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Extrapolation of Convex Functions

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Abstract. The idea of the well known Jensen inequality is to interpolate convex functions, by finding an upper bound of the function at a point in the convex hull of predefined points. In this article, we present a counterpart of this inequality by giving a lower bound of the function outside this convex hull. This inequality is then refined by finding as many refining positive terms as we wish. Some applications treating means, integrals and eigenvalues are given in the end.

Moreover, we present a MATLAB code that helps generate the parameters appearing in our results.

1. Introduction

The celebrated Jensen inequality states that

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i f(x_i), \tag{1}$$

for the convex function $f : \mathbb{I} \to \mathbb{R}$ and the convex sequence $\{p_i\}$, where $\{x_i\} \subset \mathbb{I}$; the interval of convexity of f.

Research related to this inequality includes obtaining new inequalities and refining existing ones. For example, (1) was refined and reversed in [6] as follows

Lemma 1.1. *Let* $f : \mathbb{I} \longrightarrow \mathbb{R}$ *be convex,* $\{x_1, \cdots, x_n\} \subset \mathbb{I}$ *and* $\{p_1, \cdots, p_n\} \subset (0, 1)$ *be such that* $\sum_{i=1}^{n} p_i = 1$. *Then*

$$\begin{aligned} np_{\min}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right) &\le \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \\ &\le np_{\max}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right), \end{aligned}$$

where $p_{\min} = \min\{p_1, \cdots, p_n\}$ *and* $p_{\max} = \max\{p_1, \cdots, p_n\}$.

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In the recent paper [8], a refinement and reverse of Jensen's inequality were proved, with as many refining terms as we wish.

Notice that when $\{p_i\}$ is a convex sequence and $\{x_i\} \subset \mathbb{I}$, we have $\sum_{i=1}^{n} p_i x_i \in [x_i]$; the convex hull of the set $\{x_i\}$. Thus, Jensen's inequality can be thought of as an inequality that gives an upper bound of the convex function on the convex hull of predefined points.

For the rest of the paper, we adopt the following notations. We use the notation $\{\alpha_i\} > 0$ to mean $\alpha_i > 0$ for all i. By convention, our finite sets will have n elements, for some $n \in \mathbb{N}$. We cite the following result from [10], which gives a reversed version of Jensen's inequality. A simple proof is given for completeness.

Theorem 1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be convex, $a \in \mathbb{R}$ and $\{\alpha_i\} > 0$. If $\{b_i\} \subset \mathbb{R}$ then*

$$(1+\beta)f(a) - \sum_{i=1}^{n} \alpha_i f(b_i) \le f\left((1+\beta)\,a - \sum_{i=1}^{n} \alpha_i b_i\right), \tag{2}$$

where $\beta = \sum\limits_{i=1}^{n} \alpha_i$.

Proof. Notice first that for $t \ge 0$, one has

$$a = \frac{t}{t+1} b + \frac{1}{t+1}((1+t)a - tb).$$

Convexity of f implies $f(a) \le \frac{t}{t+1} f(b) + \frac{1}{t+1} f((1+t)a - tb)$, which implies

$$(1+t)f(a) - f((1+t)a - tb) \le t f(b). \tag{3}$$

Now, applying (3), we have

$$\begin{aligned}
(1+\beta)f(a) \quad - &\quad f\left((1+\beta)a - \sum_{i=1}^{n} \alpha_i b_i\right) \\
= &\quad (1+\beta)f(a) - f\left((1+\beta)a - \beta \sum_{i=1}^{n} \frac{\alpha_i}{\beta} b_i\right) \\
\le &\quad \beta f\left(\sum_{i=1}^{n} \frac{\alpha_i}{\beta} b_i\right) \\
\le &\quad \sum_{i=1}^{n} \alpha_i f(b_i),
\end{aligned}$$

by Jensen's inequality. This completes the proof. □

But then, if Jensen's inequality can be refined as in Lemma 1.1 and as in [8], it is fair to ask whether we have a refinement of (2). This will be our main target of this paper; to find as many refining terms of (2) as we wish! The main theorem in this direction will be

Theorem 1.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be convex, $a \in \mathbb{R}$, $\{\alpha_i^{(1)} : 1 \le i \le n\} \subset (0, \infty)$, and $\{b_i^{(1)} : 1 \le i \le n\} \subset \mathbb{R}$. Then, for $N \in \mathbb{N}$,*

$$\begin{aligned}
&(1+\beta^{(1)})f(a) - \sum_{i=1}^{n} \alpha_i^{(1)} f\left(b_i^{(1)}\right) \\
&+ (n+1) \sum_{k=1}^{N} \alpha_{\min}^{(k)} \left(\frac{f(a) + \sum_{i=1}^{n} f\left(b_i^{(k)}\right)}{n+1} - f\left(\frac{a + \sum_{i=1}^{n} b_i^{(k)}}{n+1}\right)\right) \\
&\le f\left((1+\beta^{(1)})a - \sum_{i=1}^{n} \alpha_i^{(1)} b_i^{(1)}\right),
\end{aligned} \tag{4}$$

where $\beta^{(1)} = \sum_{i=1}^n \alpha_i^{(1)}$ *and for certain* $\alpha_{\min}^{(k)}, b_i^{(k)}$.

See (7) and (8) for the exact definition of the parameters appearing in this theorem and see Theorem 2.3 for the details and the proof of this result.

The above result implies and generalizes our recent main result in [7].

The organization of the paper will be as follows. In the first section we prove some negative Jensen-type inequalities leading to the main refinement (4). In the end, we present some applications of these refinements. Furthermore, we present a MATLAB code that generates the parameters appearing in our main result.

2. Main Results

In this part of the paper, we present our main results about convex functions. First, we have the following simple consequence of Theorem 1.2; an odd behavior of convex funtions.

Corollary 2.1. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be convex and* $\{\alpha_i\}>0$ *be such that* $\sum_{i=1}^n \alpha_i = \beta$. *Then*

$$(1+\beta)f(0) - \sum_{i=1}^n \alpha_i f(b_i) \le f\left(-\sum_{i=1}^n \alpha_i b_i\right).$$

In particular, if $f(0)=0$, *then*

$$\sum_{i=1}^n \alpha_i f(b_i) \ge -f\left(-\sum_{i=1}^n \alpha_i b_i\right).$$

In the following result, we present a one-term refinement of Theorem 1.2, which will be used then to prove the general refinement.

Lemma 2.2. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be convex,* $a\in\mathbb{R}$ *and* $\{\alpha_i\}>0$. *If* $\{b_i\}\subset\mathbb{R}$, *then*

$$\begin{aligned}(1+\beta)f(a) - \sum_{i=1}^n \alpha_i f(b_i) \quad &+ \quad (n+1)\alpha_{\min}\left(\frac{f(a)+\sum_{i=1}^n f(b_i)}{n+1} - f\left(\frac{a+\sum_{i=1}^n b_i}{n+1}\right)\right)\\ &\le \quad f\left((1+\beta)\,a - \sum_{i=1}^n \alpha_i b_i\right),\end{aligned}$$

where $\beta = \sum_{i=1}^n \alpha_i$ *and* $\alpha_{\min} = \min\{\alpha_i\}$.

Proof. Notice that

$$\begin{aligned} I \quad := \quad & (1+\beta)f(a) - \sum_{i=1}^n \alpha_i f(b_i) + (n+1)\alpha_{\min}\left(\frac{f(a)+\sum_{i=1}^n f(b_i)}{n+1} - f\left(\frac{a+\sum_{i=1}^n b_i}{n+1}\right)\right)\\ & = (1+\beta+\alpha_{\min})f(a) - \sum_{i=1}^n (\alpha_i - \alpha_{\min}) f(b_i) - (n+1)\alpha_{\min} f\left(\frac{a+\sum_{i=1}^n b_i}{n+1}\right). \end{aligned} \tag{5}$$

Let $\gamma_{n+1} = (n+1)\alpha_{\min}$ and for $1\le i\le n$, let $\gamma_i = \alpha_i - \alpha_{\min}$. Further, denote $\beta+\alpha_{\min}$ by λ. Then

$$\sum_{i=1}^n \gamma_i + \gamma_{n+1} = \beta + \alpha_{\min} = \lambda.$$

Therefore, we may apply Theorem 1.2 on (5) to obtain

$$\begin{aligned} I &\leq f\left((1+\beta+\alpha_{\min})a-\sum_{i=1}^{n}(\alpha_i-\alpha_{\min})b_i-(n+1)\alpha_{\min}\frac{a+\sum_{i=1}^{n}b_i}{n+1}\right) \\ &= f\left((1+\beta)\,a-\sum_{i=1}^{n}\alpha_i b_i\right). \end{aligned} \tag{6}$$

This completes the proof. □

The following notations will be adopted for the following and other results in the paper. For a given $k \in \mathbb{N}$, let $\{\alpha_i^{(k)}\} > 0$ and $\{b_i^{(k)}\} \subset \mathbb{R}$ be given. Denote $\min\left\{\alpha_i^{(k)}\right\}$ by $\alpha_{\min}^{(k)}$ and let $J_k := \left\{i : \alpha_i^{(k)} = \alpha_{\min}^{(k)}\right\}$ have cardinality $|J_k|$. Then define the new sets $\{\alpha_i^{(k+1)}\}$ and $\{b_i^{(k+1)}\}$ inductively as follows:

$$\alpha_i^{(k+1)} = \begin{cases} \alpha_i^{(k)} - \alpha_{\min}^{(k)} : & \alpha_i^{(k)} \neq \alpha_{\min}^{(k)} \\ \frac{(n+1)\alpha_{\min}^{(k)}}{|J_k|} : & \alpha_i^{(k)} = \alpha_{\min}^{(k)} \end{cases}, \tag{7}$$

and

$$b_i^{(k+1)} = \begin{cases} b_i^{(k)} : & \alpha_i^{(k)} \neq \alpha_{\min}^{(k)} \\ \frac{a+\sum_{i=1}^{n} b_i^{(k)}}{n+1} : & \alpha_i^{(k)} = \alpha_{\min}^{(k)} \end{cases}. \tag{8}$$

Now we are ready to present the main result in the paper.

Theorem 2.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be convex, $a \in \mathbb{R}$, $\{\alpha_i^{(1)}\} > 0$, and $\{b_i^{(1)}\} \subset \mathbb{R}$. Then, for $N \in \mathbb{N}$,*

$$\begin{aligned} & (1+\beta^{(1)})f(a)-\sum_{i=1}^{n}\alpha_i^{(1)}f\left(b_i^{(1)}\right) \\ + \quad & (n+1)\sum_{k=1}^{N}\alpha_{\min}^{(k)}\left(\frac{f(a)+\sum_{i=1}^{n}f\left(b_i^{(k)}\right)}{n+1}-f\left(\frac{a+\sum_{i=1}^{n}b_i^{(k)}}{n+1}\right)\right) \\ & \leq f\left((1+\beta^{(1)})a-\sum_{i=1}^{n}\alpha_i^{(1)}b_i^{(1)}\right), \end{aligned} \tag{9}$$

where $\beta^{(1)} = \sum_{i=1}^{n}\alpha_i^{(1)}$.

Proof. We prove this by induction on N. When $N = 1$, the result has been shown in Lemma 2.2. Now assume the inequality is valid for a certain $N \in \mathbb{N}$. We prove the statement for $N + 1$. Notice that

$$\begin{aligned} I &:= (1+\beta^{(1)})f(a)-\sum_{i=1}^{n}\alpha_i^{(1)}f(b_i^{(1)}) \\ &\quad +(n+1)\alpha_{\min}^{(1)}\left(\frac{f(a)+\sum_{i=1}^{n}f(b_i^{(1)})}{n+1}-f\left(\frac{a+\sum_{i=1}^{n}b_i^{(1)}}{n+1}\right)\right) \qquad (10) \\ &= (1+\beta^{(1)}+\alpha_{\min}^{(1)})f(a)-\sum_{i=1}^{n}(\alpha_i^{(1)}-\alpha_{\min}^{(1)})f(b_i^{(1)})-(n+1)\alpha_{\min}^{(1)}f\left(\frac{a+\sum_{i=1}^{n}b_i^{(1)}}{n+1}\right) \\ &= (1+\beta^{(2)})f(a)-\sum_{i=1}^{n}\alpha_i^{(2)}f(b_i^{(2)}), \qquad (11) \end{aligned}$$

where $\beta^{(2)} = \beta^{(1)} + \alpha^{(1)}_{\min}$ and $\{\alpha^{(2)}_i\}$ and $\{b^{(2)}_i\}$ are as in (7) and (8), respectively. Now notice that

$$\begin{aligned}\sum_{i=1}^{n}\alpha_i^{(2)} &= \sum_{i=1}^{n}\left(\alpha_i^{(1)}-\alpha_{\min}^{(1)}\right)+\sum_{i\in J_1}\frac{(n+1)\alpha_{\min}^{(1)}}{|J_1|}\\ &= \beta^{(1)}-n\alpha_{\min}^{(1)}+(n+1)\alpha_{\min}^{(1)}\\ &= \beta^{(1)}+\alpha_{\min}^{(1)}\\ &= \beta^{(2)}.\end{aligned}\tag{12}$$

Now apply the inductive step on (11). For convenience, we denote $\alpha^{(2)}_i$ by $\gamma^{(1)}_i$ and $b^{(2)}_i$ by $c^{(1)}_i$. Then (11) becomes

$$\begin{aligned}I &= (1+\beta^{(2)})f(a)-\sum_{i=1}^{n}\gamma_i^{(1)}f(c_i^{(1)})\ \text{ (apply the inductive step now)}\\ &\le f\left((1+\beta^{(2)})a-\sum_{i=1}^{n}\gamma_i^{(1)}c_i^{(1)}\right)\\ &\quad -(n+1)\sum_{k=1}^{N}\gamma_{\min}^{(k)}\left(\frac{f(a)+\sum_{i=1}^{n}f\left(c_i^{(k)}\right)}{n+1}-f\left(\frac{a+\sum_{i=1}^{n}c_i^{(k)}}{n+1}\right)\right),\end{aligned}\tag{13}$$

where $\{\gamma^{(k)}_i\}$ and $\{c^{(k)}_i\}$ are generated from $\{\gamma^{(1)}_i\}$ and $\{c^{(1)}_i\}$ according to (7) and (8) respectively.
Now notice that

$$\begin{aligned}&(1+\beta^{(2)})a-\sum_{i=1}^{n}\gamma_i^{(1)}c_i^{(1)}\\ &=(1+\beta^{(1)}+\alpha_{\min}^{(1)})a-\sum_{i=1}^{n}(\alpha_i^{(1)}-\alpha_{\min}^{(1)})b_i^{(1)}-(n+1)\alpha_{\min}^{(1)}\frac{a+\sum_{i=1}^{n}b_i^{(1)}}{n+1}\\ &=(1+\beta^{(1)})a-\sum_{i=1}^{n}\alpha_i^{(1)}b_i^{(1)}.\end{aligned}\tag{14}$$

Moreover, it is clear that $\gamma^{(k)}_i = \alpha^{(k+1)}_i$ and $c^{(k)}_i = b^{(k+1)}_i$ by definition. This fact together with (14) implemented in (13) and recalling the definition of I in (10), (13) becomes

$$\begin{aligned}&(1+\beta^{(1)})f(a)-\sum_{i=1}^{n}\alpha_i^{(1)}f(b_i^{(1)})\\ &+(n+1)\alpha_{\min}^{(1)}\left(\frac{f(a)+\sum_{i=1}^{n}f(b_i^{(1)})}{n+1}-f\left(\frac{a+\sum_{i=1}^{n}b_i^{(1)}}{n+1}\right)\right)\\ &\le f\left((1+\beta^{(1)})a-\sum_{i=1}^{n}\alpha_i^{(1)}b_i^{(1)}\right)\\ &-(n+1)\sum_{k=1}^{N}\alpha_{\min}^{(k+1)}\left(\frac{f(a)+\sum_{i=1}^{n}f\left(b_i^{(k+1)}\right)}{n+1}-f\left(\frac{a+\sum_{i=1}^{n}b_i^{(k+1)}}{n+1}\right)\right),\end{aligned}$$

which is equivalent to saying

$$\begin{aligned}&(1+\beta^{(1)})f(a)-\sum_{i=1}^{n}\alpha_i^{(1)}f(b_i^{(1)})\\&+(n+1)\sum_{k=1}^{N+1}\alpha_{\min}^{(k)}\left(\frac{f(a)+\sum_{i=1}^{n}f\left(b_i^{(k)}\right)}{n+1}-f\left(\frac{a+\sum_{i=1}^{n}b_i^{(k)}}{n+1}\right)\right)\\&\le f\left((1+\beta^{(1)})a-\sum_{i=1}^{n}\alpha_i^{(1)}b_i^{(1)}\right).\end{aligned}$$

This completes the proof. □

This entails the following log-convex version. Recall that a function $f:\mathbb{R}\to\mathbb{R}^+$ is said to be log-convex if the function $g=\log f$ is convex.

Corollary 2.4. *Let $f:\mathbb{R}\to\mathbb{R}^+$ be log-convex, $a\in\mathbb{R}$, $\{\alpha_i^{(1)}\}>0, N\in\mathbb{N}$ and $\{b_i^{(1)}\}\subset\mathbb{R}$. Then*

$$\begin{aligned}&\frac{f^{1+\beta^{(1)}}(a)}{\prod_{i=1}^{n}f^{\alpha_i^{(1)}}\left(b_i^{(1)}\right)}\prod_{k=1}^{N}\left[\frac{\left(f(a)\prod_{i=1}^{n}f\left(b_i^{(k)}\right)\right)^{\frac{1}{n+1}}}{f\left(\frac{a+\sum_{i=1}^{n}b_i^{(k)}}{n+1}\right)}\right]^{(n+1)\alpha_{\min}^{(k)}}\\&\qquad\le f\left((1+\beta^{(1)})a-\sum_{i=1}^{n}\alpha_i^{(1)}b_i^{(1)}\right).\end{aligned}$$

3. Applications

3.1. The case $n=1$ and $N\to\infty$

When $n=1$, the above parameters become easy, and we obtain the following.

Corollary 3.1. *Let $f:\mathbb{R}\to\mathbb{R}$ be convex and let $\nu>0$. Then for $a,b\in\mathbb{R}$ and $N\in\mathbb{N}$, the following holds*

$$\begin{aligned}&(1+\nu)f(a)-\nu f(b)+\sum_{j=1}^{N}2^j\nu\left[\frac{f(a)+f\left(\frac{(2^{j-1}-1)a+b}{2^{j-1}}\right)}{2}-f\left(\frac{(2^j-1)a+b}{2^j}\right)\right]\\&\qquad\le f\left((1+\nu)a-\nu b\right).\end{aligned}\tag{15}$$

This is the main inequality of [7]. However, the applications of this inequality are given in details in this reference.

It is natural to ask about the behavior of our results as $N\to\infty$. An ambitious claim would be to have the equality attained in these inequalities. Unfortunately, this is not the case. This can be easily seen by observing that the left hand side of (15), which is a special case of the main result, is linear in ν, while the right hand side is not.

However, the limiting inequality turns out to be an interesting result. We prove the formula next, then we comment on its geometric meaning, which will be a known fact about convex functions! First, we present the following auxiliary lemma, which helps compute the required limit.

Lemma 3.2. *Let $f:\mathbb{R}\to\mathbb{R}$ be a given function and let $a,b,\nu\in\mathbb{R}$. Then for $N\in\mathbb{N}$,*

$$\begin{aligned}&(1+\nu)f(a)-\nu f(b)+\sum_{j=1}^{N}2^j\nu\left[\frac{f(a)+f\left(\frac{(2^{j-1}-1)a+b}{2^{j-1}}\right)}{2}-f\left(\frac{(2^j-1)a+b}{2^j}\right)\right]\\&\qquad=f(a)+2^N\left[f(a)-f\left(a+\frac{b-a}{2^N}\right)\right]\nu.\end{aligned}\tag{16}$$

Proof. We proceed by induction on N. For $N = 1$, easy computations show the required inequality. Assuming the truth of (16) for some $N \in \mathbb{N}$, we have

$$\begin{aligned}
(1+\nu)f(a) - \nu f(b) + \sum_{j=1}^{N+1} 2^j\nu\left[\frac{f(a)+f\left(\frac{(2^{j-1}-1)a+b}{2^{j-1}}\right)}{2} - f\left(\frac{(2^j-1)a+b}{2^j}\right)\right]\\
= f(a) + 2^N\left[f(a) - f\left(a+\frac{b-a}{2^N}\right)\right]\nu +\\
+ 2^{N+1}\left[\frac{f(a)+f\left(\frac{(2^N-1)a+b}{2^N}\right)}{2} - f\left(\frac{(2^{N+1}-1)a+b}{2^{N+1}}\right)\right]\nu\\
= f(a) + 2^{N+1}\left[f(a) - f\left(a+\frac{b-a}{2^{N+1}}\right)\right]\nu,
\end{aligned}$$

where we have used the inductive step together with simple manipulations to obtain the above equations. This completes the proof. □

Now letting $N \to \infty$, and assuming differentiability of f at a, we have

$$\begin{aligned}
\lim_{N\to\infty} 2^N\left[f(a) - f\left(a+\frac{b-a}{2^N}\right)\right] &= \lim_{N\to\infty}\frac{f(a) - f\left(a+(b-a)2^{-N}\right)}{2^{-N}}\\
&= -\lim_{h\to 0}\frac{f(a+(b-a)h) - f(a)}{h}\\
&= (a-b)f'(a).
\end{aligned}$$

Consequently, Lemma 3.2 implies the following.

Proposition 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$. Then for $a, b, \nu \in \mathbb{R}$ the following holds*

$$\begin{aligned}
&(1+\nu)f(a) - \nu f(b) +\\
&\quad + \lim_{N\to\infty}\sum_{j=1}^{N} 2^j\nu\left[\frac{f(a)+f\left(\frac{(2^{j-1}-1)a+b}{2^{j-1}}\right)}{2} - f\left(\frac{(2^j-1)a+b}{2^j}\right)\right]\\
&\quad = f(a) + f'(a)(a-b)\nu, \qquad (17)
\end{aligned}$$

provided that the derivative $f'(a)$ exists.

Therefore, taking the limit of Corollary 3.1, we obtain the following inequality for convex functions.

Corollary 3.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be convex and let $a, b \in \mathbb{R}$. Then for $\nu > 0$,*

$$f(a) + f'(a)(a-b)\nu \le f\left((1+\nu)a - \nu b\right),$$

provided that the derivative $f'(a)$ exists.

Notice that the equation $y = f(a) + f'(a)(a-b)\nu$ is nothing but the tangent line equation of the convex function $y = f\left((1+\nu)a - \nu b\right)$. Thus, the above inequality states that the tangent line of a convex function is below the function; a well known fact about convex functions.

3.2. The arithmetic-geometric mean inequality

Recall that the arithmetic-geometric mean (AM-GM) inequality states that

$$\prod_{i=1}^{n} a_i^{pi/p} \leq \frac{1}{p}\sum_{i=1}^{n} p_i a_i,$$

for the positive numbers a_i, p_i, where $p = \sum_{i=1}^{n} p_i$. Searching the literature, we find a one-term refinement and reverse of this inequality. One can see [5] for such treatments. Letting $f(x) = -\log x$ in Theorem 1.2 we obtain the following negative version of the AM-GM inequality

$$a^{1+\beta}\prod_{i=1}^{n} b_i^{-\alpha_i} \geq (1+\beta)a - \sum_{i=1}^{n}\alpha_i b_i, \tag{18}$$

for the positive numbers a, b_i, α_i and $\beta = \sum_{i=1}^{n}\alpha_i$. In particular, when $n = 1$, this result reduces to $(1+\nu)a - \nu b \leq a^{1+\nu}b^{-\nu}, a, b, \nu > 0$; which is the well known negative version of Young's inequality. This last inequality, when $n = 1$, has been treated in details in [2, 7]. Applying the refinement in Theorem 2.3, one can obtain a general refinements of (18). To avoid complexity, we leave this to the interested reader.

We refer the reader to [9] for inequalities related to (18).

3.3. Corresponding inequalities in vector spaces

An interesting discussion of convex mappings between vector spaces has been presented in some details in [4]. In this context, let X be a vector spaces. An order $\leq$ on X is a binary relation on X satisfying the properties

1. $x \leq x$ for all $x \in X$.
2. If $x_1 \leq x_2$ and $x_2 \leq x_3$, then $x_1 \leq x_3$, for $x_1, x_2, x_3 \in X$.
3. If $x_1 \leq x_2$ and $x_2 \leq x_1$, then $x_1 = x_2$.

Among the most well known orderings on vector spaces is the cone ordering. In this context, let C be a subset of the vector space X. We say that C is a cone if

$$C + C \subset C \text{ and } \alpha C \subset C, \forall \alpha \in \mathbb{R}^+.$$

Then an ordering $\leq_C$ on X induced by C can be defined as follows

$$x \leq_C y \Leftrightarrow y - x \in C.$$

This ordering on X satisfies the two additional properties

$$x_1 \leq x_2 \Rightarrow x_1 + x_3 \leq x_2 + x_3 \text{ and } x_1 \leq_C x_2 \Rightarrow \alpha x_1 \leq_C \alpha x_2,$$

for $x_1, x_2, x_3 \in X$ and $\alpha \geq 0$.

The notation X_{C_X} will be used to emphasize that the vector space X has an ordering induced by the cone C_X.

Now given two vector spaces X and Y, we say that $f : X \to Y_{C_Y}$ is convex if

$$f((1-\alpha)a + \alpha b) \leq_{C_Y} (1-\alpha)f(a) + \alpha f(b), a, b \in X, \alpha \in [0, 1].$$

Using induction, one can easily show the truth of Jensen's inequality for such functions. Notice that convexity condition of f does not require any ordering on X. In the following result, we present a negative inequality similar to Theorem 1.2 when $n = 1$. In the following discussion, X and Y will be vector spaces.

Proposition 3.5. *Let $f : X \to Y_{C_Y}$ be a convex function. If $a, b \in X$, then for $t \geq 0$,*

$$(1+t)f(a) - tf(b) \leq_{C_Y} f((1+t)a - tb).$$

Proof. Notice that for the given parameters,

$$a = \frac{1}{1+t}((1+t)a - tb) + \frac{t}{1+t}b.$$

Then the definition of convexity implies the required inequality, noting that $x \leq_{C_Y} y$ implies $\alpha x \leq_{C_Y} \alpha y$, when $\alpha > 0$. □

Now following the ideas of Theorem 1.2, one can prove a negative version for convex functions $f : X \to Y_{C_Y}$. Then the refinement of Theorem 2.3 will be valid for such mappings.

3.4. An integral version

Let $(X, \mathcal{M}, \mu)$ be a finite measure space normalized so that $\mu(X) = 1$ and let $f : X \to \mathbb{R}$ be $\mu-$integrable. The well known Jensen inequality states that

$$\varphi\left(\int_X f\, d\mu\right) \leq \int_X \varphi(f)\, d\mu$$

for the convex function $\varphi : \mathbb{R} \to \mathbb{R}$. This is a continuous version of the original Jensen inequality (1). Now using Theorem 1.2 we have the following variant of this inequality.

Theorem 3.6. *Let $(X, \mathcal{M}, \mu)$ be a probability space, $f : X \to \mathbb{R}$ be $\mu-$integrable and $\varphi : \mathbb{R} \to \mathbb{R}$ be convex. Then for any real number A,*

$$2\varphi(A) - \varphi\left(2A - \int_X f\, d\mu\right) \leq \int_X \varphi(f)\, d\mu.$$

Proof. Let $\{f_k\}$ be a sequence of simple functions on X such that

$$f_k = \sum_{i=1}^{n_k} b_{i,k}\chi_{E_{i,k}},\ \{E_{i,k}\}_i \text{ are disjoint},\ f_k \to f, \sum_{i=1}^{n_k} \mu(E_{i,k}) = 1.$$

Notice that since f is bounded, X is of finite measure and $f_k \to f$, we have $\int_X f_k\, d\mu \to \int_X f\, d\mu$. Moreover, since φ is convex, it is continuous and $\varphi \circ f$ is bounded. Therefore, $\varphi \circ f_k \to \varphi \circ f$ and $\int_X \varphi \circ f_k\, d\mu \to \int_X \varphi \circ f\, d\mu$. Now applying Theorem 1.2 to the convex function φ, with $\alpha_i = \mu(E_{i,k}), \beta = \sum_{i=1}^{n_k} \alpha_i = 1, a = A$ and $b_i = b_{i,k}$ we get

$$\begin{aligned}\int_X \varphi \circ f_k\, d\mu &= \sum_{i=1}^{n_k} \varphi(b_{i,k})\mu(E_{i,k}) \\ &\geq 2\varphi(A) - \varphi\left(2A - \sum_{i=1}^{n_k} b_{i,k}\mu(E_{i,k})\right). \qquad (19)\end{aligned}$$

Taking the limit of (19) as $k \to \infty$ and noting continuity of φ, we have

$$\begin{aligned}\int_X \varphi \circ f\, d\mu &\geq 2\varphi(A) - \varphi\left(2A - \lim_{k\to\infty}\int_X f_k\, d\mu\right) \\ &= 2\varphi(A) - \varphi\left(2A - \int_X f\, d\mu\right),\end{aligned}$$

which completes the proof. □

3.5. Weak Majorization

Let $\mathbb{M}_n$ denote the algebra of $n \times n$ complex matrices. For a Hermitian $A \in \mathbb{M}_n$, we denote by $\lambda_i(A)$ the $i-$th eigenvalue of A, when arranged in a decreasing order. Among many approaches to compare between Hermitian matrices is the so called weak majorization. More precisely, if A and B are Hermitian, we say that A is weakly majorized by B and write $A \prec_w B$ if

$$\sum_{i=1}^{k} \lambda_i(A) \le \sum_{i=1}^{k} \lambda_i(B), 1 \le k \le n.$$

It is proved in [1] that if $f : \mathbb{I} \to \mathbb{R}$ is a convex function and $A, B \in \mathbb{H}_n(\mathbb{I})$, the set of Hermitian matrices in $\mathbb{M}_n$ with eigenvalues in the interval $\mathbb{I}$, then

$$f((1-\nu)A + \nu B) \prec_w (1-\nu)f(A) + \nu f(B), 0 \le \nu \le 1.$$

That is,

$$\sum_{i=1}^{k} \lambda_i \left(f((1-\nu)A + \nu B)\right) \le \sum_{i=1}^{k} \lambda_i \left((1-\nu)f(A) + \nu f(B)\right), 1 \le k \le n. \tag{20}$$

The proof of this interesting result was based on the following basic inequality for the convex function $f : \mathbb{I} \to \mathbb{R}$, $A \in \mathbb{H}_n(\mathbb{I})$ and the unit vector $\mathbf{x} \in \mathbb{C}^n$, [3], p. 281,

$$f\langle A\mathbf{x}, \mathbf{x}\rangle \le \langle f(A)\mathbf{x}, \mathbf{x}\rangle. \tag{21}$$

Our target is to prove a negative version of (20). For this, we prove first the following reverse of (21).

Lemma 3.7. *Let $f : \mathbb{I} \to \mathbb{R}$ be convex and let $A \in \mathbb{H}_n(\mathbb{I})$. If $\mathbf{x}$ is a unit vector in $\mathbb{C}^n$, then*

$$\begin{aligned} f\langle A\mathbf{x}, \mathbf{x}\rangle &\le \langle f(A)\mathbf{x}, \mathbf{x}\rangle \\ &\le f\langle A\mathbf{x}, \mathbf{x}\rangle + n\left(\frac{trf(A)}{n} - f\left(\frac{trA}{n}\right)\right). \end{aligned}$$

Proof. Let $A = U^*DU$ be the spectral decomposition of A. Then

$$\begin{aligned} f\langle A\mathbf{x}, \mathbf{x}\rangle &= f\langle U^*DU\mathbf{x}, \mathbf{x}\rangle \\ &= f\langle DU\mathbf{x}, U\mathbf{x}\rangle \\ &= f\left(\sum_{i=1}^{n} \lambda_i(A)|[U\mathbf{x}]_i|^2\right), \end{aligned} \tag{22}$$

where $[U\mathbf{x}]_i$ is the $i-$th component of the vector $U\mathbf{x}$. Since U is unitary and $\mathbf{x}$ is a unit vector, we have

$$\sum_{i=1}^{n} |[U\mathbf{x}]_i|^2 = \langle U\mathbf{x}, U\mathbf{x}\rangle = \langle \mathbf{x}, U^*U\mathbf{x}\rangle = \|\mathbf{x}\|^2 = 1.$$

Now since f is convex and $\sum_{i=1}^{n} |[U\mathbf{x}]_i|^2 = 1$, we may apply the second inequality of Lemma 1.1 on (22) to obtain

$$\begin{aligned} & f\langle A\mathbf{x}, \mathbf{x}\rangle \ge \sum_{i=1}^{n} |[U\mathbf{x}]_i|^2 f(\lambda_i(A)) \\ & -n \max_{1\le k\le n} |[U\mathbf{x}]_i|^2 \left(\frac{\sum_{i=1}^{n} f(\lambda_i(A))}{n} - f\left(\frac{\sum_{i=1}^{n} \lambda_i(A)}{n}\right)\right) \\ & \ge \sum_{i=1}^{n} |[U\mathbf{x}]_i|^2 f(\lambda_i(A)) - n\left(\frac{\sum_{i=1}^{n} f(\lambda_i(A))}{n} - f\left(\frac{\sum_{i=1}^{n} \lambda_i(A)}{n}\right)\right) \\ & = \langle f(A)\mathbf{x}, \mathbf{x}\rangle - n\left(\frac{\mathrm{tr} f(A)}{n} - f\left(\frac{\mathrm{tr} A}{n}\right)\right). \end{aligned}$$

Notice that the fact $\sum_{i=1}^{n}|[U\mathbf{x}]_i|^2 = 1$ implies $\max\limits_{1\le k\le n}|[U\mathbf{x}]_i|^2 \le 1$, which then implies the last inequality above. This completes the proof. □

Moreover, we need to recall the following basic result for $A \in \mathbb{H}_n$, the set of Hermitian matrices in $\mathbb{M}_n$, [3], p. 35,

$$\sum_{i=1}^{k}\lambda_i(A) = \max\sum_{i=1}^{k}\langle A\mathbf{x}_i, \mathbf{x}_i\rangle, \tag{23}$$

where the maximum is taken over orthonormal vectors $\{\mathbf{x}_i\} \subset \mathbb{C}^n$.
Now we are ready to prove our negative version of (20).

Theorem 3.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be convex and let $A, B \in \mathbb{H}_n$ Then for $\nu \ge 0$ and $1 \le k \le n$,*

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_i\left((1+\nu)f(A) - \nu f(B)\right)\\ &\qquad\le \sum_{i=1}^{k}\lambda_i\left(f((1+\nu)A - \nu B)\right) + (1+\nu)nkT_n(f,A),\end{aligned}$$

where

$$T_n(f,A) = \left(\frac{trf(A)}{n} - f\left(\frac{trA}{n}\right)\right).$$

Proof. Let $\{\mathbf{x}_i\}$ be orthonormal eigenvectors of $(1+\nu)f(A) - \nu f(B)$ corresponding to its eigenvalues λ_i. Then

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_i\left(f((1+\nu)A - \nu B)\right)\\
&\ge \sum_{i=1}^{k}\left\langle f((1+\nu)A - \nu B)\mathbf{x}_i, \mathbf{x}_i\right\rangle \text{ (by (23))}\\
&\ge \sum_{i=1}^{k} f\left\langle((1+\nu)A - \nu B)\mathbf{x}_i, \mathbf{x}_i\right\rangle \text{ (by (21))}\\
&= \sum_{i=1}^{k} f\left((1+\nu)\langle A\mathbf{x}_i, \mathbf{x}_i\rangle - \nu\langle B\mathbf{x}_i, \mathbf{x}_i\rangle\right) \text{ (now apply (2))}\\
&\ge \sum_{i=1}^{k}\left[(1+\nu)f\left(\langle A\mathbf{x}_i, \mathbf{x}_i\rangle\right) - \nu f\left(\langle B\mathbf{x}_i, \mathbf{x}_i\rangle\right)\right] \text{ (now apply Lemma 3.7)}\\
&\ge \sum_{i=1}^{k}\left[(1+\nu)\left\langle f(A)\mathbf{x}_i, \mathbf{x}_i\right\rangle - \nu\left\langle f(B)\mathbf{x}_i, \mathbf{x}_i\right\rangle\right] - (1+\nu)knT_n(f,A)\\
&= \sum_{i=1}^{k}\left\langle((1+\nu)f(A) - \nu f(B))\mathbf{x}_i, \mathbf{x}_i\right\rangle - (1+\nu)knT_n(f,A)\\
&= \sum_{i=1}^{k}\lambda_i\left((1+\nu)f(A) - \nu f(B)\right) - (1+\nu)knT_n(f,A),\end{aligned}$$

which completes the proof. □

The following is a related inequality.

Proposition 3.9. *Let $f:\mathbb{R}\to\mathbb{R}$ be convex. If $A\in\mathbb{H}_n$ and $\lambda=\lambda_1(A)$ or $\lambda=\lambda_n(A)$, then*

$$\begin{aligned} f(\lambda) &\le \frac{\langle f(A)\mathbf{x},\mathbf{x}\rangle + f\langle(2\lambda I-A)\mathbf{x},\mathbf{x}\rangle}{2}\\ &\le \frac{\langle f(A)\mathbf{x},\mathbf{x}\rangle + \langle f(2\lambda I-A)\mathbf{x},\mathbf{x}\rangle}{2}, \end{aligned}$$

for any unit vector $\mathbf{x}\in\mathbb{C}^n$.

Proof. Following the computations of Lemma 3.7, we have

$$\begin{aligned} \langle f(A)\mathbf{x},\mathbf{x}\rangle &= \sum_{i=1}^n \alpha_i f(\lambda_i(A)),\ \alpha_i = |[U\mathbf{x}]_i|^2\\ &\ge 2f(\lambda) - f\left(2\lambda - \sum_{i=1}^n \alpha_i\lambda_i(A)\right)\ \text{(by Theorem 1.2)}\\ &= 2f(\lambda) - f\left(\sum_{i=1}^n \alpha_i(2\lambda-\lambda_i(A))\right)\left(\text{because } \sum_{i=1}^n \alpha_i = 1\right)\\ &= 2f(\lambda) - f\langle(2\lambda I-A)\mathbf{x},\mathbf{x}\rangle\\ &\ge 2f(\lambda) - \langle f(2\lambda I-A)\mathbf{x},\mathbf{x}\rangle\ \text{(by (21))}. \end{aligned}$$

This completes the proof. □

3.6. An algorithmic approach

We have seen that our refinements depend on generating certain parameters from some initial values.

We present the following MATLAB code that generates all the parameters of Theorem 2.3. We built the code so that the initial values are all generated randomly. Of course, this can be fixed if the user wants to start with specific values. The possible outcomes of this code are:

- a matrix p of all the coefficients $\left\{\alpha_i^{(k)}\right\}$. For a fixed k, the $k-$th column of the matrix p is $\left\{\alpha_i^{(k)}\right\}$.
- a matrix b of the sequence $\left\{x_i^{(k)}\right\}$. For a fixed k, the $k-$th column of the matrix b is $\left\{x_i^{(k)}\right\}$.
- a matrix "fmat" whose $k-$th column is $\left\{f\left(x_i^{(k)}\right)\right\}$.
- the numerical value "LsideW" which is the left hand side in Theorem 1.2.
- the numerical value "LsideR" which is the refining sum in the left hand side in Theorem 2.3.
- the numerical value "Lside" which is the the left hand side in Theorem 2.3. That is Lside=LsideW+LsideR.
- the numerical value "Rside" which is the right hand side in Theorem 2.3 or Theorem 1.2.

Algorithm 3.10. *(Code generating the parameters of Theorem 2.3)*

```
n =?
N =?
f =?
a=rand(1,1);
p = zeros(n,N);
b=zeros(n,N);
for i = 1 : n do
 | p(i,1)=rand(1,1); b(i,1)=rand(1,1);
end
The user inputs f, n and N ≥ 2
for k = 1 : N − 1 do
 | m = min(p); m = m(1,k); s = sum(b); s = s(1,k); j = 0;
 | for i = 1 : n do
 |  | p(i,k) ≤ m
 |  | j = j + 1;
 | end
 | for i = 1 : n do
 |  | if p(i,k) ≤ m then
 |  |  | p(i,k+1) = (n+1) * m/j;
 |  |  | b(i,k+1) = (a+s)/(n+1);
 |  | else
 |  |  | p(i,k+1) = p(i,k) − m;
 |  |  | b(i,k+1) = b(i,k);
 |  | end
 | end
end
fmat=f(b);beta=sum(p(:,1));
LsideW=(1+beta)*f(a)-sum(dot(p(:,1),fmat(:,1)));
Rside=f((1+beta)*a-sum(dot(p(:,1),b(:,1))));
LsideR=0;
for k = 1 : N do
 | alpha=min(p);alpha=alpha(1,k);
 | LsideR=LsideR+(n+1)*alpha*((f(a)+sum(fmat(:,k)))/(n+1)-f((a+sum(b(:,k)))/(n+1)));
end
Lside=LsideW+LsideR;
```

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