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Simulation Type Functions and Coincidence Points

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Abstract. In this paper, we obtain some sufficient conditions for the existence and uniqueness of point of coincidence by using simulation functions in the context of metric spaces and prove some interesting results. Our results generalize the corresponding results of [5, 8, 13, 14, 16] in several directions. Also, we provide an example which shows that our main result is a proper generalization of the result of Jungck [American Math. Monthly 83(1976) 261-263], L-de-Hierro et al. [J. Comput. Appl. Math 275(2015) 345-355] and of Olgun et al. [Turk. J. Math. (2016) 40:832-837].

1. Introduction and Preliminaries

To begin with, we have the following definitions, notations and results which will be used in the sequel.

Definition 1.1. *[3] A mapping $G : [0,+\infty)^2 \to \mathbb{R}$ is called a C-class function if it is continuous and satisfies the following conditions:*
*(**1**) $G(s,t) \le s$;*
*(**2**) $G(s,t) = s$ implies that either $s = 0$ or $t = 0$, for all $s,t \in [0,+\infty)$.*

For C-class functions see also [4, 7, 14].
In [14], the authors generalized the simulation function introduced by Khojasteh et al. ([13]) using the function of C-class as follows:

Definition 1.2. *A mapping $G : [0,+\infty)^2 \to \mathbb{R}$ has the property C_G, if there exists an $C_G \ge 0$ such that*
*(**3**) $G(s,t) > C_G$ implies $s > t$;*
*(**4**) $G(t,t) \le C_G$, for all $t \in [0,+\infty)$.*

Some examples of C-class functions that have property C_G are as follows:
a) $G(s,t) = s - t$, $C_G = r$, $r \in [0,+\infty)$;
b) $G(s,t) = s - \frac{(2+t)t}{1+t}$, $C_G = 0$;
c) $G(s,t) = \frac{s}{1+kt}$, $k \ge 1$, $C_G = \frac{r}{1+k}$, $r \ge 2$.

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For more examples of C-class functions that have property C_G see [5, 7, 14] .

Recently, Khojasteh et al. ([13]) (also see [2, 8, 15]) introduced a new approach in the fixed point theory by using the following:

Definition 1.3. *A simulation function is a mapping $\zeta : [0,\infty)^2 \to \mathbb{R}$ satisfying the following:*
(5) $\zeta(t,s) < s - t$ for all $t,s > 0$;
(6) if $\{t_n\}, \{s_n\}$ are sequences in $(0,+\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, and $t_n < s_n$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Definition 1.4. *(see [14]) A C_G-simulation function is a mapping $\zeta : [0,+\infty)^2 \to \mathbb{R}$ satisfying the following:*
(7) $\zeta(t,s) < G(s,t)$ for all $t,s > 0$, where $G : [0,+\infty)^2 \to \mathbb{R}$ is a C-class function;
(8) if $\{t_n\}, \{s_n\}$ are sequences in $(0,+\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, and $t_n < s_n$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < C_G$.

Some examples of simulation functions and C_G-simulation functions are:
d) $\zeta(t,s) = \frac{s}{s+1} - t$ for all $t,s \geq 0$.
e) $\zeta(t,s) = s - \varphi(s) - t$ for all $t,s \geq 0$, where $\varphi : [0,+\infty) \to [0,+\infty)$ is a lower semi continuous function and $\varphi(t) = 0$ if and only if $t = 0$.

For more examples of simulation functions and C_G-simulation functions see [5, 8, 13–15, 18].

Let $\mathcal{Z}_G$ be the family of all C_G-simulation functions $\zeta : [0,+\infty)^2 \to \mathbb{R}$. Each simulation function as in Definition 1.3 is also a C_G-simulation function as in Definition 1.4, but the converse is not true.

For this claim see Example 3.3 of [8] using the C-class function $G(s,t) = s - t$.

Let f and g be self maps of a set X. Recall that if $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. The pair (f,g) is weakly compatible if f and g commute at their coincidence points. A sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}} \subseteq X$ is a Picard-Jungck sequence of the pair (f,g) (based on x_0) if $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N}\cup\{0\}$ (see also [8, Definition 4.4]).

Now, we recall the following result of Abbas and Jungck [1] to be used in the sequel.

Proposition 1.5. *Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is a unique common fixed point of f and g.*

Assertions similar to the following result of Radenović et al. [17] were used (and proved) in the proofs of several fixed point results in various papers. Here, we formulate and prove an improved version of this result.

Lemma 1.6. *Let (X,d) be a metric space and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{1.1}$$

If $\{x_n\}$ is not a Cauchy sequence in X, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following sequences tend to ε^+ when $k \to +\infty$:

$$d\left(x_{m(k)}, x_{n(k)}\right),\ d\left(x_{m(k)}, x_{n(k)+1}\right),\ d\left(x_{m(k)-1}, x_{n(k)}\right), \tag{1.2}$$
$$d\left(x_{m(k)-1}, x_{n(k)+1}\right),\ d\left(x_{m(k)+1}, x_{n(k)+1}\right).$$

Proof. If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that

$$n_k > m_k > k,\ d(x_{m_k}, x_{n_k-1}) < \varepsilon,\ d(x_{m_k}, x_{n_k}) \geq \varepsilon$$

for all positive integers k. Then

$$\begin{aligned} \varepsilon \leq d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \varepsilon + d(x_{n_k-1}, x_{n_k}). \end{aligned}$$

Using (1.1), we conclude that

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \varepsilon. \tag{1.3}$$

Further,

$$d\left(x_{m(k)}, x_{n(k)}\right) \le d\left(x_{m(k)}, x_{n(k)+1}\right) + d\left(x_{n(k)+1}, x_{n(k)}\right)$$

as well as

$$d\left(x_{m(k)}, x_{n(k)+1}\right) \le d\left(x_{m(k)}, x_{n(k)}\right) + d\left(x_{n(k)}, x_{n(k)+1}\right).$$

Passing to the limit when $k \to \infty$ and using (1.1) and (1.3) we obtain that

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)+1}\right) = \varepsilon.$$

Also,

$$d\left(x_{m(k)+1}, x_{n(k)+1}\right) \le d\left(x_{m(k)+1}, x_{m(k)}\right) + d\left(x_{m(k)}, x_{n(k)+1}\right) \tag{1.4}$$

and

$$d\left(x_{n(k)}, x_{m(k)}\right) \le d\left(x_{n(k)}, x_{n(k)+1}\right) + d\left(x_{n(k)+1}, x_{m(k)+1}\right). \tag{1.5}$$

Now, from (1.4) and (1.5) it follows that

$$\lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) = \varepsilon.$$

That the remaining two sequences in (1.2) tend to ε can be proved similarly. □

2. Main Results

In this section, we establish some results on the existence and uniqueness of coincidence point by using simulation functions in the framework of metric spaces. We begin with the following definition.

Definition 2.1. *(see [5, 8, 14]) Let (X, d) be a metric space and $f, g : X \to X$ be self-mappings. A mapping f is called a (Z_G, g)-contraction if there exists $\zeta \in Z_G$ such that*

$$\zeta\left(d\left(fx, fy\right), d\left(gx, gy\right)\right) \ge C_G \tag{2.1}$$

for all $x, y \in X$ with $gx \ne gy$.

In the case, $g = i_X$ (identity mapping on X) and $C_G = 0$ we get $\mathcal{Z}-$contraction of [13, Definition 2.3].

Now, we state our first new result for the notion of (Z_G, g)-contraction. It generalizes the corresponding results of [5, 8, 13, 16] in several directions.

Theorem 2.2. *Let (X, d) be a metric space, $f, g : X \to X$ be self-mappings and f be a (Z_G, g)-contraction. Suppose that there exists a Picard-Jungck sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ of (f, g). Also assume that at least one of the following conditions hold:*

(i) *$(f(X), d)$ or $(g(X), d)$ is complete;*

(ii) *(X, d) is complete, g is continuous and (f, g) is compatible.*

Then f and g have a unique point of coincidence.

Proof. First of all we shall prove that the point of coincidence of f and g is unique (if it exists). Suppose that z_1 and z_2 are distinct points of coincidence of f and g. From this it follows that there exist two points v_1 and v_2 ($v_1 \neq v_2$) such that $fv_1 = gv_1 = z_1$ and $fv_2 = gv_2 = z_2$. Then (2.1) implies that

$$C_G \leq \zeta\left(d\left(fv_1, fv_2\right), d\left(gv_1, gv_2\right)\right) = \zeta\left(d\left(z_1, z_2\right), d\left(z_1, z_2\right)\right) < G\left(d\left(z_1, z_2\right), d\left(z_1, z_2\right)\right) \leq C_G,$$

which is a contradiction.

In order to prove that f and g have a point of coincidence, suppose that there is a Jungck sequence $\{y_n\}$ such that $y_n = fx_n = gx_{n+1}$ where $n \in \mathbb{N} \cup \{0\}$.

If $y_k = y_{k+1}$ for some $k \in \mathbb{N} \cup \{0\}$, then $gx_{k+1} = y_k = y_{k+1} = fx_{k+1}$ and f and g have a point of coincidence. Therefore, suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

Substituting $x = x_{n+1}$, $y = x_{n+2}$ in (2.1) we obtain that

$$\begin{aligned} C_G &\leq \zeta\left(d\left(fx_{n+1}, fx_{n+2}\right), d\left(gx_{n+1}, gx_{n+2}\right)\right) = \zeta\left(d\left(y_{n+1}, y_{n+2}\right), d\left(y_n, y_{n+1}\right)\right) \\ &< G\left(d\left(y_n, y_{n+1}\right), d\left(y_{n+1}, y_{n+2}\right)\right). \end{aligned}$$

Using (3) of Definition 1.2, we have $d\left(y_n, y_{n+1}\right) > d\left(y_{n+1}, y_{n+2}\right)$. Hence, for all $n \in \mathbb{N} \cup \{0\}$ we get that $d\left(y_{n+1}, y_{n+2}\right) < d\left(y_n, y_{n+1}\right)$. Therefore there exists $D \geq 0$ such that $\lim_{n\to\infty} d\left(y_n, y_{n+1}\right) = D \geq 0$. Suppose that $D > 0$. Since $d\left(y_{n+1}, y_{n+2}\right) < d\left(y_n, y_{n+1}\right)$ and both $d\left(y_{n+1}, y_{n+2}\right)$ and $d\left(y_n, y_{n+1}\right)$ tend to D, using (8) of Definition 1.4, we get

$$C_G \leq \limsup_{n\to\infty} \zeta\left(d\left(y_{n+1}, y_{n+2}\right), d\left(y_n, y_{n+1}\right)\right) < C_G,$$

which is a contradiction. Hence $\lim_{n\to\infty} d\left(y_n, y_{n+1}\right) = D = 0$.

Further we have to prove that $y_n \neq y_m$ for $n \neq m$. Indeed, suppose that $y_n = y_m$ for some $n > m$. Then we choose $x_{n+1} = x_{m+1}$ (which is obviously possible by the definition of Jungck sequence y_n) and hence also $y_{n+1} = y_{m+1}$. Then following the previous arguments, we have

$$d\left(y_n, y_{n+1}\right) < d\left(y_{n-1}, y_n\right) < \cdots < d\left(y_m, y_{m+1}\right) = d\left(y_n, y_{n+1}\right),$$

which is a contradiction.

Now, we have to show that $\{y_n\}$ is a Cauchy sequence. Suppose, to the contrary, that it is not. Putting $x = x_{m(k)+1}$, $y = y_{n(k)+1}$ in (2.1), we obtain

$$C_G \leq \zeta\left(d\left(y_{m(k)+1}, y_{n(k)+1}\right), d\left(y_{m(k)}, y_{n(k)}\right)\right) < G\left(d\left(y_{m(k)}, y_{n(k)}\right), d\left(y_{m(k)+1}, y_{n(k)+1}\right)\right). \tag{2.2}$$

Using (3) of Definition 1.2, it follows that $d\left(y_{m(k)}, y_{n(k)}\right) > d\left(y_{m(k)+1}, y_{n(k)+1}\right)$.

Now, since the sequence $\{y_n\}$ is not a Cauchy sequence, then by Lemma 1.6, we have $d\left(y_{m(k)}, y_{n(k)}\right)$ and $d\left(y_{m(k)+1}, y_{n(k)+1}\right)$ tend to $\varepsilon > 0$, as $k \to \infty$. Therefore, using (2.2), we have

$$C_G \leq \limsup_{n\to\infty} \zeta\left(d\left(y_{m(k)+1}, y_{n(k)+1}\right), d\left(y_{m(k)}, y_{n(k)}\right)\right) < C_G,$$

which is a contradiction. Therefore, the Jungck sequence $\{y_n\}$ is a Cauchy sequence.

Suppose that (i) holds, i.e., $(g(X), d)$ is complete. Then there exists $v \in X$ such that $gx_n \to gv$ as $n \to \infty$. We shall prove that $fv = gv$. It is clear that we can suppose $y_n \neq fv, gv$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore, by (2.1), we have

$$C_G \leq \zeta\left(d\left(fx_n, fv\right), d\left(gx_n, gv\right)\right) < G\left(d\left(gx_n, gv\right), d\left(fx_n, fv\right)\right).$$

Using (3) of Definition 1.2, we get $d\left(fx_n, fv\right) < d\left(gx_n, gv\right)$. It implies that $fx_n \to fv$ as $n \to \infty$. Hence, $fv = gv$ is a (unique) point of coincidence of f and g.

Similarly, we can prove that $fv = gv$ is a (unique) point of coincidence of f and g, when $(f(X), d)$ is complete.

Finally, suppose that (ii) holds. Since (X, d) is complete, then there exists $v \in X$ such that $fx_n \to v$, when $n \to \infty$. As g is continuous, $g(fx_n) \to gv$ when $n \to \infty$. Consider

$$C_G \le \zeta(d(f(gx_n), fv), d(g(fx_n), gv)) < G(d(g(fx_n), gv), d(f(gx_n), fv)).$$

Using (3) of Definition 1.2 and continuity of g, we have $d(f(gx_n), fv) < d(g(fx_n), gv) \to 0$, as $n \to \infty$. It implies that $d(f(gx_n), fv) \to 0$, as $n \to \infty$. Further, as f and g are compatible, we have

$$d(fv, gv) \le d(fv, f(gx_n)) + d(f(gx_n), g(fx_n)) + d(g(fx_n), gv) \to 0 + 0 + 0 = 0.$$

Hence, the result is proved in both cases, i.e., the mappings f and g have a unique point of coincidence. □

Theorem 2.3. *Let (X, d) be a metric space, $f, g : X \to X$ be self-mappings and f be a (Z_G, g)-contraction. Suppose that there exists a Picard-Jungck sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ of (f, g). Also assume that $(f(X), d)$ or $(g(X), d)$ is complete and f and g are weakly compatible. Then f and g have a unique common fixed point in X.*

Proof. Using Theorem 2.2, f and g have a unique point of coincidence. Further, since f and g are weakly compatible, then according to Proposition 1.5, they have a unique common fixed point. □

Remark 2.4. *It is obvious that inequality (2.1) implies that f is continuous if g is continuous. Indeed, since $d(gx, gy) > 0$ for all $x, y \in X$ for which $gx \neq gy$, we have that $d(fx, fy) < d(gx, gy)$. Hence, the assumptions in [5] and [8] that f is continuous are superfluous.*

In the sequel we introduce the following generalized (Z_G, g)-contraction.

Definition 2.5. *Let (X, d) be a metric space and $f, g : X \to X$ be self-mappings. A mapping f is called a generalized (Z_G, g)-contraction if there exists $\zeta \in Z_G$ such that*

$$\zeta\left(d(fx, fy), \max\left\{d(gx, gy), d(gx, fx), d(gy, fy), \frac{d(gx, fy) + d(gy, fx)}{2}\right\}\right) \ge C_G$$

for all $x, y \in X$ with $gx \neq gy$.

In the case that $g = i_X$ (identity mapping on X) and $C_G = 0$ we get $\mathcal{Z}$-contraction of [16] (Definition 2, Theorem 2).

The next result also generalizes several ones in the existing literature. Since its proof is similar to the proof of Theorem 2.2, we omit it.

Theorem 2.6. *Let (X, d) be a metric space, $f, g : X \to X$ be self-mappings and f be a generalized (Z_G, g)-contraction. Suppose that there exists a Picard-Jungck sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ of (f, g). Also assume that at leas, one of the following conditions hold:*

*(**i**) $(f(X), d)$ or $(g(X), d)$ is complete;*

*(**ii**) (X, d) is complete and fand g are continuous and compatible.*

Then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible, then they have a unique common fixed point in X.

Remark 2.7. *Theorems 2.2 and 2.6 hold true if, in particular, (X, d) is complete, g is continuous and f and g are commuting.*

It is also worth noting that the two examples given in [8] are not suitable to support their main result. Neither of these examples is a proper generalization of the corresponding result of Jungck [11]. In other words, there is a $\lambda \in (0, 1)$ such that $d(fx, fy) \le \lambda d(gx, gy)$ for all $x, y \in X = [0, +\infty)$, $fx = x + 10$, $gx = \frac{10}{9}x + e^x + \sin\frac{\pi x}{2(1+x)} + 1$, $d(x, y) = \left|x - y\right|$, ([8, Example 5.11]), resp. $fx = \arctan(x + 2)$, $gx = \log(x + 3)$ ([8, Example 5.12]). Also, [5, Example 20], in the same metric space, where $fx = x + 2$, $gx = 4x + e^{2x}$ is such.

In the sequel, we describe how to use our Theorem 2.2 in order to guarantee existence and uniqueness of a solution for a nonlinear equation.

Example 2.8. *Let $X = [0, +\infty)$ be endowed with the usual metric $d(x, y) = \left|x - y\right|$ for all $x, y \in [0, +\infty)$, and consider the mappings $f, g : [0, +\infty) \to [0, +\infty)$ given, for all $x \in [0, +\infty)$, by*

$$fx = x + 2, \quad gx = 4x + e^{2x}.$$

In order to solve the nonlinear equation

$$x + 2 = 4x + e^{2x},$$

Theorem 2.2 can be applied using the simulation function $\zeta(t, s) = \frac{9}{10}\left(s - \frac{(2+t)t}{1+t}\right)$ for $s, t \in [0, +\infty)$ and $C_F = 0$, $F(s, t) = s - \frac{(2+t)t}{1+t}$. Now, we have that

$$\begin{aligned}\zeta(d(fx, fy), d(gx, gy)) &= \frac{9}{10}\left(d(gx, gy) - \frac{(2 + d(fx, fy))\, d(fx, fy)}{1 + d(fx, fy)}\right) \\ &= \frac{9}{10}\left(\left|4(x-y) + \left(e^{2x} - e^{2y}\right)\right| - \frac{\left(2 + \left|x-y\right|\right)\left|x-y\right|}{1 + \left|x-y\right|}\right) \\ &\geq 0.\end{aligned}$$

Since $f(X) = [2, +\infty)$, $g(X) = [1, +\infty)$, using Theorem 2.2(i) the result follows.

The following example shows that our Theorem 2.2 is a proper generalization of the corresponding results of Jungck [11], L.-de-Hierro et al. [8] and of Olgun et al. [16].

Example 2.9. *Let $X = [0, 1]$ and $d : X \times X \to [0, +\infty)$ be defined by $d(x, y) = \left|x - y\right|$. Then (X, d) is a complete metric space. Define $f, g : X \to X$ as $fx = \frac{x}{2+x}$, $gx = \frac{x}{2}$. Then, f is not Jungck's contraction in the sense that there is $\lambda \in (0, 1)$ such that $d(fx, fy) \leq \lambda d(gx, gy)$ for all $x, y \in X$. However, putting $\zeta(t, s) = \frac{s}{s+1} - t$, $G(s, t) = s - t$, $C_G = 0$, we have that f is a $(\mathcal{Z}, g)$-contraction with respect to ζ. Indeed, we obtain*

$$\zeta(d(fx, fy), d(gx, gy)) \geq C_G = 0 \Leftrightarrow \frac{d(gx, gy)}{1 + d(gx, gy)} - d(fx, fy) \geq 0$$

$$\begin{aligned}\Leftrightarrow &\quad \frac{\frac{1}{2}\left|x-y\right|}{1 + \frac{1}{2}\left|x-y\right|} - \left|\frac{x}{x+2} - \frac{y}{y+2}\right| \\ = &\quad \frac{\left|x-y\right|}{2 + \left|x-y\right|} - \frac{2\left|x-y\right|}{(x+2)(y+2)} \geq 0,\end{aligned}$$

whenever $x, y \in X$. Further, since $f(X) = \left[0, \frac{1}{3}\right] \subseteq \left[0, \frac{1}{2}\right] = g(X)$ there exists a Picard-Jungck sequence $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ of (f, g). As both $(f(X), d)$ or $(g(X), d)$ are complete, this means that all the conditions of Theorem 2.3 are satisfied, i.e., the mappings f and g have a coincidence point $x = 0$. In other words, they have a unique common fixed point, which is the only solution of equation $\frac{x}{x+2} = \frac{x}{2}$, $x \in [0, 1]$.

Finally, we introduce the following:

Definition 2.10. *Let (X, d) be a metric space and $f, g : X \to X$ be self-mappings. A mapping f is called a (Z_G, g)-quasi-contraction of Ćirić-Das-Naik type if there exist $\zeta \in Z_G$, $\lambda \in (0, 1)$ such that*

$$\zeta(d(fx, fy), \lambda \max\{d(gx, gy), d(gx, fx), d(gy, fy), d(gx.fy), d(gy, fx)\}) \geq C_G$$

for all $x, y \in X$ with $gx \neq gy$.

In the case that $g = i_X$ (identity mapping on X) and $C_G = 0$ we get a $\mathcal{Z}$-quasi-contraction of Ćirić type.

Finally, we have the following open question: Does the following claim hold?

Claim. *Let f be a (Z_G, g)-quasi-contraction of of Ćirić-Das-Naik type in a metric space (X, d) and suppose that there exists a Picard-Jungck sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ of (f, g). Also assume that at least one of the following conditions hold:*

*(**i**) $(f(X), d)$ or $(g(X), d)$ is complete;*

*(**ii**) (X, d) is complete and f and g are continuous and compatible.*

Then f and g have unique point of coincidence. Moreover, if f and g are weakly compatible, then they have a unique common fixed point in X.

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