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R-P-Spaces and Subrings of $C(X)$

Mehdi Parsinia[a]

[a]*Shahid Chamran University of Ahvaz, Iran*

Abstract. A Tychonoff space X is called a P-space if $M^p = O^p$ for each $p \in \beta X$. For a subring R of $C(X)$, we call X an R-P-space, if $M^p \cap R = O^p \cap R$ for each $p \in \beta X$. Various characterizations of R-P-spaces are investigated some of which follows from constructing the smallest invertible subring of $C(X)$ in which R is embedded, $\mathcal{S}_R^{-1}R$. Moreover, we study R-P-spaces when R is an intermediate ring or an intermediate C-ring. We follow a new approach to some results of [W. Murray, J. Sack and S. Watson, P-spaces and intermediate rings of continuous functions, *Rocky Mount. J. Math.*, to appear]. Also, some algebraic characterizations of P-spaces via intermediate rings are given. Finally, we establish some characterizations of $C(X)$ among intermediate C-rings which are of the form $I + C^*(X)$, where I is an ideal in $C(X)$.

1. Introduction

Throughout this paper all topological spaces are assumed to be Tychonoff. Moreover, all ideals are assumed to be proper and, unless otherwise mentioned, all subrings are assumed to be unital. We denote by $C(X)$ the algebra of all real-valued continuous functions on a given topological space X and by $C^*(X)$ the subalgebra of $C(X)$ consisting of all bounded elements. For each $f \in C(X)$, $Z(f) = \{x : f(x) = 0\}$ denotes the zero-set of f. The collection of all zero-sets of elements of R is denoted by $Z(R)$ for each subring R of $C(X)$. We use $Z(X)$ instead of $Z(C(X))$. For a topological space X, βX denotes the Stone-Čech compactification of X and υX denotes the Hewitt-realcompactification of X. It is well-known that every $f \in C(X)$ has a continuous extension f^* from βX to $\mathbb{R}^*$ (the one-point compactification of $\mathbb{R}$) and a continuous extension f^υ from υX to $\mathbb{R}$. For each $f \in C(X)$, we denote by $\upsilon_f X$ the set $\{p \in \beta X : f^*(p) < \infty\}$. For a subset A of $C(X)$, we set $\upsilon_A X = \{p \in \beta X : f^*(p) < \infty, \forall f \in A\} = \bigcap_{f \in A} \upsilon_f X$. Evidently, $\upsilon_C X = \upsilon X$ and $\upsilon_{C^*} X = \beta X$. Also, $\upsilon X \subseteq \upsilon_A X$ for each subset A of $C(X)$. By a realcompactification of X, we mean a realcompact space containing X as a dense subspace. It follows from [11, 8B, 3] that $\upsilon_A X$ is a realcompactification of X for each $A \subseteq C(X)$. The reader is referred to [11] for terms and notations not defined here.

A subring $A(X)$ of $C(X)$ is called an intermediate ring, if $C^*(X) \subseteq A(X)$. An intermediate ring $A(X)$ is called a C-ring, if $A(X)$ is isomorphic to $C(Y)$ for some topological space Y. Following [17], for each element f of an intermediate ring $A(X)$ of $C(X)$, we set $S_A(f) = \{p \in \beta X : (fg)^*(p) = 0, \forall g \in A(X)\}$. A routine reasoning shows that $S_A(fg) = S_A(f) \cup S_A(g)$, $S_A(f^2+g^2) = S_A(f) \cap S_A(g)$ and $S_A(f^n) = S_A(f)$ for each $f, g \in A(X)$ and each $n \in \mathbb{N}$. Furthermore, $\mathrm{cl}_{\beta X} Z(f) \subseteq S_A(f) \subseteq Z(f^*)$ and thus $S_A(f) \cap X = Z(f)$. Moreover, $S_C(f) = \mathrm{cl}_{\beta X} Z(f)$ for each $f \in C(X)$ and $S_{C^*}(f) = Z(f^\beta)$ for each $f \in C^*(X)$. Also, we use M_A^p to denote the set $\{f \in A(X) : p \in S_A(f)\}$

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Email address: parsiniamehdi@gmail.com (Mehdi Parsinia)

for each $p \in \beta X$. Clearly, $M^p_C = M^p = \{f \in C(X) : p \in \mathrm{cl}_{\beta X} Z(f)\}$ and $M^p_{C^*} = M^{*p} = \{f \in C^*(X) : p \in Z(f^\beta)\}$. From [17, Theorem 2.9] it follows that the collection of all the maximal ideals of an intermediate ring $A(X)$ is $\{M^p_A : p \in \beta X\}$. This implies that $f \in A(X)$ is invertible in $A(X)$ if and only if $S_A(f) = \emptyset$.

An ideal I of a subring R of $C(X)$ is called growing, if it contains no unit element of $C(X)$. A growing ideal I of R is called fixed, if $\bigcap_{f\in I} Z(f) = \emptyset$, otherwise, is called free. A growing ideal I in R is called maximal growing, if it is maximal in the set of all growing ideals of R. If every ideal of a subring R is growing, then R is called invertible. It is easy to observe that R is an invertible subring of $C(X)$ if and only if f is invertible in R whenever $f \in R$ with $Z(f) = \emptyset$. An ideal I of R is called a z_R-ideal, if whenever $Z(f) \subseteq Z(g)$, $f \in I$ and $g \in R$, then $g \in I$. It is easy to see that every z_R-ideal is a growing ideal. However, the converse of this statement does not hold, in general. For example, the principal ideal in $C(\mathbb{R})$ generated by the identity function on $\mathbb{R}$ is a growing ideal in $C(\mathbb{R})$ which is not a z_C-ideal. For a subring R of $C(X)$ and each $p \in \beta X$, we denote by $M^p(R)$ (resp., $O^p(R)$) the set $\{f \in A(X) : p \in \mathrm{cl}_{\beta X} Z(f)\}$ (resp., $\{f \in A(X) : p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)\}$). We use M^p (resp., O^p) instead of $M^p(C(X))$ (resp., $O^p(C(X))$). Clearly, whenever $x \in X$, then $M^x(R)$ (resp., $O^x(R)$) coincides with $M_x(R) = \{f \in R : x \in Z(f)\}$ (resp., $O_x(R) = \{f \in R : x \in \mathrm{int}_X Z(f)\}$).

Remember that an ideal I of a commutative ring Q is called a z-ideal if whenever $f \in I$, then $M_f(Q) \subseteq I$ in which $M_f(Q)$ denotes the intersection of all the maximal ideals of Q containing f. We use M_f instead of $M_f(C(X))$ for each $f \in C(X)$. It is well-known that $M_f = \{g \in C(X) : Z(f) \subseteq Z(g)\}$ for each $f \in C(X)$. Thus, z-ideals of $C(X)$ coincide with z_C-ideals. Moreover, from [15, Proposition 2.7] it follows that $M_f(A(X)) = \{g \in A(X) : S_A(f) \subseteq S_A(g)\}$ for each element f of an intermediate ring $A(X)$. Hence, an ideal I of an intermediate ring $A(X)$ is a z-ideal if and only if whenever $S_A(f) \subseteq S_A(g)$ where $f \in I$ and $g \in A(X)$, then $g \in I$. This implies that z-ideals of an intermediate ring $A(X)$ do not coincide with z_A-ideals. For example, M^{*p}, for each $p \in \beta\mathbb{R} \setminus \mathbb{R}$, is a z-ideal in $C^*(\mathbb{R})$ which is not a z_{C^*}-ideal. A commutative ring Q is called a regular ring (in the sense of Von-Neuman), if for every $a \in Q$, there exists some $b \in Q$ such that $a^2 b = a$, equivalently, every prime ideal in Q is maximal; see [12] for more details. It is shown in [13, Theorem 1] that a commutative ring Q is a regular ring if and only if every ideal in Q is a z-ideal.

A topological space X is called a P-space, if $O^p = M^p$ for every $p \in \beta X$, equivalently, $C(X)$ is a regular ring; see [11, 14.29] for more details. The close relation between P-spaces and ring properties of $C(X)$, inspired us to check this notion when $C(X)$ is replaced by one of its subrings. The aim of this paper is to introduce and study the notion of R-P-spaces as a generalization of the notion of P-spaces via subrings of $C(X)$. Associated to a subring R of $C(X)$, we call X an R-P-space, if $M^p(R) = O^p(R)$ for every $p \in \beta X$. It is evident that every P-space is an R-P-space. However, the converse of this statement does not hold, in general. This paper consists 4 sections. In Section 1, some necessary notations and terminologies are introduced. In Section 2, we introduce the notion of R-P-spaces. By constructing the smallest invertible subring of $C(X)$ in which a given subring R is embedded, $\mathcal{S}_R^{-1}R$, and characterizing growing maximal ideals of R, various characterizations of R-P-spaces are investigated. In Section 3, we study R-P-spaces for the case that R is an intermediate ring of $C(X)$. We will observe that whenever R is an intermediate ring of $C(X)$, then R-P-spaces coincide with P-spaces. Some algebraic characterizations of P-spaces via intermediate rings follows from this fact. Moreover, by establishing a characterization of $C(X)$ among its intermediate rings, some results of [14] are proved by a different way. In Section 4, we study R-P-spaces with the restriction that R is an intermediate C-ring of $C(X)$. As an special case, we consider intermediate C-rings of the form $I + C^*(X)$ where I is an ideal of $C(X)$. This is done by identifying the realcompactifications that ideals of $C(X)$ generate.

2. *R*-*P*-Spaces

Recall that a topological space X is called an R-P-space, where R is a subring of $C(X)$, if $M^p(R) = O^p(R)$ for every $p \in \beta X$. Clearly, every P-space is an R-P-space. However, the converse of this statement does not hold, in general. For example, every topological space X is trivially an $\mathbb{R}$-P-space; see also Example 2.6 and Example 2.10 in below. For a subring R of $C(X)$, we use $\mathcal{S}_R$ to denote the set of all elements of R which are invertible in $C(X)$. It is evident that $\mathcal{S}_R$ is a multiplicative subset of R. The ring of fractions of R with respect to $\mathcal{S}_R$ is denoted by $\mathcal{S}_R^{-1}R$. For an ideal I of R and ideal $\mathcal{J}$ in $\mathcal{S}_R^{-1}R$, we denote by I^e and $\mathcal{J}^c$, the extension of I

and contraction of $\mathcal{J}$ with respect to the identity mapping of R into $\mathcal{S}_R^{-1}R$, respectively; see [24]. Hereafter, we denote $\mathcal{S}_R^{-1}R$ by R^e.

Proposition 2.1. *For a subring R of $C(X)$, R^e is the smallest invertible subring of $C(X)$ in which R is embedded.*

Proof. It is easy to see that $Z(\frac{f}{g}) = Z(f)$ for each $f \in R$ and $g \in \mathcal{S}_R$. Therefore, R^e is an invertible subring. Also, clearly, R^e contains a copy of R. Now, let Q be an invertible subring of $C(X)$ in which R is embedded and $\frac{f}{s} \in R^e$. Hence, $f \in R$ and $s \in \mathcal{S}_R$ such that $Z(s) = \emptyset$. As Q is an invertible subring in which R is embedded and $s \in Q$, we have $f, s, \frac{1}{s} \in Q$ and hence $\frac{f}{s} \in Q$. Thus, $R^e \subseteq Q$ which completes the proof. □

For a subring R of $C(X)$, we use $\mathrm{cl}_{\beta X}Z(R)$ to denote $\{\mathrm{cl}_{\beta X}Z(f) : f \in R\}$. Clearly, $\mathrm{cl}_{\beta X}Z(R)$ is an open base for a topology on βX which we call $\mathrm{cl}_{\beta X}Z(R)$-topology. Using this topology, we characterize maximal growing ideals of R by the next proposition. Note that this proposition is first stated in [3] and we prove it here for the sake of completeness.

Proposition 2.2. *The following statements hold for a subring R of $C(X)$:*
(a) Every maximal growing ideal of R is of the form $M^p(R)$ for some $p \in \beta X$.
(b) An ideal I of R is a growing ideal if and only if there exists $p \in \beta X$ such that $I \subseteq M^p(R)$.
(c) $M^p(R)$ is a maximal growing ideal for every $p \in \beta X$ if and only if $\mathrm{cl}_{\beta X}Z$ is clopen in the $\mathrm{cl}_{\beta X}Z(R)$-topology for every $Z \in Z(R)$.

Proof. (a) Suppose that M is a maximal growing ideal of R. Hence, $\mathrm{cl}_{\beta X}Z(M)$ has the finite intersection property. Thus, there exists $p \in \cap_{f\in M}\mathrm{cl}_{\beta X}Z(f)$ and hence $M \subseteq M^p(R)$. Also, clearly, $M^p(R)$ is a growing ideal. Consequently, $M = M^p(R)$.

(b⇒) This is obvious by using part (a) and the fact that every growing ideal is contained in a maximal growing ideal.

(b⇐) By part (a), $M^p(R)$ is a maximal growing ideal in R for each $p \in \beta X$. Also, if I is an ideal in R such that $I \subseteq M^p(R)$, then for each $f \in I$, we will have $Z(f) \neq \emptyset$, since, $f \in M^p(R)$. This means that I is a growing ideal.

(c⇒) Suppose that $p \in \beta X \setminus \mathrm{cl}_{\beta X}Z(f)$ for some $f \in R$. Thus, $f \notin M^p(R)$ and since $M^p(R)$ is a maximal growing ideal, there exists $g \in M^p(R)$ such that $Z(f)\cap Z(g) = \emptyset$. Hence, $p \in \mathrm{cl}_{\beta X}Z(g)$ and $\mathrm{cl}_{\beta X}Z(f)\cap\mathrm{cl}_{\beta X}Z(g) = \emptyset$ which implies that $\mathrm{cl}_{\beta X}Z(f)$ is clopen in the $\mathrm{cl}_{\beta X}Z(R)$-topology.

(c⇐) Assume that $p \in \beta X$ and $f \notin M^p(R)$. Thus, $p \notin \mathrm{cl}_{\beta X}Z(f)$ and by our hypothesis, there exists $g \in M^p(R)$ such that $\mathrm{cl}_{\beta X}Z(f) \cap \mathrm{cl}_{\beta X}Z(g) = \emptyset$; i.e., $Z(f) \cap Z(g) = \emptyset$. Therefore, $M^p(R)$ is a maximal growing ideal. □

Proposition 2.3. *For a subring R of $C(X)$, we have $(M^p(R))^e = M^p(R^e)$ and $(O^p(R))^e = O^p(R^e)$ for each $p \in \beta X$.*

Proof. If $\frac{f}{s} \in (M^p(R))^e$, then $f \in M^p(R)$ and $s \in \mathcal{S}_R$. Hence, $p \in \mathrm{cl}_{\beta X}Z(f) = \mathrm{cl}_{\beta X}Z(\frac{f}{s})$. This means that $\frac{f}{s} \in M^p(R^e)$; i.e., $(M^p(R))^e \subseteq M^p(R^e)$. Conversely, if $\frac{f}{s} \in M^p(R^e)$, then $p \in \mathrm{cl}_{\beta X}Z(\frac{f}{s}) = \mathrm{cl}_{\beta X}Z(f)$. This implies that $f \in M^p(R)$ and thus $\frac{f}{s} \in (M^p(R))^e$; i.e., $M^p(R^e) \subseteq (M^p(R))^e$ and hence the first equality follows. The second equality could be proved by a similar reasoning. □

A subring R of $C(X)$ is called β-determining, if $\{Z(f^*) : f \in R\}$ constitutes a base for the closed sets of βX. This class of subrings is first introduced in [17]. It is evident that every intermediate ring of $C(X)$ is a β-determining subring, however, the converse of this statement does not hold, in general. For example, it follows from [17, Theorem 2.8] and [19, Remark 1.7] that $M^p + \mathbb{R}$, for each $p \in \beta\mathbb{R} \setminus \mathbb{R}$, is a β-determining subring of $C(\mathbb{R})$ which, by [17, Theorem 2.9], [18, Theorem 5.7] and [19, Remark 2.13], is not an intermediate ring; see also [15]. It should be noted that whenever R is a β-determining subring, then $Z(R)$ is a base for the closed sets of X. However, the converse of this fact does not hold, in general. For example, the mentioned subring $M^p + \mathbb{R}$ of $C(\mathbb{R})$, where $p \in \beta\mathbb{R} \setminus \mathbb{R}$, is not a β-determining ring. However, by [19, Remark 4.3], $Z(M^p + \mathbb{R})$ constitutes a base for the closed subsets of $\mathbb{R}$.

The next statement generalizes [11, 7.12 (b) and 7.13] to the β-determining subrings of $C(X)$.

Lemma 2.4. *Suppose that R is a β-determining subring of $C(X)$. Then the following statements hold:*

(a) If $f \in O^p(R)$, then there exists $g \notin M^p(R)$ such that $fg = 0$.

(b) If P is a prime ideal of R contained in $M^p(R)$, then $O^p(R) \subseteq P$.

(c) If P is a growing prime ideal in R, then there exists a unique $p \in \beta X$ such that $O^p(R) \subseteq P \subseteq M^p(R)$.

Proof. (a) Assume that $f \in O^p(R)$. Thus, $p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z(f)$ and, by our hypothesis, there exists $g \in R$ such that $p \in (\beta X \setminus Z(g^*)) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z(f)$. Thus, $p \notin Z(g^*)$ which means $g \notin M^p(R)$ and $X \setminus Z(g) \subseteq \mathrm{int}_X Z(f)$ which implies that $fg = 0$.

(b) Assume that $f \in O^p(R)$, by part (a), there exists $g \notin M^p(R)$ such that $fg = 0 \in P$ and consequently $f \in P$.

(c) As P is a growing ideal, by Proposition 2.2, there exists $p \in \beta X$ such that $P \subseteq M^p(R)$. Thus, by part (b), $O^p(R) \subseteq P$. Now, let $P \subseteq M^p(R)$ and $P \subseteq M^q(R)$ for two distinct points $p, q \in \beta X$. As R is β-determining, there exist $f, g \in R$ such that $p \in \beta X \setminus Z(f^*)$, $q \in \beta X \setminus Z(g^*)$ and $(\beta X \setminus Z(f^*)) \cap (\beta X \setminus Z(g^*)) = \emptyset$. It follows that $p \in \beta X \setminus Z(f^*) \subseteq Z(g^*)$ which means $p \in \mathrm{int}_{\beta X}Z(g^*)$. Thus, $p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z(g)$, since, as $p \in \mathrm{int}_{\beta X}Z(g^*)$, there exists an open set U in βX such that $p \in U \subseteq Z(g^*)$ which implies $p \in U \subseteq \mathrm{cl}_{\beta X}U = \mathrm{cl}_{\beta X}(U \cap X) \subseteq \mathrm{cl}_{\beta X}(Z(g^*) \cap X) = \mathrm{cl}_{\beta X}Z(g)$. Hence, $g \in O^p(R) \subseteq P \subseteq M^q(R)$ which is a contradiction. □

Theorem 2.5. *Let R be a subring of $C(X)$. The following statements are equivalent:*

(a) X is an R-P-space.

(b) X is an R^e-P-space.

(c) $\mathrm{cl}_{\beta X}Z = \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z$ for every $Z \in Z(R)$.

(d) $Z = \mathrm{int}_X Z$ for every $Z \in Z(R)$.

(e) $M_x(R) = O_x(R)$ for every $x \in X$.

In addition, if R is a β-determining subring, then the following statements are equivalent to the above:

(f) Every fixed maximal growing ideal of R is a minimal prime ideal.

(g) Every maximal growing ideal of R is a minimal prime ideal.

(h) R^e is a regular ring.

Proof. The implications (a⇔c), (c⇒d) and (d⇔e) are easy to prove.

(a⇒b) This is evident by Proposition 2.3.

(b⇒a) Let $f \in M^p(R)$. Hence, $p \in \mathrm{cl}_{\beta X}Z(f)$. As $\frac{f}{1} \in M^p(R^e)$ and $Z(f) = Z(\frac{f}{1})$, by our hypothesis, we would have $p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z(\frac{f}{1}) = \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z(f)$; i.e., $f \in O^p(R)$.

(d⇒c) By our hypothesis, each $Z \in Z(R)$ is clopen in X and thus, by [11, 6.9 (c)], $\mathrm{cl}_{\beta X}Z$ is clopen in βX for each $Z \in Z(R)$. This implies that $\mathrm{cl}_{\beta X}Z = \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z$ for each $Z \in Z(R)$.

(e⇒f) By Proposition 2.2, every maximal growing ideal in R is of the form $M^p(R)$ for some $p \in \beta X$ and clearly, $M^p(R)$ is fixed if and only if $p \in X$. Thus, every fixed maximal growing ideal in R is of the form $M_x(R)$ for some $x \in X$.

Now, if $M_x(R)$ is not a minimal prime ideal in R, then there exists a prime ideal P properly contained in $M_x(R)$. By Lemma 2,4, we would have $O_x(R) \subseteq P$ which contradicts our hypothesis.

(f⇒e) Assume on the contrary that $M_x(R) \neq O_x(R)$ for some $x \in X$, then there exists some $f \in M_x(R) \setminus O_x(R)$. This means that $f \in C(X)$ and $f \in M_x \setminus O_x$. Thus, by [11, 4I, 6], there exists s prime ideal P in $C(X)$ such that $O_x \subseteq P$ and $f \notin P$. It follows that $P \cap R$ is a prime ideal in R which is properly contained in $M^p(R)$. This contradicts our hypothesis.

(c⇒g) By Proposition 2.2, every maximal growing ideal in R is of the form $M^p(R)$ for some $p \in \beta X$. Assume on the contrary that $M^p(R)$ is not a minimal prime ideal for some $p \in \beta X$. Thus, there exists a prime ideal P in R which properly contained in $M^p(R)$. By Lemma 2.4, $O^p(R) \subseteq P$. Thus, there exists some $f \in M^p(R) \setminus O^p(R)$. This means that $\mathrm{cl}_{\beta X}Z(f) \neq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z(f)$ which is a contradiction.

(g⇒c) Let $\mathrm{cl}_{\beta X}Z(f) \neq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z(f)$ for some $f \in R$. Hence, $M^p(R) \neq O^p(R)$ for some $p \in \mathrm{cl}_{\beta X}Z(f) \setminus \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z(f)$. Thus, there exists some $f \in M^p \setminus O^p$ and hence, by [11, 7H, 6], there exists a prime ideal P in R such that $O^p \subseteq P$ and $f \notin P$. This implies that $M^p(R)$ is not a minimal prime ideal in R, however, is a maximal growing ideal.

(b⇒h) As R^e is an invertible subring, maximal ideals of R^e coincide with its maximal growing ideals and are of the form $M^p(R^e)$ for $p \in \beta X$. Now, we show that every prime ideal in R^e is maximal. Let P be a prime ideal in R^e. Then P^c is evidently a growing prime ideal in R. Thus, by Lemma 2.4, there exists a unique $p \in \beta X$ such that $O^p(R) \subseteq P^c \subseteq M^p(R)$. It follows that $P^{ce} = P = (M^p(R))^e = M^p(R^e)$ which means P is a maximal ideal in R^e. Therefore, by [12, Theorem 1.16], R^e is a regular ring.

(h⇒b) Assume on the contrary that $M^p(R^e) \neq O^p(R^e)$. As, clearly, $O^p(R^e)$ is a semiprime ideal (is an intersection of prime ideals) in R^e , there exists a prime ideal P in R^e such that $O^p(R^e) \subseteq P$ and $P \neq M^p(R^e)$. Thus, P^c is a growing prime ideal in R and hence, by Lemma 2.4, there exists a unique $q \in \beta X$ such that $O^q(R) \subseteq P^c \subseteq M^q(R)$. We claim that $p = q$. If $p \neq q$, similar to the proof of part (c) of Lemma 2.4, there exist $f, g \in R$ such that $p \in \beta X \setminus Z(f^*)$, $q \in \beta X \setminus Z(g^*)$ and $(\beta X \setminus Z(f^*)) \cap (\beta X \setminus Z(g^*)) = \emptyset$. It follows that $p \in \beta X \setminus Z(f^*) \subseteq Z(g^*)$. Hence, $p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z(g)$ and $q \notin Z(g^*)$. These imply $\frac{g}{1} \in O^p(R^e) \subseteq P \subseteq M^q(R^e)$ which is a contradiction, since, $q \notin Z(g^*) = Z((\frac{g}{1})^*)$. Therefore, P is a prime ideal in R^e which is not maximal. This contradicts regularity of R^e. □

It follows from Theorem 2.5 that in studying R-P-spaces, without lose of generality, we could consider R as an invertible subring. By the following examples, we investigate non-trivial instances of R-P-spaces which are not P-space.

Example 2.6. For a topological space X, we denote by $C_c(X)$ the subring of $C(X)$ consisting of all functions with countable image; see [10] for details about this subring. Now, let Y be a connected space with more than one point, Z be a P-space and $X = Y \bigoplus Z$ be the free union of the spaces Y and Z. It clearly follows that $C_c(X) \cong C_c(Y) \times C_c(Z) \cong \mathbb{R} \times C_c(Z)$, since, as Y is a connected space, $C_c(Y) = \mathbb{R}$. Also, as Z is a P-space, it is a C_c-P-space which, by [10, Theorem 5.8], implies that $C_c(Z)$ and thus $C_c(X)$ is a regular ring. Hence, by [10, Theorem 5.8], X is a C_c-P-space. However, X is not a P-space, since, Y is a subspace of X which is not a P-space; see [11, 4K, 4]. Therefore, X is a C_c-P-space that is not a P-space.

Remark 2.7. In Theorem 2.5, the condition that R is a β-determining subring is not necessary for R^e to be a regular ring whenever X is an R-P-space. For instance, consider the space X constructed in Example 2.6. From [10, Remark 2.3] it follows that $C_c(X)$ is an invertible subring of $C(X)$. Thus, $\mathcal{S}_{C_c}^{-1}C_c(X) = C_c(X)$. Also, X is a C_c-P-space and $C_c(X)$ is a regular ring. However, $C_c(X)$ is not β-determining, since, otherwise, it is inferred from [10, Proposition 4.4] that X is a zero-dimensional space and thus Y is a zero-dimensional space which is impossible.

Remark 2.8. In Theorem 2.5, in general, the condition that R is a β-determining subring could not be removed for X to be an R-P-space whenever R^e is a regular ring. For example, let $X = \mathbb{R}$ and R be the subring of $C(X)$ consisting of one-variable polynomials. It is evident that R is not a β-determining subring of $C(X)$. Also, we could observe that $f = x \in M_0(R) \setminus O_0(R)$ which implies that X is not an R-P-space. However, R^e is a field and hence is a regular ring.

Subrings of the form $I + \mathbb{R}$ where I is an ideal of $C(X)$ are first introduced in [19] and more studied in [4]. Using these subrings, we will construct another example of an R-P-space which is not a P-space. We remind that an ideal I in $C(X)$ is said to be a P-ideal if whenever I is considered as a non-unital subring, then each prime ideal of I is maximal in I. This class of ideals is first introduced in [20] and more studied in [2].

Proposition 2.9. *For an ideal I of $C(X)$, let $R_I = I + \mathbb{R}$. Then the following statements are equivalent:*

(a) X is an R_I-P-space.

(b) I is a P-ideal.

(c) $I + \mathbb{R}$ is a regular ring.

Proof. (a⇒b) As X is an R_I-P-space, by Theorem 2.5, $Z(f)$ is an open set in X for each $f \in R_I$. Thus, by [4, Proposition 2.10], I is a P-ideal.

(b⇒a) As I is a P-ideal, by [4, Proposition 2.10], $Z(f)$ is an open set in X for each $f \in R_I$ which, by Theorem 2.5, implies that X is an R_I-P-space.

(b⇔c) refer to [4, Proposition 2.10]. □

Example 2.10. Let $X = \mathbb{R} \bigoplus \{\sigma\}$ be the free union of the topological space $\mathbb{R}$ endowed with the usual topology and the one-point discrete space $\{\sigma\}$. Also, let $I = \{f \in C(X) : \mathbb{R} \subseteq Z(f)\}$ and $R = I + \mathbb{R}$. By [2, Theorem 2.2], I is a P-ideal in $C(X)$. Thus, by Proposition 2.9, X is an R-P-space. However, evidently, X is not a P-space. It is worthwhile to note that I is the unique P-ideal of $C(X)$.

3. *P*-Spaces and Intermediate Rings of *C*(*X*)

In this section, we study R-P-spaces with the restriction that R is an intermediate ring. Throughout this section we use $A(X)$ to denote an intermediate ring. It is obvious that $\mathcal{S}_A^{-1}A(X) = C(X)$. Moreover, $A(X)$ is a β-determining subring and $\mathrm{cl}_{\beta X}Z(f)$ is clopen in the $\mathrm{cl}_{\beta X}Z(A)$-topology for each $f \in A(X)$. Thus, by Proposition 2.2, growing maximal ideals of $A(X)$ are precisely the ideals $M^p(A)$ for $p \in \beta X$. We need the following lemma which shows that $O_A^p = \{f \in A(X) : p \in int_{\beta X}S_A(f)\}$ equals to $O^p(A)$ for each $p \in \beta X$.

Lemma 3.1. *Let $A(X)$ be an intermediate ring of $C(X)$. Then for each $f \in A(X)$ we have $int_{\beta X}S_A(f) = int_{\beta X}cl_{\beta X}Z(f)$.*

Proof. As $\mathrm{cl}_{\beta X}Z(f) \subseteq S_A(f)$, clearly, $\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z(f) \subseteq \mathrm{int}_{\beta X}S_A(f)$. Conversely, if $p \in \mathrm{int}_{\beta X}S_A(f)$, then there exists an open subset U in βX such that $p \in U \subseteq S_A(f)$. Hence, $U \cap X$ is a non-empty open subspace of X such that $U \cap X \subseteq S_A(f) \cap X = Z(f)$. Thus, $p \in U \subseteq \mathrm{cl}_{\beta X}Z(f)$ which means that $p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z(f)$. □

By the following statement, we give some algebraic characterizations of P-spaces via intermediate rings of $C(X)$. This statement is also a generalization of [14, Theorem 2.5]. Recall that we use $M_x(A)$ (resp., $O_x(A)$) instead of $M^x(A)$ (resp., $O^x(A)$) for each $x \in X$. Also, $M_A^x = M_x(A)$ and $O_A^x = O_x(A)$ for each $x \in X$.

Proposition 3.2. *Let $A(X)$ be an intermediate ring of $C(X)$. Then the following statements are equivalent:*
a) X is a P-space.
b) $M^p(A) = O^p(A)$ for each $p \in \beta X$.
c) $M_x(A) = O_x(A)$ for each $x \in X$.
d) $M_x(A)$ is a minimal prime ideal in $A(X)$ for each $x \in X$.
e) $M^p(A)$ is a minimal prime ideal in $A(X)$ for each $p \in \beta X$.

Proof. An easy consequence of Theorem 2.5. □

In [14, Theorem 3.10] a characterization of $C(X)$ among its intermediate rings is stated for the case that X is a P-space. By the next statement, we investigate a characterization of $C(X)$ among its intermediate rings even when X is not a P-space which gives [14, Theorem 3.10] as a corollary. It should be noted that this result is first stated in [5, Theorem 2.2] and we prove it here for the sake of completeness.

Theorem 3.3. *The following statements are equivalent for an intermediate ring $A(X)$.*
(a) $S_A(f) = cl_{\beta X}Z(f)$ for each $f \in A(X)$.
(b) $M_A^p = M^p(A)$ for each $p \in \beta X$.
(c) $M_f(A(X)) = M_f \cap A(X)$ for each $f \in A(X)$.
(d) Every z-ideal in $A(X)$ is a z_A-ideal.
(e) $A(X) = C(X)$.

Proof. (a⇒b) This is evident.

(b⇒c) Let $g \in M_f \cap A(X)$. Then $Z(f) \subseteq Z(g)$. Thus, if $p \in S_A(f)$, then $f \in M_A^p$ and hence, by our hypothesis, $f \in M^p(A(X))$; i.e, $p \in \mathrm{cl}_{\beta X}Z(f)$ which implies that $p \in \mathrm{cl}_{\beta X}Z(g)$. It follows that $g \in M^p(A) = M_A^p$ which menas $p \in S_A(g)$. Thus, $S_A(f) \subseteq S_A(g)$. Hence, by [15, Proposition 2.7], $g \in M_f(A(X))$. Therefore, $M_f \cap A(X) \subseteq M_f(A(X))$. Conversely, let $g \in M_f(A(X))$, then, by [15, Proposition 2.7], we have $S_A(f) \subseteq S_A(g)$ and thus $Z(f) \subseteq Z(g)$. It follows that $g \in M_f \cap A(X)$ and thus the equality follows.

(c⇒d) Suppose that I is a z-ideal in $A(X)$ and $Z(f) \subseteq Z(g)$ where $f \in I$ and $g \in A(X)$. We are to show that $g \in I$. As $Z(f) \subseteq Z(g)$, it follows that $g \in M_f \cap A(X)$. Thus, by our hypothesis and the fact that I is a z-ideal, we have $g \in M_f \cap A(X) = M_f(A(X)) \subseteq I$.

(d⇒e) Assume on the contrary that $A(X) \neq C(X)$. Hence, there exists some $f \in C(X) \setminus A(X)$. Set $g = \frac{1}{1+|f|}$. It is clear that $g \in C^*(X) \subseteq A(X)$, $Z(g) = \emptyset$ and $g^{-1} = 1 + |f| \notin A(X)$; i.e., g is a non-unit of $A(X)$. Therefore, $S_A(g) \neq \emptyset$. Now, choose some $p \in S_A(g)$. It follows that M_A^p is a z-ideal in $A(X)$ which is not a z_A-ideal, since, $g \in M_A^p$ and $Z(g) = \emptyset$.

(e⇒a) This is evident, since, $S_C(f) = \mathrm{cl}_{\beta X} Z(f)$ for each $f \in C(X)$. □

Proposition 3.4. ([14, Theorem 3.10]) *Let X be a P-space and $A(X)$ be an intermediate ring of $C(X)$. Then $A(X) = C(X)$ if and only if $M_A^p = O_A^p$ for each $p \in \beta X$.*

Proof. (⇒) As X is a P-space, clearly, we have $M_C^p = M^p = O^p = O_C^p$ for each $p \in \beta X$.

(⇐) Evidently, $M^p(A) \subseteq M_A^p$ for each $p \in \beta X$. Thus, by our hypothesis, we have $M^p(A) \subseteq M_A^p = O_A^p = O^p(A) \subseteq M^p(A)$. Therefore, $M_A^p = M^p(A)$ for each $p \in \beta X$. Hence, by Theorem 3.3, we have $A(X) = C(X)$. □

In [14, Proposition 3.6], it is shown that whenever $A(X)$ is an intermediate ring of $C(X)$ such that $A(X) \neq C(X)$, then there exists a non-maximal prime ideal in $A(X)$. Using Theorem 3.3, we give a different proof to this statement. Moreover, we specify that non-maximal prime ideal.

Proposition 3.5. *[14, Proposition 3.6] Let $A(X)$ be an intermediate ring of $C(X)$ such that $A(X) \neq C(X)$. Then there exists a non-maximal prime ideal in $A(X)$.*

Proof. As $A(X) \neq C(X)$, by Theorem 3.3, there exists some $p \in \beta X$ such that $M_A^p \neq M^p(A)$. Thus, clearly, $M^p(A)$ is a prime ideal in $A(X)$ which is not maximal. □

Using Proposition 3.5, a different proof is stated for [14, Proposition 3.2] by the next statement.

Proposition 3.6. *[14, Proposition 3.2] Let $A(X)$ be an intermediate ring of $C(X)$. If $A(X) \neq C(X)$, then $A(X)$ is not a regular ring.*

Proof. Since $A(X) \neq C(X)$, by Proposition 3.5, there exists some non-maximal prime ideal in $A(X)$. Thus, by [12, Theorem 1.16], $A(X)$ is not a regular ring. □

4. *P*-spaces and Intermediate *C*-Rings

In this section, we study R-P-spaces where R is assumed to be an intermediate C-ring of $C(X)$. The following theorem is a a characterization of intermediate C-rings of $C(X)$ which reveals the role of the mapping S_A in intermediate C-rings of $C(X)$ and investigates another approach to [21, Theorem 8]. Note that we use f^{υ_A} to denote the extension of f to the space $\upsilon_A X$ for each element f of an intermediate ring $A(X)$. In fact, $f^{\upsilon_A} = f^*|_{\upsilon_A X}$.

Theorem 4.1. *Let $A(X)$ be an intermediate ring of $C(X)$. Then $A(X)$ is a C-ring if and only if $S_A(f) = cl_{\beta X} Z(f^{\upsilon_A})$ for each $f \in A(X)$.*

Proof. Refer to [16, Theorem 2.2]. □

By the following example we give an instance of an intermediate ring $A(X)$ which is not a C-ring and specify an element $f \in A(X)$ for which $S_A(f) \neq \mathrm{cl}_{\beta X} Z(f^{\upsilon_A})$. In this example, we use the notion of singly generated intermediate rings over a given intermediate ring which is first introduced in [7] and more studied in [9]. For an intermediate ring $A(X)$ and $f \in C(X)$, we use $A(X)[f]$ to denote the singly generated intermediate ring of $C(X)$ over $A(X)$ generated by f, which is clearly the smallest intermediate ring of $C(X)$ containing both $A(X)$ and f. It is easy to see that $A(X)[f] = \{\sum_{i=0}^{n} f^i g_i : g_i \in A(X), n \in \mathbb{N} \cup \{0\}\}$. Moreover, if $f \geq c > 0$ for some $c \in \mathbb{R}$, then $C^*(X)[f] = \{g \in C(X) : |g| \leq f^n, \text{for some } n \in \mathbb{N}\}$. An intermediate ring $A(X)$ of $C(X)$ is called a singly generated intermediate ring over $C^*(X)$, if there exists $f \in C(X)$ such that $A(X) = C^*(X)[f]$. We could easily observe that $\upsilon_{A[f]} X = \upsilon_A X \cap \upsilon_f X$. It is stated in [9] that whenever f is a non-negative unbounded element of $C(X)$, then $e^f \notin C^*(X)[f]$ which implies that $C^*(X)[f]$ is not a C-ring, since, it is not closed under composition with elements of $C(\mathbb{R})$. Furthermore, by [7, Corollary 3.4], $C^*(X)[f] = C^*(X)[1 + |f|]$. Hence, every singly generated intermediate ring over $C^*(X)$ is not a C-ring.

Example 4.2. Let $i : \mathbb{R} \longrightarrow \mathbb{R}$ be the identity mapping and $f = 1+|i|$. Evidently, $f^*(p) = \infty$ for each $p \in \beta\mathbb{R}\setminus\mathbb{R}$. Now, let $A(\mathbb{R}) = C^*(\mathbb{R})[f]$. As stated above, $A(\mathbb{R})$ is an intermediate ring of $C(\mathbb{R})$ which is not a C-ring and $A(\mathbb{R}) = \{h \in C(\mathbb{R}) : |h| \leq f^n, \text{ for some } n \in \mathbb{N}\}$. Hence, $e^f \notin A(\mathbb{R})$. If we set $g = \frac{1}{1+e^f}$, then $g \in C^*(\mathbb{R}) \subseteq A(\mathbb{R})$. We claim that $S_A(g) = \beta\mathbb{R} \setminus \mathbb{R}$. Let $p \in \beta\mathbb{R} \setminus \mathbb{R}$ be given. Then, for each $h \in A(\mathbb{R})$, as $|h| \leq f^n$ for some $n \in \mathbb{N}$, we would have $|h| \leq e^f$ and thus $(gh)^*(p) = 0$; i.e., $p \in S_A(g)$. This proves our claim. Also, $\upsilon_A\mathbb{R} = \mathbb{R}$. Thus, $Z(g^{\upsilon_A}) = Z(g) = \emptyset$ and $S_A(g) = \beta\mathbb{R} \setminus \mathbb{R}$; i.e., $S_A(g) \neq \mathrm{cl}_{\beta X}Z(g^{\upsilon_A})$.

From Theorem 4.1, a characterization of maximal ideals in intermediate C-rings follows which is similar to the Gelfand-Kolmogoroff theorem about maximal ideals of $C(X)$.

Corollary 4.3. *[21, Theorem 8] For the maximal ideals of an intermediate C-ring $A(X)$, we have $M_A^p = \{f \in A(X) : p \in cl_{\beta X}Z(f^{\upsilon_A})\}$ for $p \in \beta X$.*

It is well-known that, for each $f \in C^*(X)$, $\mathrm{int}_{\beta X}Z(f^\beta) = \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z(f)$. Using Lemma 3.1 and Theorem 4.1, we generalize this fact to the intermediate C-rings.

Corollary 4.4. *Let $A(X)$ be an intermediate C-ring. Then for each $f \in A(X)$ we have $int_{\beta X}cl_{\beta X}Z(f^{\upsilon_A}) = int_{\beta X}cl_{\beta X}Z(f)$.*

The following proposition gives some characterizations of $C(X)$ among intermediate C-rings whenever X is a P-space.

Proposition 4.5. *Let X be a P-space and $A(X)$ be an intermediate C-ring of $C(X)$. The following statements are equivalent:*

(a) $A(X) = C(X)$

(b) $Z(f^{\upsilon_A})$ is open in $\upsilon_A X$ for each $f \in A(X)$.

(c) $\upsilon_A X$ is a P-space.

Proof. (a⇒b) This is evident, since, as X is a P-space, by [11, 14.29], $\upsilon X = \upsilon_C X$ is also a P-space.

(b⇒c) Using Theorem 4.1, it is easy to observe that $M_A^p = O^p(A)$ for each $p \in \beta X$ and thus by Proposition 3.4 we are done.

(c⇒a) As $\upsilon_A X$ is a P-space, $C(\upsilon_A X)$ is a regular ring and thus, by using [6, Theorem 1.3], $A(X)$ is a regular ring. Hence, by Proposition 3.6, we have $A(X) = C(X)$. □

The next example presents an instance of an intermediate ring $A(X)$ for which $\upsilon_A X$ is a P-space, however, $A(X) \neq C(X)$.

Example 4.6. Let $i : \mathbb{N} \longrightarrow \mathbb{R}$ be the identity mapping. Evidently, $i^*(p) = \infty$ for each $p \in \beta\mathbb{N} \setminus \mathbb{N}$. Set $A(\mathbb{N}) = C^*(\mathbb{N})[i]$. It follows that $A(\mathbb{N})$ is an intermediate ring of $C(\mathbb{N})$ which is not a C-ring and $\upsilon_A\mathbb{N} = \mathbb{N}$. Thus, $\upsilon_A\mathbb{N}$ is a P-space. Also, $Z(f^{\upsilon_A})$ is open in $\upsilon_A\mathbb{N}$ for each $f \in A(\mathbb{N})$. However, clearly, $A(\mathbb{N}) \neq C(\mathbb{N})$.

In the final part of this section, we consider the class of intermediate C-rings of $C(X)$ of the form $I + C^*(X)$ where I is an ideal of $C(X)$. This class of intermediate rings is first introduced in [7] and more studied in [3]. As stated in [7, 2.4], for each ideal I of $C(X)$, $I + C^*(X)$ is the smallest intermediate ring which contains the ideal I. Moreover, $I + C^*(X) = Z^{-1}(Z[I]) + C^*(X)$ where $Z^{-1}(Z[I])$ is the smallest z-ideal in $C(X)$ containing I. Furthermore, it follows from [7, 2.5] that each subring $I + C^*(X)$ is an intermediate C-ring, in fact, $I + C^*(X)$ is isomorphic to $C(X \cup \theta(I))$ where $\theta(I) = \bigcap_{f\in I} \mathrm{cl}_{\beta X}Z(f)$. The next statement determines the real compact spaces induced by ideals of $C(X)$ which is an special case of [3, Proposition 2.3].

Proposition 4.7. *For each ideal I in $C(X)$, $\upsilon_I X = \upsilon X \cup \theta(I)$.*

Proof. Clearly, $\upsilon X \cup \theta(I) \subseteq \upsilon_I X$. Now, whenever $p \notin \upsilon X \cup \theta(I)$, there exist $f \in C(X)$ and $g \in I$ such that $f^*(p) = \infty$ and $p \notin \mathrm{cl}_{\beta X}Z(g)$. Thus, there exists $h \in C^*(X)$ such that $p \notin Z(h^\beta)$ and $\mathrm{cl}_{\beta X}Z(g) \subseteq \mathrm{int}_{\beta X}Z(h^\beta)$. It follows from the later inclusion that there exists $k \in C(X)$ such that $h = gk$. It clearly follows that $fh = fgk \in I$ and $(fh)^*(p) = \infty$ which means that $p \notin \upsilon_I X$ and completes the proof. □

It follows from Proposition 4.7 that for an ideal I in $C(X)$, we have

$$\upsilon_I X = \upsilon_{I+C^*(X)} X = \upsilon X \cup \theta(I).$$

It is well-known that $M^{*p} = M^p \cap C^*(X)$ if and only if $p \in \upsilon X$. Using Proposition 4.7, we generalize this fact to the subrings of the form $I + C^*(X)$ by the following statement.

Theorem 4.8. *Let $A_I = I + C^*(X)$ where I is an ideal of $C(X)$. Then $M^p_{A_I} = M^p(A_I)$ if and only if $p \in (\beta X \setminus \upsilon_{A_I} X) \cup \upsilon X$.*

Proof. Refer to [3, Proposition 4.7]. □

The next statement easily follows from Theorem 4.8 and investigates a characterization of $C(X)$ among intermediate C-rings of the form $I + C^*(X)$ where I is an ideal of $C(X)$.

Corollary 4.9. *Let $A_I(X) = I + C^*(X)$ where I is an ideal in $C(X)$. The following statements are equivalent:*
(a) $I + C^*(X) = C(X)$.
(b) $\theta(I) \subseteq \upsilon X$.
(c) $\upsilon_I X = \upsilon X$.
Moreover, if X is a P-space, then the above conditions are equivalent to the following.
(d) $\upsilon_I X$ is a P-space

Proof. Using Proposition 4.7, we can easily observe that (a) to (c) are equivalent.
(c⇒d) This is clear by Proposition 4.7.
(d⇒b) By our hypothesis, υX is a P-space. Hence, as $\upsilon X \cup \theta(I)$ is also a P-space, we have $\theta(I) \subseteq \upsilon X$ and thus we are done. □

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