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Trace Inequalities for a Block Hadamard Product

Saliha Pehlivan[a], Mustafa Özel[b], Cenap Özel[c]

[a]*Department of Biomedical Engineering, Faculty of Engineering and Architecture*
Gediz University
Izmir, Turkey
[b]*Department of Geophysics Engineering, Faculty of Engineering*
Dokuz Eylul University
Izmir, Turkey
[c]*Department of mathematics, Faculty of Sciences*
King Abdulaziz University
Jeddah, Saudi Arabia
And
Department of mathematics, Faculty of Science
Dokuz Eylul University
Izmir, Turkey

Abstract. In this paper, some inequalities for the trace and eigenvalues of a block Hadamard product of positive semidefinite matrices are investigated. In particular, a Hölder type inequality and inequalities related to norm and determinants of block matrices are obtained. Additionally, the relation between the trace of block Hadamard product and the usual Kronecker product is established.

1. Introduction

Block Kronecker and block Hadamard products seem to be firstly defined and used by Horn, Mathias and Nakamura in 1991([4]). Günther and Klotz presented a survey focusing on the study of a block Hadamard and block Kronecker products of positive semidefinite matrices in 2012([2]). In that paper, some properties of such products were discussed and an inequality (a lower bound) on the determinant of the block Hadamard product was given. In 2014, Lin provided an Oppenheim type inequality for the determinant of the block Hadamard product([5]), which confirms a conjecture in [2]. So far, there have not been quite many results on the trace of such products in the literature. In this paper, we shall be mainly interested in the inequalities for the trace of the block Hadamard product.

For the rest of the paper, we first introduce the definitions and terminology that are used throughout the paper. In Section 3, we give some lemmas that play important roles in our results. In Section 4, we give our main results on the upper and lower bounds for the trace of the block Hadamard product of positive definite matrices.

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Email addresses: saliha.pehlivan@gediz.edu.tr (Saliha Pehlivan), mustafa.ozel@deu.edu.tr (Mustafa Özel), cenap.ozel@gmail.com (Cenap Özel)

2. Preliminaries

Let $M_{m\times n}$ be the linear space of $m \times n$ matrices with complex entries and $\mathcal{M}_{p,q}(M_{m,n})$ be the space of $p \times q$ block matrices, and write $M_{n,n} := M_n$ and $\mathcal{M}_p := \mathcal{M}_{p,p}(M_{n,n})$. The identity matrix in $\mathcal{M}_p$ is denoted by $\mathcal{I}_p = diag(I_n, \ldots, I_n)$ where $I_n \in M_n$.

A matrix $\mathcal{A} \in \mathcal{M}_p$ is Hermitian if $\mathcal{A}^* = \mathcal{A}$ where $\mathcal{A}^*$ is the conjugate transpose of $\mathcal{A}$. A Hermitian matrix $\mathcal{A}$ is said to be positive definite (positive semidefinite), denoted by $\mathcal{A} > 0$ ($\mathcal{A} \geq 0$), if $x^*\mathcal{A}x > 0$ ($x^*\mathcal{A}x \geq 0$) for all nonzero $x \in \mathbb{C}$.

The eigenvalues and singular values of $\mathcal{A} \in \mathcal{M}_p$ are denoted by $\lambda_1(\mathcal{A}), \lambda_2(\mathcal{A}), \ldots, \lambda_{pn}(\mathcal{A})$ and $\sigma_1(\mathcal{A}), \sigma_2(\mathcal{A}), \ldots, \sigma_{pn}(\mathcal{A})$, respectively. They are arranged in decreasing order: for a Hermitian matrix $\mathcal{A} \in \mathcal{M}_p$, $\lambda_1(\mathcal{A}) \geq \lambda_2(\mathcal{A}) \geq \ldots \geq \lambda_{pn}(\mathcal{A})$, and $\sigma_1(\mathcal{A}) \geq \sigma_2(\mathcal{A}) \geq \ldots \geq \sigma_{pn}(\mathcal{A})$. Note that $\sigma_i(\mathcal{A}) = \lambda_i(\mathcal{A})$ for a positive semidefinite matrix $\mathcal{A}$ for $i = 1, \ldots, pn$ since $\sigma_i(\mathcal{A}) = \lambda_i(\mathcal{A}^*\mathcal{A})^{1/2}$.

Let $\mathcal{A} = (A_{ij}) \in \mathcal{M}_{p,q}$ and $\mathcal{B} = (B_{ij}) \in \mathcal{M}_{s,t}$. Then we call matrices $\mathcal{A}$, $\mathcal{B}$ block commuting if every $n \times n$ block of $\mathcal{A}$ commutes with every $n \times n$ block of $\mathcal{B}$.

Let $\mathcal{A} = (A_{ij}) \in \mathcal{M}_{p,q}(M_{m,l})$ and $\mathcal{B} = (B_{ij}) \in \mathcal{M}_{p,q}(M_{l,n})$. Then the block Hadamard product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $\mathcal{A}\Box\mathcal{B} = (A_{ij}B_{ij})$. If $\mathcal{A}$ and $\mathcal{B}$ are both positive definite(positive semidefinite) and block commuting matrices, then $\mathcal{A}\Box\mathcal{B}$ is positive definite (positive semidefinite) as in [2] Corollary 3.3.

Let $A \in M_{m,l}$ and $\mathcal{B} = (B_{ij}) \in \mathcal{M}_{s,t}(M_{l,n})$. Then the block Kronecker product of A and $\mathcal{B}$ is defined by $A \boxtimes \mathcal{B} = (AB_{ij})_{i=1,\ldots,s}^{j=1,\ldots,t}$ where AB_{ij} is the usual matrix product of A and B_{ij}. For $\mathcal{A} = (A_{ij}) \in \mathcal{M}_{p,q}(M_{m,l})$, the block Kronecker product is given by $\mathcal{A} \boxtimes \mathcal{B} = (A_{ij} \boxtimes \mathcal{B})_{i=1,\ldots,p}^{j=1,\ldots,q}$.

For real vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in decreasing order, it is said that $\mathbf{x}$ is majorized by $\mathbf{y}$ ($\mathbf{x} \prec \mathbf{y}$) if

$$\sum_{i=k}^{n} x_i \geq \sum_{i=k}^{n} y_i \quad \text{for } k = 2, \ldots, n \quad \text{and} \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

A linear map Φ from M_n to M_n is called doubly stochastic if it is positive ($A \geq 0 \to \Phi(A) \geq 0$), unital ($\Phi(I) = I$) and trace preserving ($tr(\Phi(A)) = trA$). Then, for $\mathcal{A} \in M_{pn}$ and $\Phi : M_{pn} \to M_{pn}$, it is clear that $\Phi(\mathcal{A}) = \mathcal{A}\Box\mathcal{I}$ is doubly stochastic. Thus, by Lemma 2.14 in [8], we have

$$\lambda(\mathcal{A}\Box\mathcal{I}) \prec \lambda(\mathcal{A}). \tag{1}$$

3. Lemmas

In this section, we give lemmas that are fundamental in our main results.

Lemma 3.1. *[3] For any two matrices A and B,*

$$\sigma(AB) \prec_{log} \sigma(A)\sigma(B).$$

Lemma 3.2. *[6] Let $\alpha_i > 0$ ($i = 1, \ldots, n$) and $\sum_{i=1}^{n} \alpha_i \geq 1$. Let $a_{ij} > 0$ for $j = 1, \ldots, m$. Then*

$$\sum_{j=1}^{m} a_{1j}^{\alpha_1} a_{2j}^{\alpha_2} \cdots a_{nj}^{\alpha_n} \leq \Big(\sum_{j=1}^{m} a_{1j}\Big)^{\alpha_1} \Big(\sum_{j=1}^{m} a_{2j}\Big)^{\alpha_2} \cdots \Big(\sum_{j=1}^{m} a_{nj}\Big)^{\alpha_n}.$$

Lemma 3.3. *[7] Let A_i, $B_i \in M_n(\mathbb{C})$ for $i = 1, \ldots, m$ be positive semidefinite matrices and p, q be positive real numbers such that $1/p + 1/q = 1$. Then*

$$tr\Big(\sum_{i=1}^{m} A_i B_i\Big) \leq \Big(tr\Big(\sum_{i=1}^{m} A_i^p\Big)\Big)^{1/p} \Big(tr\Big(\sum_{i=1}^{m} B_i^q\Big)\Big)^{1/q}.$$

For any $\mathcal{A}$, $\mathcal{B} \in \mathcal{M}_p$, $\mathcal{A}\Box\mathcal{B} = (\mathcal{A} \boxtimes \mathcal{B})[\alpha]$ where α is a block index set such that $\alpha = \{1, p+2, 2p+3, \cdots, p^2\}$, there exists a unital positive linear map Φ from M_{p^2} to M_p such that $\Phi(\mathcal{A} \boxtimes \mathcal{B}) = \mathcal{A}\Box\mathcal{B}$ for all $\mathcal{A}$, $\mathcal{B} \in \mathcal{M}_p$ as in [8] Lemma 1.9.

Lemma 3.4. *For any* $\mathcal{A}, \mathcal{B} > 0$, *we have*

$$(\log \mathcal{A} + \log \mathcal{B})\square I \leq \log(\mathcal{A}\square\mathcal{B}) \quad \mathcal{A}, \mathcal{B} > 0. \tag{2}$$

Proof. First note that $\Phi(\mathcal{A} \boxtimes \mathcal{I}) = \mathcal{A}\square\mathcal{I} = \mathcal{I}\square\mathcal{A} = \Phi(\mathcal{I} \boxtimes \mathcal{A})$ and $(\mathcal{A} \boxtimes \mathcal{I})^k = (\mathcal{A}^k \boxtimes \mathcal{I})$ by Proposition 2.3 of [2] so for any analytic function f we have $f(\mathcal{A} \boxtimes \mathcal{I}) = f(\mathcal{A}) \boxtimes \mathcal{I}$ by power series representation of f. Then, since Φ is linear, we get

$$\begin{aligned}
(\log \mathcal{A} + \log \mathcal{B})\square\mathcal{I} &= \Phi\Big((\log \mathcal{A} + \log \mathcal{B}) \boxtimes \mathcal{I}\Big) && (3)\\
&= \Phi\Big(\log \mathcal{A} \boxtimes \mathcal{I} + \log \mathcal{B} \boxtimes \mathcal{I}\Big) && (4)\\
&= \Phi\Big(\log \mathcal{A} \boxtimes \mathcal{I}\Big) + \Phi\Big(\log \mathcal{B} \boxtimes \mathcal{I}\Big) && (5)\\
&= \Phi\Big(\log \mathcal{A} \boxtimes \mathcal{I}\Big) + \Phi\Big(\mathcal{I} \boxtimes \log \mathcal{B}\Big) && (6)\\
&= \Phi\Big(\log \mathcal{A} \boxtimes \mathcal{I} + \mathcal{I} \boxtimes \log \mathcal{B}\Big) && (7)\\
&= \Phi\Big(\log(\mathcal{A} \boxtimes \mathcal{I}) + \log(\mathcal{I} \boxtimes \mathcal{B})\Big) && (8)\\
&= \Phi\Big(\log((\mathcal{A} \boxtimes \mathcal{I})(\mathcal{I} \boxtimes \mathcal{B}))\Big) && (9)\\
&= \Phi\Big(\log(\mathcal{A} \boxtimes \mathcal{B}\Big) && (10)\\
&\leq \log\Big(\Phi(\mathcal{A} \boxtimes \mathcal{B}\Big) && (11)\\
&= \log(\mathcal{A}\square\mathcal{B}) && (12)
\end{aligned}$$

where (10) and (11) follow from the facts, respectively, that $(\mathcal{A} \boxtimes \mathcal{I})(\mathcal{I} \boxtimes \mathcal{B}) = \mathcal{A} \boxtimes \mathcal{B}$ (see [2]) and $\Phi(\log \mathcal{A}) \leq \log(\Phi(\mathcal{A}))$ for a unital positive linear map Φ and $\mathcal{A} > 0$ (see [8]).

□

4. Main Results

In this section, we first introduce an inequality on the product of eigenvalues of the block Hadamard product of positive definite matrices that generalize Oppenheim's inequality. Then we give some upper and lower bounds on the trace of block Hadamard product of positive definite (positive semidefinite) matrices.

Theorem 4.1. *Let* $\mathcal{A}, \mathcal{B} \in \mathcal{M}_p$ *be block commuting and positive definite matrices. Then*

$$\prod_{j=k}^{pn} \lambda_j(\mathcal{A}\square\mathcal{B}) \geq \prod_{j=k}^{pn} \lambda_j(\mathcal{A}\mathcal{B}), \quad k = 1, 2, \ldots, pn. \tag{13}$$

Proof. By (2) and (1), respectively, for all $k = 1, 2, \ldots, pn$ we have

$$\begin{aligned}
\log\Big(\prod_{j=k}^{pn} \lambda_j(\mathcal{A}\Box\mathcal{B})\Big) &= \sum_{j=k}^{pn} \lambda_j\Big(\log(\mathcal{A}\Box\mathcal{B})\Big) \\
&\geq \sum_{j=k}^{pn} \lambda_j\Big((\log \mathcal{A} + \log \mathcal{B})\Box I\Big) \\
&\geq \sum_{j=k}^{pn} \lambda_j\Big(\log \mathcal{A} + \log \mathcal{B}\Big) \\
&\geq \sum_{j=k}^{pn} \lambda_j\Big(\log \mathcal{A}^{1/2}\mathcal{B}\mathcal{A}^{1/2}\Big) \\
&= \log\Big(\prod_{j=k}^{pn} \lambda_j(\mathcal{A}\mathcal{B})\Big)
\end{aligned}$$

where the last inequality follows from $\log \mathcal{A} + \log \mathcal{B} \prec \log(\mathcal{A}^{1/2}\mathcal{B}\mathcal{A}^{1/2})$ (see [8] pg. 22). $\Box$

We remark here that, by Theorem 4.1 for k=1, we get Openheim's inequality for the block Hadamard product: $\det(A\Box B) \geq \det A \det B$.

The following result is a Hölder type inequality for the trace of block Hadamard product.

Theorem 4.2. *Let $\mathcal{A} = (A_{ij})$, $\mathcal{B} = (B_{ij}) \in \mathcal{M}_p$ be positive semidefinite block matrices and $q_1, q_2 > 1$ with $1/q_1 + 1/q_2 = 1$. Then*

(i) $tr(\mathcal{A}\Box\mathcal{B}) \leq \Big(\sum_{i=1}^{p} trA_{ii}^{q_1}\Big)^{1/q_1}\Big(\sum_{i=1}^{p} trB_{ii}^{q_2}\Big)^{1/q_2}.$

(ii) $tr(\mathcal{A}\Box\mathcal{B}) \leq \sum_{i=1}^{p} \Big(trA_{ii}^{q_1}\Big)^{1/q_1}\Big(trB_{ii}^{q_2}\Big)^{1/q_2}.$

Proof. i) By Lemma 3.3 and the linearity of the trace we have

$$\begin{aligned}
tr(\mathcal{A}\Box\mathcal{B}) &= \sum_{i=1}^{p} tr(A_{ii}B_{ii}) = tr\Big(\sum_{i=1}^{p} A_{ii}B_{ii}\Big) \\
&\leq \left(tr\left(\sum_{i=1}^{p} A_{ii}^{q_1}\right)\right)^{1/q_1} \left(tr\left(\sum_{i=1}^{p} B_{ii}^{q_2}\right)\right)^{1/q_2} \\
&= \left(\sum_{i=1}^{p} trA_{ii}^{q_1}\right)^{1/q_1} \left(\sum_{i=1}^{p} trB_{ii}^{q_2}\right)^{1/q_2}.
\end{aligned}$$

ii) For $m = 1$ in Lemma 3.3, we obtain

$$\begin{aligned}
tr(\mathcal{A}\Box\mathcal{B}) &= \sum_{i=1}^{p} (trA_{ii}B_{ii}) \\
&\leq \sum_{i=1}^{p} \Big(trA_{ii}^{q_1}\Big)^{1/q_1}\Big(trB_{ii}^{q_2}\Big)^{1/q_2}.
\end{aligned}$$

□

The relation between the trace of block Hadamard product and the trace of Kronecker product is given in the following form:

Corollary 4.3. *Let $\mathcal{A}$, $\mathcal{B} \in \mathcal{M}_2$ be block commuting and positive semidefinite matrices. Then*

$$tr(\mathcal{A}\Box\mathcal{B}) \leq \Big(tr\,((\mathcal{A}\Box\mathcal{A}) \otimes (\mathcal{B}\Box\mathcal{B}))\Big)^{1/2}.$$

Proof. Let $p = q_1 = q_2 = 2$ in Theorem 4.2(i). Then we get

$$tr(\mathcal{A}\Box\mathcal{B}) \leq \Big(\sum_{i=1}^{2} trA_{ii}^2\Big)^{1/2}\Big(\sum_{i=1}^{2} trB_{ii}^2\Big)^{1/2}$$

Moreover, we have

$$\begin{aligned}\Big(tr\Big(\sum_{i=1}^{2} A_{ii}^2\Big)\Big)^{1/2}\Big(tr\Big(\sum_{i=1}^{2} B_{ii}^2\Big)\Big)^{1/2} &= \Big(tr\Big(A_{11}^2 + A_{22}^2\Big)\Big)^{1/2}\Big(tr\Big(B_{11}^2 + B_{22}^2\Big)\Big)^{1/2}\\ &= \Big(tr(\mathcal{A}\Box\mathcal{A})\Big)^{1/2}\Big(tr(\mathcal{B}\Box\mathcal{B})\Big)^{1/2}\\ &= \Big(tr(\mathcal{A}\Box\mathcal{A})tr(\mathcal{B}\Box\mathcal{B})\Big)^{1/2}\\ &= \Big(tr\,((\mathcal{A}\Box\mathcal{A}) \otimes (\mathcal{B}\Box\mathcal{B})])\Big)^{1/2},\end{aligned}$$

where the last equality follows from the fact that $tr(N \otimes M) = trNtrM$. Thus, we obtain the result. □

This can also be generalized to the following:

$$tr(\mathcal{A}\Box\mathcal{B}) \leq \Big(tr\Big((\Box_{j=1}^{p}\mathcal{A}) \otimes (\Box_{j=1}^{p}\mathcal{B})\Big)\Big)^{1/2} \quad \text{for } \mathcal{A}, \mathcal{B} \in \mathcal{M}_p \text{ and } \mathcal{A}, \mathcal{B} > 0,$$

where $\Box_{j=1}^{p}\mathcal{A}$ denotes for the block Hadamard product of $\mathcal{A}$ p times.

Now, an inequality that depends on the norm and trace of block matrices is given as below.

Theorem 4.4. *Let $\mathcal{A} = (A_{ij})$, $\mathcal{B} = (B_{ij}) \in \mathcal{M}_p$ be block commuting and positive semidefinite matrices. Then for any natural number $k \geq 1$, we have*

$$tr\Big(\Box_{i=1}^{k}(\mathcal{A}\Box\mathcal{B})\Big) \leq \min\Big\{\sum_{i=1}^{p} \|A_{ii}\|^k trB_{ii}^k, \sum_{i=1}^{p} \|B_{ii}\|^k trA_{ii}^k\Big\},$$

where $\|\cdot\|$ is a matrix norm. In particular,

$$tr(\mathcal{A}\Box\mathcal{B}) \leq \min\Big\{\sum_{i=1}^{p} \|A_{ii}\|trB_{ii}, \sum_{i=1}^{p} \|B_{ii}\|trA_{ii}\Big\}.$$

Proof. First note that we have $A_{ii}^kB_{ii}^k = (A_{ii}B_{ii})^k$ since $\mathcal{A}$ and $\mathcal{B}$ are block commuting. By Lemma 3.1 and

$\lambda_j(A) \leq \|A\|$, we get

$$\begin{aligned} tr\Big(\square_{i=1}^{k}(\mathcal{A}\square\mathcal{B})\Big) &= tr\Big(\sum_{i=1}^{p}(A_{ii}B_{ii})^{k}\Big) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{n}\lambda_j^{k}(A_{ii}B_{ii}) \\ &\leq \sum_{i=1}^{p}\sum_{j=1}^{n}\|A_{ii}\|^{k}\lambda_j^{k}(B_{ii}) \\ &= \sum_{i=1}^{p}\|A_{ii}\|^{k}\sum_{j=1}^{n}\lambda_j^{k}(B_{ii}) \\ &= \sum_{i=1}^{p}\|A_{ii}\|^{k} trB_{ii}^{k}. \end{aligned}$$

Similarly, we have $tr\Big(\square_{i=1}^{k}(\mathcal{A}\square\mathcal{B})\Big) \leq \sum_{i=1}^{p}\|B_{ii}\|^{k} trA_{ii}^{k}$. □

Corollary 4.5. *Let $\mathcal{A} = (A_{ij})$, $\mathcal{B} = (B_{ij}) \in \mathcal{M}_p$ be block commuting and positive semidefinite matrices. Then we have the followings:*

(i) For $q_1 + q_2 \geq 1$ and $q_1, q_2 > 0$,

$$tr\Big(\square_{i=1}^{k}(\mathcal{A}\square\mathcal{B})\Big)$$

$$\leq \min\Big\{\Big(\sum_{i=1}^{p}\|A_{ii}\|^{q_1 k}\Big)^{1/q_1}\Big(\sum_{i=1}^{p}\Big(trB_{ii}^{k}\Big)^{q_2}\Big)^{1/q_2}, \Big(\sum_{i=1}^{p}\|B_{ii}\|^{q_1 k}\Big)^{1/q_1}\Big(\sum_{i=1}^{p}\Big(trA_{ii}^{k}\Big)^{q_2}\Big)^{1/q_2}\Big\}$$

(ii) $tr\Big[\square_{i=1}^{k}\mathcal{A}\Big] \leq \min\Big\{n\sum_{i=1}^{p}\|A_{ii}\|^{k}, \sum_{i=1}^{n} trA_{ii}^{k}\Big\}$

Proof. (i) follows from Theorem 4.4 and Theorem 2.8 of [7]. Letting $\mathcal{B} = \mathcal{I}$ in Theorem 4.4, we get (ii). □

Finally, we shall discuss a lower bound on the trace of block Hadamard product in terms of determinants of block matrices.

Theorem 4.6. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{M}_p$ be positive definite block matrices. Then*

$$tr\Big(\square_{i=1}^{k}(\mathcal{A}\square\mathcal{B})\Big) \geq n\sum_{i=1}^{p}\Big(det(A_{ii}B_{ii})\Big)^{k/n}.$$

In particular, $tr(\mathcal{A}\square\mathcal{B}) \geq n\sum_{i=1}^{p}\Big(det(A_{ii}B_{ii})\Big)^{1/n}$.

Proof. For each A_{ii} we can find a unitary matrix U_i such that $U_i\Lambda_{ii}U_i^{*} = A_{ii}$ since $\mathcal{A} > 0$ where $\Lambda_{ii} = diag(\lambda_1(\Lambda_{ii}), \ldots, \lambda_n(\Lambda_{ii}))$. Define $b_{jj}^{(i)}$ as the diagonal elements of $U_i^{*}B_{ii}^{k}U_i$ for $j = 1, \ldots, n$. Then by arithmeticgeometric mean inequality we get

$$
\begin{aligned}
tr\Big(\square_{i=1}^{k}(\mathcal{A}\square\mathcal{B})\Big) &= \sum_{i=1}^{p}(tr(A_{ii}B_{ii})^{k}) \\
&= \sum_{i=1}^{p} tr(U_i\Lambda_{ii}^{k}U_i^{*}B_{ii}^{k}) \\
&= \sum_{i=1}^{p} tr(\Lambda_{ii}^{k}(U_i^{*}B_{ii}U_i)^{k}) \\
&= \sum_{i=1}^{p} \lambda_1^{k}(\Lambda_{ii})b_{11}^{(i)} + \cdots + \lambda_n^{k}(\Lambda_{ii})b_{nn}^{(i)} \\
&\geq n\sum_{i=1}^{p}\Big(\lambda_1^{k}(\Lambda_{ii})b_{11}^{(i)}\cdots\lambda_n^{k}(\Lambda_{ii})b_{nn}^{(i)}\Big)^{1/n} \\
&\geq n\sum_{i=1}^{p}\Big(det(\Lambda_{ii}^{k})\Big)^{1/n}\Big(det(U_i^{*}B_{ii}U_i)^{k}\Big)^{1/n} \\
&= n\sum_{i=1}^{p}\Big(detA_{ii}detB_{ii}\Big)^{k/n}
\end{aligned}
$$

where the last inequality follows from the fact that $detC \leq c_{11}c_{22}\cdots c_{nn}$ for any matrix $C \in M_n$ with diagonal elements c_{ii}, $i = 1, \ldots, n$. $\square$

5. Example

Let $\mathcal{A} = \mathcal{B}$ be positive definite matrix such that

$$
\begin{pmatrix} 4 & 1 & \cdot & 1 & 0 \\ 2 & 3 & \cdot & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & 6 & 1 \\ 0 & 1 & \cdot & 0 & 2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.
$$

Then 36.0804, 24.9328, 4.9058, 3.0810 are the eigenvalues of

$$
\mathcal{A}\square\mathcal{A} = \begin{pmatrix} 18 & 7 & 1 & 0 \\ 14 & 11 & 0 & 1 \\ 1 & 0 & 36 & 8 \\ 0 & 1 & 0 & 4 \end{pmatrix}.
$$

For the result of Theorem 4.1, we have the following table:

	k=1	k=2	k=3	k=4
$\prod_{i=k}^{4}\lambda_i(\mathcal{A}\square\mathcal{A})$	13597.0262	376.8535	15.1148	3.0810
$\prod_{i=k}^{4}\lambda_i(\mathcal{A}\mathcal{A})$	7913.8618	183.3899	8.8305	1.1143

In Theorem 4.2, for $q_1 = q_2 = 2$, we have the equality in both cases. On the other hand, for $q_1 = 3$, $q_2 = 3/2$ we have 70.8071 and 70.6389, respectively, as an upper bound for $tr(\mathcal{A}\square\mathcal{A}) = 69$.

For $k = 1$ and $k = 2$ in Theorem 4.4, we have the following inequalities $69 < 83$ and $1953 < 2165$, respectively.

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