q transform to a class of third q-Bessel functions.

Theorem 4. Let $J^{(3)}_{2\mu_1}\left(\sqrt{q^{-1}a_1t};q\right),...,J^{(3)}_{2\mu_n}\left(\sqrt{q^{-1}a_nt};q\right)$ be n q-Bessel functions, $f(t)=t^{\Delta-1}\prod_{j=1}^{n}q^{\mu_j(3)}_{2\mu_j}\left(\sqrt{q^{-1}a_nt};q\right)$ and $B_{\hat{\Delta}}=\frac{u^{\Delta-1}}{v^\Delta}\frac{(1-q)^{\Delta-1}}{(q;q)_\infty}$. Then, we have

$$N_q\left(f(t)\right)(u;v) = B_{\hat{\Delta}}\prod_{j=1}^{n}\sum_{m_j=0}^{\infty}(-1)^{m_j}\frac{q^{m_j\left(\frac{m_j-1}{2}\right)}a_j^{\mu_j+m_j}}{(q;q)_{m_j}}(q^{2\mu_j+m_j+1};q)_\infty(1-q)^{\mu_j+m_j}\Gamma_q\left(\Delta+\mu_j+m_j\right)\left(\frac{u}{v}\right)^{\mu_j+m_j}.$$

Proof Similarly, by Equation 30 and Equation 6 and direct computations we write

$$\begin{aligned}N_q\left(f(t)\right)(u;v) &= \frac{(q;q)_\infty}{v}\sum_{k=0}^{\infty}\frac{q^k}{(q;q)_k}f\left(q^k\frac{u}{v}\right)\\ &= \frac{(q;q)_\infty}{v}\sum_{k=0}^{\infty}\frac{q^k}{(q;q)_k}\left(\frac{u}{v}q^k\right)^{\Delta-1}\prod_{j=1}^{n}q^{\mu_j}J^{(3)}_{2\mu_j}\left(\sqrt{a_jq^{k-1}};q\right)\\ &= (q;q)_\infty\frac{u^{\Delta-1}}{v^\Delta}\sum_{k=0}^{\infty}\frac{q^{k\Delta}}{(q;q)_k}\prod_{j=1}^{n}q^{\mu_j}\left(a_j\frac{u}{v}q^{k-1}\right)^{\mu_j}\sum_{m_j=0}(-1)\frac{\left(a_j\frac{u}{v}q^{k-1}\right)^{\mu_j}}{(q;q)_{2\mu_j+m_j}(q;q)_{m_j}}\\ &= (q;q)_\infty\frac{u^{\Delta-1}}{v^\Delta}\sum_{k=0}^{\infty}\frac{q^{k\Delta}}{(q;q)_k}\prod_{j=1}^{n}\left(a_j\frac{u}{v}\right)^{\mu_j}q^{k\mu_j}\sum_{m_j=0}^{\infty}\frac{(-1)^{m_j}q^{m_j\left(\frac{m_j-1}{2}\right)}\left(a_j\frac{u}{v}q^k\right)^{m_j}}{(q;q)_{2\mu_j+m_j}(q;q)_{m_j}}.\end{aligned}$$

Hence

$$N_q\left(f\left(t\right)\right)\left(u;v\right)=\left(q;q\right)_\infty\frac{u^{\Delta-1}}{v^{\Delta}}\prod_{j=1}^{n}\sum_{m_j=0}^{\infty}\frac{\left(-1\right)^{m_j}q^{m_j\left(\frac{m_j-1}{2}\right)}\left(a_j\frac{u}{v}\right)^{\mu_j+m_j}}{\left(q;q\right)_{2\mu_j+m_j}\left(q;q\right)_{m_j}}\sum_{k=0}^{\infty}\frac{q^{k\left(\Delta\mu_j+m_j\right)}}{\left(q;q\right)_k}. \tag{33}$$

By further use of Equation 8 and Equation 26, Equation 33 finally yields

$$N_q\left(f\left(t\right)\right)\left(u;v\right)=B_{\hat{\Delta}}\prod_{j=1}^{n}\sum_{m_j=0}\left(-1\right)^{m_j}\frac{q^{m_j\left(\frac{m_j-1}{2}\right)}a_j^{\mu_j+m_j}}{\left(q;q\right)_{m_j}}\left(q^{2\mu_j+m_j+1};q\right)_\infty\left(1-q\right)^{\mu_j+m_j}\Gamma_q\left(\Delta+\mu_j+m_j\right)\left(\frac{u}{v}\right)^{\mu_j+m_j},$$

where

$$B_{\hat{\Delta}}=\frac{u^{\Delta-1}}{v^{\Delta}}\frac{\left(1-q\right)^{\Delta-1}}{\left(q;q\right)_\infty}.$$

This completes the proof of the theorem.

4. ${}_qN$ Transform of q-Bessel Functions

In this section of this article, we focuss our attention to the second type q-Natural transform. The series representation of ${}_qN$ transform can be derived from Equation 21 and Equation 11 as

$${}_qN\left(f\left(t\right)\right)\left(u;v\right)=\sum_{k\in\mathbb{Z}}\frac{q^kf\left(q^k\right)}{\left(-\frac{u}{v}q^k;q\right)_\infty}.$$

Hence, by Equation 8, the above representation can be expressed as follows

$${}_qN\left(f\left(t\right)\right)\left(u;v\right)=\frac{1}{\left(-\frac{u}{v};q\right)_\infty}\sum_{k\in\mathbb{Z}}\left(-\frac{u}{v};q\right)_k q^kf\left(q^k\right). \tag{34}$$

We establish the following theorem.

Theorem 5. Let $J_{2\mu_j}^{(2)}\left(2\sqrt{a_jt};q\right),...,J_{2\mu_n}^{(2)}\left(2\sqrt{a_nt};q\right)$ be n q-Bessel functions of type 2, $B_{\Delta}^{q}=\dfrac{\left(1-q\right)^{\Delta-1}u^{\Delta}}{\left(q;q\right)_\infty v^{\Delta}}$ and $f\left(t\right)=t^{\Delta-1}\prod_{j=1}^{n}J_{2\mu_j}^{(2)}\left(2\sqrt{a_jt};q\right)$. Then, we have

$$\begin{aligned}{}_qN\left(f\left(t\right)\right)\left(u;v\right)&=B_{\Delta}^{q}\prod_{j=1}^{n}\sum_{m_j=0}\left(-1\right)^{m_j}\frac{a_j^{\mu_j+m_j}q^{m_j\left(\mu_j+m_j\right)}\left(1-q\right)^{\left(\mu_j+m_j\right)}}{k\left(\frac{u}{v};\Delta+\mu_j+m_j\right)}\\&\quad\times_q\Gamma\left(\Delta+\mu_j+m_j\right)\left(\frac{u}{v}\right)^{\mu_j+m_j}.\end{aligned}$$

Proof Taking into account Equation 34 and Equation 3, one can write

$$\begin{aligned}{}_qN\left(f\left(t\right)\right)\left(u;v\right)&=\frac{1}{\left(-\frac{v}{u};q\right)_\infty}\sum_{k\in\mathbb{Z}}q^k\left(-\frac{u}{v};q\right)_k f\left(q^k\right)\\&=\frac{1}{\left(-\frac{v}{u};q\right)_\infty}\sum_{k\in\mathbb{Z}}q^k\left(-\frac{u}{v};q\right)_k\left(q^k\right)^{\Delta-1}\prod_{j=1}^{n}J_{2\mu_j}^{(2)}\left(2\sqrt{a_jq^k};q\right)\\&=\frac{1}{\left(-\frac{v}{u};q\right)_\infty}\sum_{k\in\mathbb{Z}}\left(-\frac{u}{v};q\right)_k q^{k\Delta}\prod_{j=1}^{n}\left(a_jq^k\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{q^{m_j\left(\mu_j+m_j\right)}\left(-a_jq^k\right)^{m_j}}{\left(q;q\right)_{2\mu_j+m_j}\left(q;q\right)_{m_j}}.\end{aligned}$$

Hence, by using Equation 8, we obtain

$$ {}_qN\left(f(t)\right)(u;v)=\frac{1}{\left(-\frac{v}{u};q\right)_\infty}\prod_{j=1}^{n}\sum_{m_j=0}^{\infty}(-1)^{m_j}\frac{\left(a_j\right)^{\left(\mu_j+m_j\right)}q^{m_j\left(\mu_j+m_j\right)}}{(q;q)_{m_j}(q;q)_\infty}\left(q^{2\mu_j+m_j+1};q\right)_\infty\sum_{k\in\mathbb{Z}}q^{k\left(\Delta+\mu_j+m_j\right)}\left(-\frac{v}{u};q\right)_k. \quad (35) $$

By aid of Equation 27 and by setting $A=\dfrac{u}{v}$ and $\alpha=\Delta+\mu_j+m_j$, we put Equation 35 into the form

$$ \begin{aligned} {}_qN\left(f(t)\right)(u;v)&=\frac{1}{\left(-\frac{v}{u};q\right)_\infty}\prod_{j=1}^{n}\sum\nolimits_{m_j=0}^{\infty}(-1)^{m_j}\frac{\left(a_j\right)^{\left(\mu_j+m_j\right)}q^{m_j\left(\mu_j+m_j\right)}}{(q;q)_{m_j}(q;q)_\infty}\\ &\quad\times\frac{\left(q^{m_j\left(\mu_j+m_j\right)};q\right)_\infty(1-q)^{\Delta+\mu_j+m_j-1}\left(-\frac{u}{v};q\right)_\infty\Gamma\left(\Delta+\mu_j+m_j\right)\left(\frac{v}{u}\right)^{\Delta+\mu_j+m_j}}{K\left(\frac{u}{v};\Delta+\mu_j+m_j\right)}\\ &=\frac{(1-q)^{\Delta-1}u^{\Delta}}{(q;q)_\infty v^{\Delta}}\prod_{j=1}^{n}\sum\nolimits_{m_j=0}^{\infty}(-1)^{m_j}\frac{\left(a_j\right)^{\left(\mu_j+m_j\right)}q^{m_j\left(2m_j+2\mu_j\right)}(1-q)^{\left(\mu_j+m_j\right)}\Gamma\left(\Delta+\mu_j+m_j\right)}{K\left(\frac{u}{v};\Delta+\mu_j+m_j\right)}\left(\frac{u}{v}\right)^{\mu_j+m_j}. \end{aligned} $$

Hence the theorem is proved.

Theorem 6. Let $J^{(3)}_{2\mu_1}\left(2\sqrt{q^{-1}a_jt};q\right),\ldots,J^{(3)}_{2\mu_n}\left(2\sqrt{q^{-1}a_nt};q\right)$ be n class of third q-Bessel functions, $f(t)=t^{\Delta-1}\prod_{j=1}^{n}q^{m_j}J^{(3)}_{2\mu_j}\left(2\sqrt{q^{-1}a_jt};q\right)$ and $B^q_\Delta=\dfrac{(1-q)^{\Delta-1}u^{\Delta}}{(q;q)_\infty v^{\Delta}}$. Then, we have

$$ {}_qN\left(f(t)\right)(u;v)=B^q_\Delta\prod_{j=1}^{n}\sum_{m_j=0}^{\infty}(-1)^{m_j}\left(a_j\right)^{\left(\mu_j+m_j\right)}q^{\frac{m_j\left(\mu_j+m_j\right)}{2}}\left(q^{m_j\left(\mu_j+m_j\right)};q\right)_\infty\frac{(1-q)^{\left(\mu_j+m_j\right)}\Gamma\left(\Delta+\mu_j+m_j\right)}{K\left(\frac{u}{v};\Delta+\mu_j+m_j\right)}\left(\frac{u}{v}\right)^{\mu_j+m_j}. $$

Proof On taking account of Equation 34 and Equation 6 we write

$$ \begin{aligned} {}_qN\left(f(t)\right)(u;v)&=\frac{1}{\left(-\frac{v}{u};q\right)_\infty}\sum_{k\in\mathbb{Z}}q^k\left(-\frac{v}{u};q\right)_k f\left(q^k\right)\\ &=\frac{1}{\left(-\frac{v}{u};q\right)_\infty}\sum_{k\in\mathbb{Z}}q^k\left(-\frac{v}{u};q\right)_k q^{k\Delta-k}\prod_{j=1}^{n}q^{\mu_j}J^{(3)}_{2\mu_j}\left(2\sqrt{a_jq^{k-1}};q\right)_\infty\\ &=\frac{1}{\left(-\frac{v}{u};q\right)_\infty}\sum_{k\in\mathbb{Z}}q^{k\Delta}\left(-\frac{v}{u};q\right)_k\prod_{j=1}^{n}q^{\mu_j}\left(a_jq^{k-1}\right)\mu_j\sum_{m_j=0}^{\infty}(-1)^{m_j}\frac{q^{\frac{m_j\left(m_j-1\right)}{2}}\left(q^ka_j\right)^{m_j}}{(q;q)_{2\mu_j+m_j}(q;q)_{m_j}}. \end{aligned} $$

In view of Equation 8, the above equation gives

$$ {}_qN\left(f(t)\right)(u;v)=\frac{1}{\left(-\frac{v}{u};q\right)_\infty(q;q)_\infty}\prod_{j=1}^{n}\sum_{m_j=0}^{\mu_j}(-1)^{m_j}\frac{\left(a_j\right)^{\left(\mu_j+m_j\right)}\left(q^{2\mu_j+m_j+1};q\right)}{(q;q)_{m_j}}q^{\frac{m_j\left(m_j-1\right)}{2}}\sum_{k\in\mathbb{Z}}q^{k\left(\Delta+\mu_j+m_j\right)}\left(-\frac{v}{u};q\right)_k $$

As in the previous theorem, use Equation 27, for $A=\frac{u}{v}$ and $\alpha=\Delta+\mu_j+m_j$, to get

$$ {}_qN\left(f(t)\right)(u;v)=B^q_\Delta\prod_{j=1}^{n}\sum_{m_j=0}^{\infty}(-1)^{m_j}\frac{\left(a_j\right)^{\left(\mu_j+m_j\right)}q^{\frac{m_j\left(m_j-1\right)}{2}}\left(q^{2\mu_j+m_j+1};q\right)}{K\left(\frac{u}{v};\Delta+\mu_j+m_j\right)}(1-q)^{\left(\mu_j+m_j\right)}\Gamma\left(\Delta+\mu_j+m_j\right)\times\left(\frac{v}{u}\right)^{\mu_j+m_j}, $$

where

$$B_{\Delta}^{q}=\frac{(1-q)^{\Delta-1}\,u^{\Delta}}{(q;q)_{\infty}\,v^{\Delta}}.$$

This completes the proof of the theorem.

5. Examples

In this final section we mainly give some corollaries and applications to q-Sumudu and q-Laplace transforms.
Corollary 7. Let $J_1^{(1)}$ be a Bessel function of first type. Then, we have

$$N_q\left(t^{\Delta-1}J_1^{(1)}\left(2\sqrt{at};q\right)\right)(u;v)=\frac{(1-q)^{\Delta}\,u^{\Delta-1}}{v^{\Delta}}\sum_{m=0}^{\infty}\frac{(1-q)^m}{(q^m;q)_2\,(q^m;q)_m}\left(\frac{au}{v}\right)^m.$$

Proof While putting $n=1$ and $\mu=\frac{1}{2}$ in Theorem 2, we get

$$N_q\left(t^{\Delta-1}J_1^{(1)}\left(2\sqrt{at};q\right)\right)(u;v)=B_{\Delta}\left(\frac{u}{v}\right)^{\frac{1}{2}}\sum_{m=0}^{\infty}\frac{(1-q)^m\left(q^{m+2};q\right)_{\infty}}{(q;q)_m}\Gamma_q(m)\left(\frac{au}{v}\right)^m.$$

By using Equation 13, we obtain

$$N_q\left(t^{\Delta-1}J_1^{(1)}\left(2\sqrt{at};q\right)\right)(u;v)=B_{\Delta}\left(\frac{u}{v}\right)^{\frac{1}{2}}\sum\frac{(1-q)^m\,(q^m;q)_{\infty}}{(q^m;q)\,(q;q)_m}\Gamma_q(m)\left(\frac{au}{v}\right)^m.$$

By Equation 26, we assert

$$\begin{aligned}N_q\left(t^{\Delta-1}J_1^{(1)}\left(2\sqrt{at};q\right)\right)(u;v) &= B_{\Delta}\left(\frac{au}{v}\right)^{\frac{1}{2}}\sum_{m=0}^{\infty}\frac{(1-q)^m}{(q^m;q)_2\,(q;q)_m}(q;q)_{\infty}(1-q)^{m-1}\left(\frac{au}{v}\right)^m\\ &= B_{\Delta}\left(\frac{au}{v}\right)^{\frac{1}{2}}(q;q)_{\infty}\sum_{m=0}^{\infty}\frac{\left((1-q)^2\right)^m}{(q^m;q)_2\,(q;q)_m}\\ &= \frac{\left((1-q)^{\Delta-2}\right)u^{\Delta-1}}{v^{\Delta}}\sum_{m=0}^{\infty}\frac{\left((1-q)^2\right)^m}{(q^m;q)_2\,(q;q)_m}\left(\frac{au}{v}\right)^m.\end{aligned}$$

This completes the proof of the theorem.

Hence from Corollary 7 we state without proof the following result.

Corollary 8. Let $J_1^{(1)}$ be a Bessel function of first type. Then, we have

(i) $L_q\left(t^{\Delta-1}J_1^{(1)}\left(2\sqrt{at};q\right)\right)(v)=\frac{(1-q)^{\Delta}}{v^{\Delta}}\sum_{m=0}^{\infty}\frac{(1-q)^{2m}}{(q^m;q)_2\,(q;q)_m}\left(\frac{a}{v}\right)^m,$

(ii) $S_q\left(t^{\Delta-1}J_1^{(1)}\left(2\sqrt{at};q\right)\right)(u)=(1-q)^{\Delta}\,u^{\Delta-1}\sum_{m=0}^{\infty}\frac{(1-q)^{2m}}{(q^m;q)_2\,(q;q)_m}(au)^m.$

Proof follows from Corollary 7 and Equation 22 and Equation 33. Hence, details are omitted.

Corollary 9. Let $J_1^{(2)}$ be a Bessel function of second type. Then we have

$N_q\left(t^{\Delta-1}J_1^{(2)}\left(2\sqrt{at};q\right)\right)(u;v)=\left(\frac{au}{v}\right)^{\frac{1}{2}}\frac{u^{\Delta-1}}{(1-q)^{\frac{1}{2}}v^{\Delta}}\sum_{m=0}^{\infty}\frac{q^{m(m+1)}(q^m;q)_\infty}{(q^m;q)_2(q^{\Delta+m+1};q)_\infty}\left(\frac{u}{v}\right)^m.$

Proof By assuming $n=1,\mu=\frac{1}{2}$ in Theorem 4, we find that

$$\begin{aligned}N_q\left(t^{\Delta-1}J_1^{(2)}\left(2\sqrt{at};q\right)\right)(u;v) &= B_\Delta\sum_{m=0}^{\infty}q^{m(m+1)}q^{m+\frac{1}{2}}\left(q^{m+2},q\right)_\infty\frac{(1-q)^{\frac{1}{2}+m}\Gamma_q(\Delta+m+1)}{(q;q)_m}\left(\frac{u}{v}\right)^{\frac{1}{2}+m}\\ &= (1-q)^{\frac{1}{2}}\left(\frac{u}{v}\right)^{\frac{1}{2}}B_\Delta\sum_{m=0}^{\infty}q^{m(m+1)}\left(q^{m+2};q\right)_\infty\frac{(1-q)^{\frac{1}{2}+m}\Gamma_q(\Delta+m+1)}{(q;q)_m}\left(\frac{u}{v}\right)^m.\end{aligned}$$

A direct use of Equation 13 and Equation 26 gives

$$\begin{aligned}N_q\left(t^{\Delta-1}J_1^{(2)}\left(2\sqrt{at};q\right)\right)(u;v) &= (1-q)^{\frac{1}{2}}\left(\frac{au}{v}\right)^{\frac{1}{2}}B_\Delta\sum_{m=0}^{\infty}q^{m(m+1)}\frac{(q^m;q)_\infty(1-q)^m(q;q)_\infty(1-q)^{-(\Delta+m)}}{(q^m;q)_2(q^{\Delta+m+1};q)_\infty}\left(\frac{u}{v}\right)^m\\ &= \left(\frac{au}{v}\right)^{\frac{1}{2}}\frac{u^{\Delta-1}}{(1-q)^{\frac{1}{2}}v^{\Delta}}\sum_{m=0}^{\infty}\frac{q^{m(m+1)}(q^m,q)_\infty}{(q^m,q)_2}\left(q^{\Delta+m+1};q\right)_\infty\left(\frac{u}{v}\right)^m.\end{aligned}$$

This completes the proof of the corollary.

In terms of q-Sumudu and q-Laplace transforms we state without proof the following result.

Corollary 10. The following hold.

(i) $L_q\left(t^{\Delta-1}J_1^{(2)}\left(2\sqrt{at};q\right)\right)(v)=\frac{a^{\frac{1}{2}}}{(1-q)^{\frac{1}{2}}v^{\Delta+\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{q^{m(m+1)}(q^m;q)_\infty}{(q^m;q)_2(q^{\Delta+m+1};q)_\infty}\left(\frac{1}{v}\right)^m,$

(ii) $S_q\left(t^{\Delta-1}J_1^{(2)}\left(2\sqrt{at};q\right)\right)(u)=\frac{a^{\frac{1}{2}}u^{\Delta-\frac{1}{2}}}{(1-q)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{q^{m(m+1)}(q^m;q)_\infty}{(q^m;q)_2(q^{\Delta+m+1};q)_\infty}u^m.$

This corollary follows from Corollary 9. Hence, we omit the details.

Following results are straightforward corollaries from Theorem 5 and 6, respectively. We therefore omit the details.

Corollary 11. Let $J_{2\mu_j}^{(2)}\left(2\sqrt{a_jt};q\right),...,J_{2\mu_n}^{(2)}\left(2\sqrt{a_nt};q\right)$ be n q-Bessel functions of type 2, $f(t)=t^{\Delta-1}\prod_{j=1}^{n}J_{2\mu_j}^{(2)}\left(2\sqrt{a_jt};q\right)$ and $B_\Delta^q=\frac{(1-q)^{\Delta-1}}{(q;q)_\infty v^{\Delta}}$. Then, the following hold .

(i) ${}_qL(f(t))(v)=B_\Delta^q\prod_{j=1}^{n}\sum_{m_j=0}^{\infty}(-1)^{m_j}\frac{a_j^{\mu_j+m_j}q^{m_j(\mu_j+m_j)}(1-q)^{(\mu_j+m_j)}}{k\left(\frac{1}{v};\Delta+\mu_j+m_j\right)}{}_q\Gamma\left(\Delta+\mu_j+m_j\right)\left(\frac{1}{v}\right)^{\mu_j+m_j},$

(ii) ${}_qS(f(t))(u)=B_\Delta^q\prod_{j=1}^{n}\sum_{m_j=0}^{\infty}(-1)^{m_j}\frac{a_j^{\mu_j+m_j}q^{m_j(\mu_j+m_j)}(1-q)^{(\mu_j+m_j)}}{k\left(u;\Delta+\mu_j+m_j\right)}{}_q\Gamma\left(\Delta+\mu_j+m_j\right)(u)^{\mu_j+m_j}.$

Corollary 12. Let $J_{2\mu_1}^{(3)}\left(2\sqrt{q^{-1}a_jt};q\right),...,J_{2\mu_n}^{(3)}\left(2\sqrt{q^{-1}a_nt};q\right)$ be n class of third q-Bessel functions, $f(t)=t^{\Delta-1}\prod_{j=1}^{n}q^{m_j}J_{2\mu_j}^{(3)}\left(2\sqrt{q^{-1}a_jt};q\right)$ and $B_\Delta^q=\frac{(1-q)^{\Delta-1}u^{\Delta}}{(q;q)_\infty v^{\Delta}}$. Then, we have

(i) ${}_qN(f(t))(u;v)=B_\Delta^q\prod_{j=1}^{n}\sum_{m_j=0}^{\infty}(-1)^{m_j}\left(a_j\right)^{(\mu_j+m_j)}q^{\frac{m_j(\mu_j+m_j)}{2}}\left(q^{m_j(\mu_j+m_j)};q\right)_\infty\frac{(1-q)^{(\mu_j+m_j)}\Gamma\left(\Delta+\mu_j+m_j\right)}{K\left(\frac{u}{v};\Delta+\mu_j+m_j\right)}\left(\frac{u}{v}\right)^{\mu_j+m_j},$

(ii) ${}_qN(f(t))(u;v)=B_\Delta^q\prod_{j=1}^{n}\sum_{m_j=0}^{\infty}(-1)^{m_j}\left(a_j\right)^{(\mu_j+m_j)}q^{\frac{m_j(\mu_j+m_j)}{2}}\left(q^{m_j(\mu_j+m_j)};q\right)_\infty\frac{(1-q)^{(\mu_j+m_j)}\Gamma\left(\Delta+\mu_j+m_j\right)}{K\left(\frac{u}{v};\Delta+\mu_j+m_j\right)}\left(\frac{u}{v}\right)^{\mu_j+m_j}.$

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