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Generalization of Darbos Fixed Point Theorem via SR_μ-Contractions with Application to Integral Equations

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Abstract. The aim of the current paper is introducing a generalization of Darbo's fixed point theorem based on $SR-$functions. In comparison with simulation function, $SR-$functions are able to cover the Meir-Keeler functions. Thus, the integral equations which are related to $L-$functions can be solved by our results. In the sequel, we find a solution for an integral equation to support our results.

1. Introduction and Preliminaries

In 1969, Meir and Keeler [9] introduced a generalization of Banach contraction principle and many authors such as Suzuki [12, 13], Lim [8] studied such type of contractions in detail. In the way of studying Meir-Keeler contraction many problems were raised which have been remained open up to now (for more details see [8, 12, 13]). Throughout this paper, let $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{N}$ denotes the set of all natural numbers and Let $\mathbb{C}$ denotes the set of all complex numbers.

Definition 1.1. *Let (X, d) be a metric space and let $T : X \to X$ be a mapping. We say that T is a Meir-Keeler contraction whenever, for all $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $x, y \in X$ which*

$$\varepsilon \le d(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) < \varepsilon.$$

Meir and Keeler [9] proved that every Meir-Keeler contraction on a complete metric space has a unique fixed point.

In 2015, Khojasteh *et al.* first introduced the concept of simulation function in order to generalize and unify some recent results in fixed point theory and continued and investigated by Karapinar later (for more details we refer to [5, 6, 11]) as follows:

Definition 1.2. *([7]) A simulation function is a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ which satisfies the following conditions:*

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(ζ_1) $\zeta(t,s) < s - t$ *for all* $t, s > 0$,

(ζ_2) *if* $\{t_n\}, \{s_n\}$ *be two sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, *then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Many examples of simulation functions can be found in [7].

Very recently, Roldán López de Hierro and Shahzad [10] introduced the following category of mappings which are connected to extend the simulation functions:

Definition 1.3. *Let* $A \subseteq \mathbb{R}$ *be a nonempty subset and let* $\varrho : A \times A \to \mathbb{R}$ *be a function. We say that* ϱ *is an R-function if it satisfies the following two conditions:*

(ϱ_1) *If* $\{a_n\} \subset (0,\infty) \cap A$ *be a sequence such that* $\varrho(a_{n+1}, a_n) > 0$, *for all* $n \in \mathbb{N}$, *then* $\{a_n\} \to 0$.

(ϱ_2) *If* $\{a_n\}, \{b_n\} \subset (0,\infty) \cap A$ *be two sequences which converge to the same limit* $L \geq 0$ *such that* $L < a_n$ *and* $\varrho(a_n, b_n) > 0$, *for all* $n \in \mathbb{N}$, *then* $L = 0$.

We denote by R_A, *the family of all R-functions whose domain is* $A \times A$.

In some cases, for given a function $\varrho : A \times A \to \mathbb{R}$, we will also consider the following property.

(ϱ_3) If $\{a_n\}, \{b_n\} \subset (0,\infty) \cap A$ be two sequences such that $\{b_n\} \to 0$ and $\varrho(a_n, b_n) > 0$, for all $n \in \mathbb{N}$, then $\{a_n\} \to 0$.

Definition 1.4. *([8]) A function* $\phi : [0,\infty) \to [0,\infty)$ *is called an L-function if*

(a) $\phi(0) = 0$,

(b) $\phi(t) > 0$ *for all* $t > 0$,

(c) *for all* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $\phi(t) \leq \varepsilon$, *for all* $t \in [\varepsilon, \varepsilon + \delta]$.

Let E be a Banach space over $\mathbb{R}$ (or $\mathbb{C}$) with respect to a certain norm $\|\,.\,\|$. For any subsets X and Y of E, we have the following notations:

(1). $\bar{X}$ denotes the closure of X,

(2). conv(X) denotes the convex hull of X,

(3). $P(X)$ denotes the set of nonempty subsets of X,

(4). $X + Y$ and λX, ($\lambda \in \mathbb{R}$) stand for algebraic operations on sets X and Y.

We denote by $\mathbf{B}_E$ the family of all nonempty bounded subsets of E. Finally, if X is a nonempty subset of E and $T : X \to X$ is a given operator, we denote by $Fix(T)$ the set of fixed points of T, that is,

$$Fix(T) = \{x \in X : Tx = x\}.$$

Banaś and Goebel [2] introduced the following axiomatic definition for measure of noncompactness.

Definition 1.5. *Let* $\mu : \mathbf{B}_E \to [0,\infty)$ *be a given mapping. We say that* μ *is a BG-measure of noncompactness (in the sense of Banaś and Gobel) on E if the following conditions are satisfied:*

(i). For every $X \in \mathbf{B}_E$, $\mu(X) = 0$ *if and only if X is precompact, i.e.* $Ker\mu \neq \emptyset$.

(ii). For every pair $(X, Y) \in \mathbf{B}_E \times \mathbf{B}_E$, *we have*

$$X \subseteq Y \quad \Rightarrow \quad \mu(X) \leq \mu(Y).$$

(iii). For every $X \in \mathbf{B}_E$, we have

$$\mu(\bar{X}) = \mu(X) = \mu(conv(X)).$$

(iv). For every pair $(X, Y) \in \mathbf{B}_E \times \mathbf{B}_E$ and $\lambda \in (0, 1)$, we have

$$\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y).$$

(v). If $\{X_n\} \subseteq \mathbf{B}_E$ is a decreasing sequence of closed sets such that $\mu(X_n) \to 0$ as $n \to \infty$, then $X_\infty := \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Theorem 1.6. ***(Schauder)*** *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Then every continuous and compact map $T : \Omega \to \Omega$ has at least one fixed point in Ω.*

Theorem 1.7. ***(Darbo)****Let Ω be a nonempty, closed, bounded and convex subset of the Banach space E, μ be a measure of noncompactness on E and $T : \Omega \to \Omega$ be a continuous function. Assume that there exists a constant $k \in (0, 1)$ such that*

$$\mu(TX) \leq k\mu(X),$$

for any nonempty subset $X \in \Omega$. Then T has a fixed point in Ω.

In current research, we introduce a generalization of Darbo's fixed point theorem via R-functions. In comparison with recent results such as Jleli *et al.* [4] and Chen and Tang [1], our results are more generalized than others. Taking in account that

(a) every simulation function is an R-function which also verifies (ϱ_3) [10],

(b) there is a Meir-Keeler contraction which is not $Z-$contraction [3],

(c) every Meir and Keeler contraction is an $R-$contraction [10],

one can conclude that the results of [1] can't cover Meir and Keeler contractions. Also, (b) yields that the results of [4] is an special case of our results. Therefore, we decide to introduce a generalization of Darbo's fixed point theorem based on $SR-$functions from which Meir-Keeler functions and $R-$functions are covered and it is more generalized than previous results.

2. Main Results

In this section, we introduce $SR_\mu-$contractions in order to generalize Darbo's fixed point theorem.

Definition 2.1. *Let $A \subseteq \mathbb{R}$ be a nonempty subset and let $\varrho : A \times A \to \mathbb{R}$ be a function. We say that ϱ is an SR-function if it satisfies the following condition:*

(SR) *If $\{a_n\} \subset (0, \infty) \cap A$ be a sequence such that $\varrho(a_{n+1}, a_n) > 0$, for all $n \in \mathbb{N}$, then $\{a_n\} \to 0$.*

We denote by SR_A, the family of all SR-functions whose domain is $A \times A$.

Note that $R_A \subseteq SR_A$ and this shows that the class of functions in SR_A is much larger than R_A.

Definition 2.2. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous operator. We say that T is SR_μ-contraction if there exists an SR-function $\varrho : A \times A \to \mathbb{R}$ such that $ran(d) \subseteq A$ and*

$$\varrho(\mu(T(X)), \mu(X)) \geq 0, \tag{1}$$

for any nonempty subset X of Ω, where μ is an arbitrary measure of non-compactness.

Theorem 2.3. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Let $T : \Omega \to \Omega$ be a continuous operator. If T is a SR_μ-contraction with respect to $\varrho : A \times A \to \mathbb{R}$, then T has at least one fixed point in Ω.*

Proof. Consider the sequence $\{\Omega_n\} \subseteq E$ which is defined by

$$\Omega_0 := \Omega, \quad \Omega_{n+1} := \overline{conv}(T\Omega_n), \tag{2}$$

for all $n \in \mathbb{N}$. Applying induction, we observe easily that

$$\Omega_{n+1} \subseteq \Omega_n, \tag{3}$$

for all $n \in \mathbb{N}$. If there is an $n_0 \in \mathbb{N}$ such that $\mu(\Omega_{n_0}) = 0$, then (i) in Definition 1.5 yields Ω_{n_0} is compact. Since $T(\Omega_{n_0}) \subseteq \Omega_{n_0}$ (from 3), Schauders' fixed point theorem implies desired result. So, without loss of the generality, we can assume that $\mu(\Omega_n) > 0$ for all $n \in \mathbb{N}$. Since $\mu(T\Omega_n) = \mu(\overline{conv}(T\Omega_n)) = \mu(\Omega_{n+1}) > 0$ by applying (ϱ_1) of Definition 1.3 we have

$$\varrho(\mu(\Omega_{n+1}), \mu(\Omega_n)) = \varrho(\mu(Conv(T\Omega_n)), \mu(\Omega_n)) = \varrho(\mu(T\Omega_n), \mu(\Omega_n)) \geq 0. \tag{4}$$

So $\lim_{n\to\infty}\{\mu(\Omega_n)\} \to 0$.

Since $\mu(\Omega_{n+1}) \leq \mu(\Omega_n)$ thus $\{\mu(\Omega_n)\}$ is a decreasing sequence by applying (v) in Definition 1.5 we have $\Omega_\infty := \bigcap_{n=1}^{\infty} \Omega_n$ is nonempty. On the other hand, $\mu(\Omega_\infty) \leq \mu(\Omega_n)$ and it means that $\mu(\Omega_\infty) = 0$. Hence, $\Omega_\infty \in Ker(\mu)$ means Ω_∞ is compact. Moreover, for every $n \in \mathbb{N}$, we have $\Omega_\infty \subseteq \Omega_n$. Since $T(\Omega_n) \subseteq \Omega_n$

$$T\Omega_\infty \subseteq T\Omega_n \subseteq \Omega_n,$$

for all $n \in \mathbb{N}$. Thus, $T(\Omega_\infty) \subset \Omega_\infty$. Therefore, $T : \Omega_\infty \to \Omega_\infty$ is well defined. Applying Schauders' fixed point theorem on the mapping T, one can obtain desired result. □

3. Corollaries

The following Corollaries are various generalizations of Darbo's fixed point theorem.

Corollary 3.1. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Let $T : \Omega \to \Omega$ be a continuous operator which satisfies the following condition*

$$\mu(TX) < \varphi(\mu(X)),$$

for any nonempty subset X of Ω, where μ is an arbitrary measure of non-compactness and $\varphi : [0, +\infty) \to [0, +\infty)$ be an L–function in a sense of Definition 1.4. Then T has at least one fixed point in Ω.

Proof. Consider $\varrho(t, s) = \varphi(s) - t$. It is sufficient to show that (SR) holds.

(SR) : Let $\{a_n\} \subset (0, +\infty)$ be a sequence and let $\varrho(a_{n+1}, a_n) > 0$. Then

$$0 < \varrho(a_{n+1}, a_n) = \varphi(a_n) - a_{n+1}.$$

It means that,

$$a_{n+1} < \varphi(a_n) \leq a_n. \tag{5}$$

Therefore, $\{a_n\}$ is a non-increasing sequence, so it converges to $L \geq 0$. On the contrary, suppose that $L > 0$. By considering $\epsilon = L$, there exists $\delta > 0$ such that, for all $t \in [L, L + \delta)$, $\varphi(t) \leq \epsilon$. Since $a_n \to L$, there exists $n_0 > 0$ such that $L < a_{n_0} < L + \delta$. Therefore, $a_{n_0+1} < \varphi(a_{n_0}) \leq L$ and this is a contradiction. So $L = 0$.

□

Lim [8] shows that every Mair Keeler contraction is equivalent to a contraction induced by an $L-$function. In other words, let $T : X \to X$ be a mapping. Then T fulfills (MK) property:

(MK): For all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon \leq d(x,y) < \epsilon + \delta \quad \Rightarrow \quad d(Tx,Ty) < \epsilon,$$

if and only if, there exists an $L-$function $\varphi : [0,+\infty) \to [0,+\infty)$ such that

$$d(Tx,Ty) \leq \varphi(d(x,y)).$$

Applying Corollary 3.1 and the fact that φ is an $L-$funtion, the following corollary can be verified which is the generalization of Darbo's fixed point theorem of Meir-Keeler type.

Corollary 3.2. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Let $T : \Omega \to \Omega$ be a continuous operator which fulfills the following condition:*

For all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\epsilon \leq \mu(X) < \epsilon + \delta \quad \Rightarrow \quad \mu(TX) < \epsilon,$$

for any nonempty subset X of Ω, where μ is an arbitrary measure of non-compactness. Then T has at least one fixed point in Ω.

Corollary 3.3. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Let $T : \Omega \to \Omega$ be a continuous operator satisfied with the following condition*

$$\mu(TX) < \varphi(\mu(X))\mu(X),$$

for any nonempty subset X of Ω, where μ is an arbitrary measure of non-compactness and $\varphi : [0,+\infty) \to [0,1)$ is a mapping such that $\limsup_{t\to s^+} \varphi(t) < 1$, for all $s \in (0,+\infty)$. Then T has at least one fixed point in Ω.

Proof. Define $\varrho(t,s) = s\varphi(s) - t$. It is sufficient to show that (SR) holds.

(SR) : Let $\{a_n\} \subset (0,+\infty)$ be a sequence and $\varrho(a_{n+1},a_n) > 0$. Then

$$0 < \varrho(a_{n+1},a_n) = a_n\varphi(a_n) - a_{n+1}.$$

It means that,

$$\frac{a_{n+1}}{a_n} < \varphi(a_n). \tag{6}$$

Taking limit on both sides of (6), we have

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < \limsup_{n\to\infty} \varphi(a_n) < 1.$$

By Ratio test, $\sum_{n=1}^{\infty} a_n < \infty$ and it concludes that $\lim_{n\to\infty} a_n = 0$.

□

Corollary 3.4. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Let $T : \Omega \to \Omega$ be a continuous operator which satisfies the following condition*

$$\mu(TX) < \varphi(\mu(X))\mu(X),$$

for any nonempty subset X of Ω, where μ is an arbitrary measure of non-compactness and $\varphi : [0,+\infty) \to [0,1)$ be a mapping such that for each sequence $\{t_n\} \subset [0,+\infty)$,

$$(Ge) \quad \lim_{n\to\infty} \varphi(t_n) = 1 \quad \textit{implies} \quad \lim_{n\to\infty} t_n = 0.$$

Then T has at least one fixed point in Ω.

Proof. Define $\varrho(t,s) = s\varphi(s) - t$. It is sufficient to show that (SR) holds.

(SR) : Let $\{a_n\} \subset (0,+\infty)$ be a sequence such that $\varrho(a_{n+1}, a_n) > 0$. Therefore,

$$0 < \varrho(a_{n+1}, a_n) = \varphi(a_n)a_n - a_{n+1}.$$

Since $a_n > 0$ and $\varphi(a_n) < 1$, one can see easily

$$a_{n+1} < \varphi(a_n)a_n < a_n.$$

Hence, a_n is a strictly decreasing sequence of nonnegative real numbers and so converges to $L \geq 0$. On the contrary, assume that $L > 0$. Therefore,

$$0 < L < a_{n+1} < \varphi(a_n)a_n < a_n. \tag{7}$$

Taking limit on both sides of (7), one deduce that $\lim_{n\to\infty} \varphi(a_n) = 1$. Since φ satisfies in (Ge), so $L = \lim_{n\to\infty} a_n = 0$ which it contradicts $L > 0$. So $L = 0$.

□

Corollary 3.5. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Let $T : \Omega \to \Omega$ be a continuous operator which it satisfies the following condition*

$$\zeta(\mu(TX), \mu(X)) \geq 0,$$

for any nonempty subset X of Ω, where μ is an arbitrary measure of non-compactness and ζ be a simulation function. Then T has at least one fixed point in Ω.

Proof. Define $\varrho(t,s) = \zeta(t,s)$. It is sufficient to show that (SR) holds.

(SR) : Let $\{a_n\} \subset (0,+\infty)$ be a sequence and let $\zeta(a_{n+1}, a_n) > 0$. Therefore, $0 < \zeta(a_{n+1}, a_n) < a_n - a_{n+1}$. Hence $\{a_n\}$ is a strictly decreasing sequence, so it converges to $L \geq 0$. On the contrary, suppose that $L > 0$. Let $t_n = a_{n+1}$ and $s_n = a_n$. Then $\{t_n\}$ and $\{s_n\}$ are convergent to L and $t_n < s_n$. By condition (ζ_3) we have

$$0 \leq \limsup_{n\to\infty} \zeta(a_{n+1}, a_n) = \limsup_{n\to\infty} \zeta(t_n, s_n) < 0,$$

and this is a contradiction. So $L = 0$.

□

Corollary 3.6. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Let $T : \Omega \to \Omega$ be a continuous operator which it satisfies the following condition*

$$\psi(\mu(TX)) < \psi(\mu(X)) - \varphi(\mu(X)),$$

for each nonempty subset X of Ω, where μ is an arbitrary measure of non-compactness and $\psi, \varphi : [0,\infty) \to [0,\infty)$ be two functions such that ψ is nondecreasing and continuous from the right and φ is lower semi continuous and $\varphi^{-1}(\{0\}) = \{0\}$. Then T has at least one fixed point in Ω.

Proof. First of all, we show that, for $t, s \in [0,\infty)$,

$$s > 0, \quad \varrho_{\psi,\varphi} \geq 0 \quad \Rightarrow \quad t < s, \tag{8}$$

Let $\varrho_{\psi,\varphi}(t,s) \geq 0$ and $t \geq s$, we are going to show that $s = 0$. As ψ is nondecreasing,

$$\psi(s) \leq \psi(t) \leq \psi(s) - \varphi(s) \leq \psi(s),$$

which implies that, $\varphi(s) = 0$. Hence $s = 0$ and this is a contradiction. So we have $t < s$. Defining $\varrho_{\varphi,\psi}(t,s) = \psi(s) - \varphi(s) - \psi(t)$. It is sufficient to show that (SR) holds.

(SR) : Let $\{a_n\} \subset (0, +\infty)$ be a sequence and let $\varrho_{\psi,\varphi}(a_{n+1}, a_n) > 0$. By (8), $a_{n+1} < a_n$. Hence $\{a_n\}$ is a strictly decreasing sequence so converges to $L \geq 0$. Then $L < a_n$, for all $n \in \mathbb{N}$. On the contrary, suppose that $L > 0$. Therefore,

$$0 < \varrho_{\psi,\varphi}(a_{n+1}, a_n) = \psi(a_n) - \varphi(a_n) - \psi(a_{n+1}),$$

and so, $0 \leq \varphi(a_n) < \psi(a_n) - \psi(a_{n+1})$. Since ψ is continuous from the right and $\lim_{n\to\infty} a_n = L$, we deduce that $\lim_{n\to\infty} \varphi(a_n) = 0$. Since φ is lower semi−continuous,

$$0 \leq \varphi(L) \leq \liminf_{r\to L} \varphi(r) = \lim_{n\to\infty} \varphi(a_n) = 0.$$

Hence $\varphi(L) = 0$, so $L = 0$.

□

Corollary 3.7. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Let $T : \Omega \to \Omega$ be a continuous operator which is satisfies the following condition*

$$\mu(TX) < \frac{\mu(X)}{\mu(TX) + 1},$$

for any nonempty subset X of Ω, where μ is an arbitrary measure of non-compactness. Then T is a R_μ-contraction.

Proof. In order to prove the result, consider $\varrho(t, s) = \frac{s}{t+1} - t$. So we have to show that (SR) holds.

(SR) : Let $\{a_n\} \subset (0, +\infty)$ be a sequence and let $\varrho(a_{n+1}, a_n) > 0$. It means that

$$0 < \varrho(a_{n+1}, a_n) = \frac{a_n}{a_{n+1} + 1} - a_{n+1}.$$

Hence,

$$0 < a_{n+1} < \frac{a_n}{a_{n+1} + 1}.$$

Therefore,

$$0 < \frac{a_{n+1}}{a_n} < \frac{1}{a_{n+1} + 1} < 1.$$

Suppose that $\lim_{n\to\infty} a_n = r$. If $r > 0$, we deduce that

$$0 < \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \leq \frac{1}{r + 1} < 1,$$

which is a contradiction. So $\lim_{n\to\infty} a_n = 0$.

□

Corollary 3.8. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Let $T : \Omega \to \Omega$ be a continuous operator which it satisfies the following condition*

$$\mu(TX) < \frac{\mu(X)}{e^{\mu(TX)}},$$

for any nonempty subset X of Ω, where μ is an arbitrary measure of non-compactness. Then T is a R_μ-contraction.

Proof. Let $\varrho(t, s) = \frac{s}{e^t} - t$. It is sufficient to show that (SR) holds.

(SR) : Let $\{a_n\} \subset (0,+\infty)$ be a sequence and let $\varrho(a_{n+1},a_n) > 0$. It means that

$$0 < \varrho(a_{n+1},a_n) = \frac{a_n}{e^{a_{n+1}}} - a_{n+1}.$$

Hence,

$$0 < a_{n+1} < \frac{a_n}{e^{a_{n+1}}}.$$

Therefore,

$$0 < \frac{a_{n+1}}{a_n} < \frac{1}{e^{a_{n+1}}} < 1. \tag{9}$$

Suppose that $\{a_n\} \to r$. If $r > 0$, then taking limit from both side of (9), we deduce that

$$0 < \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \leq \frac{1}{e^r} < 1,$$

which is a contradiction. So $\{a_n\} \to 0$.

□

Corollary 3.9. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Let $T : \Omega \to \Omega$ be a continuous operator which it satisfies with the following condition*

$$\mu(TX) \leq \ln(\mu(X) + 1),$$

for any nonempty subset X of Ω, where μ is an arbitrary measure of non-compactness. Then T is a R_μ-contraction.

Proof. Consider $\varrho(t,s) = \ln(s+1) - t$. It is sufficient to show that (SR) holds.

(SR) : Let $\{a_n\} \subset (0,+\infty)$ be a sequence and let $\varrho(a_{n+1},a_n) > 0$. It means that

$$0 < \varrho(a_{n+1},a_n) = \ln(a_n + 1) - a_{n+1}.$$

Thus,

$$0 < a_{n+1} < \ln(a_n + 1).$$

Therefore,

$$0 < \frac{a_{n+1}}{a_n} < \frac{\ln(a_n+1)}{a_n} < 1.$$

Suppose that $\lim_{n\to\infty} a_n = r$. If $r > 0$, then we deduce that

$$0 < \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \leq \frac{\ln(r+1)}{r} < 1,$$

which is a contradiction. So $\lim_{n\to\infty} a_n = 0$.

□

Example 3.10. *If $\Phi : [0,\infty) \to [0,\infty)$ be a function such that $\int_0^\epsilon \Phi(u)du$ exists and for all $\int_0^\epsilon \Phi(u)du > \epsilon$, for each $\epsilon > 0$ and define $\varrho_f : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by*

$$\varrho_f(t,s) = s - \int_0^t \Phi(u)du,$$

for all $s,t \in [0,\infty)$. Then ϱ_f is an SR-function.

Proof. It is sufficient to show that (SR) *holds.*

(SR) : *Let* $\{a_n\} \subset (0, +\infty)$ *be a sequence and let* $\varrho_f(a_{n+1}, a_n) > 0$*, for all* $n \in \mathbb{N}$*. It means that*

$$0 < \varrho_f(a_{n+1}, a_n) = a_n - \int_0^{a_{n+1}} \Phi(u)du.$$

Thus,

$$\int_0^{a_{n+1}} \Phi(u)du < a_n.$$

Therefore,

$$0 < \epsilon < \int_0^{a_{n+1}} \Phi(u)du < a_n.$$

Letting n tends to infinity, we deduce $\lim_{n\to\infty} a_n = 0$*.*

□

4. Applications

In the sequel, we denote by $BC(\mathbf{R}^+)$ the family of all bounded and continuous real function on $\mathbf{R}^+$ in which $\mathbf{R}^+$ is considered as $(0, +\infty)$. The Banach space $BC(\mathbf{R}^+)$ is endowed with the following norm:

$$\| u \| = \sup\{|u(t)| : t > 0\}.$$

Now define the measure of non-compactness on $BC(\mathbf{R}^+)$. Assume a nonempty, bounded subset U of $BC(\mathbf{R}^+)$ and a positive number $L > 0$. We denote by $\zeta_L(u, \varepsilon)$, the modulus of continuity for the function u on $[0, L]$, where $u \in U$ and $\varepsilon > 0$:

$$\zeta_L(u, \varepsilon) = \sup\{|u(t) - u(s)| : t, s \in [0, L], |t - s| \leq \varepsilon\}.$$

Also assume that

$$\zeta_L(U, \varepsilon) = \sup\{\zeta_L(u, \varepsilon) : u \in U\},$$

$$\zeta_L^0(U) = \lim_{\varepsilon\to 0} \zeta_L(U, \varepsilon),$$

and

$$\zeta^0(U) = \lim_{L\to\infty} \zeta_L^0(U).$$

Moreover, we have

$$U(t) = \{u(t) : u \in U\},$$

for all $t \in \mathbf{R}^+$ and

$$\mu(U) = \zeta^0(U) + \limsup_{t\to\infty} diam U(t),$$

and

$$diam U(t) = \sup\{|u(t) - v(t)| : u, v \in U\}.$$

In [1] the authors have shown that the function μ is a measure of non-compactness in the $BC(\mathbf{R}^+)$.

Theorem 4.1. *Let*

$$u(t) = \Delta(t, u(t)) + \int_0^t \Gamma(t, s, u(s))ds, \tag{10}$$

for all $t \in \mathbf{R}^+$ where Δ satisfies the following four conditions:

(i) : $\Delta : \mathbf{R}^+ \to \mathbf{R}^+$ *is a continuous function and the function $t \to \Delta(t, 0)$ is a member of $BC(\mathbf{R}^+)$.*

(ii) : $|\Delta(t, u) - \Delta(t, v)| \leq \phi(|u - v|)$ *for all $u, v \in \mathbb{R}$ where $\phi : [0, \infty) \to [0, \infty)$ is an L–function.*

(iii) : *The function $\Gamma : \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R} \to \mathbb{R}$ is a continuous and there exist two continuous functions $m, n : \mathbf{R}^+ \to \mathbf{R}^+$ such that*

$$\lim_{t\to\infty} m(t) \int_0^t n(s)ds = 0,$$

and

$$|\Gamma(t, s, u)| \leq m(t)n(s),$$

for any $t, s \in \mathbf{R}^+$ such that $s \leq t$ and for each $u \in \mathbb{R}$.

(iv) : *There exists a positive solution $\varrho \in \mathbf{R_A}$ and r_0 such that the following inequality holds:*

$$\varrho(r_0, r_0) + Z \leq 0,$$

where Z is a constant defined by

$$Z = \sup\{|\Delta(t, 0)| + m(t) \int_0^t n(s)ds : t \geq 0\}.$$

Then (10) has at least one solution in the $BC(\mathbf{R}^+)$.

Proof. Firstly, we define operator T on $BC(\mathbf{R}^+)$ by

$$(Tu)(t) = \Delta(t, u(t)) + \int_0^t \Gamma(t, s, u(s))ds \quad for \quad t \in \mathbf{R}^+.$$

The function Tu is continuous on $\mathbf{R}^+$. Divide the proof into four steps:

Step (1): We show that $T : BC(\mathbf{R}^+) \to BC(\mathbf{R}^+)$. For an arbitrarily fixed function $u \in BC(\mathbf{R}^+)$, we get

$$\begin{array}{rcl} |(Tu)(t)| & = & |\Delta(t, u(t)) - \Delta(t, 0) + \Delta(t, 0) + \int_0^t \Gamma(t, s, u(s))ds| \\ & \leq & |\Delta(t, u(t)) - \Delta(t, 0)| + |\Delta(t, 0)| + |\int_0^t \Gamma(t, s, u(s))ds| \\ & \leq & \phi(|u(t)|) + |\Delta(t, 0)| + m(t) \int_0^t n(s)ds \\ & = & \phi(|u(t)|) + |\Delta(t, 0)| + c(t), \end{array}$$

where

$$c(t) = m(t) \int_0^t n(s)ds.$$

Since by (ii) the function ϕ is non-decreasing, so

$$\|Tu\| \leq \phi(\|u\|) + Z,$$

where Z was defined in (iv). So

$$T : BC(\mathbf{R}^+) \to BC(\mathbf{R}^+).$$

Step (2): Next, we show that T is continuous on B_{r_0} where r_0 is the constant which was appeared in (*iv*). It is clear that there exists $\varrho \in R_A$ such that $\varrho(t,s) = \phi(s) - t$, for all $t, s > 0$. By (*iv*) we can infer that T is a self mapping from B_{r_0} into itself. Consider the arbitrary number $\varepsilon > 0$. Then, for $u, v \in B_{r_0}$ such that $\|u - v\| \le \varepsilon$ we have

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| &\le \phi(|u(t) - v(t)|) + \int_0^t |\Gamma(t,s,u(s)) - \Gamma(t,s,v(s))|ds \\ &\le \phi(|u(t) - v(t)|) + \int_0^t |\Gamma(t,s,u(s))|ds - \int_0^t |\Gamma(t,s,v(s))|ds \\ &\le \phi(\varepsilon) + 2c(t), \end{aligned} \tag{11}$$

for all $t \in \mathbf{R}^+$. Also by (*iii*) we conclude that there exists a number $L > 0$ such that

$$2m(t)\int_0^t n(s)ds \le \varepsilon, \tag{12}$$

for all $t \ge L$. Since φ is an $L-$function, considering (11) and (12) together, we have

$$|(Tu)(t) - (Tv)(t)| \le \varepsilon. \tag{13}$$

Now define

$$\zeta_L(\Gamma, \varepsilon) = \sup\{|\Gamma(t,s,u) - \Gamma(t,s,v)| : t, s \in [0, L], u, v \in [-r_0, r_0], |u - v| \le \varepsilon\}.$$

Since the function $\Gamma(t,s,u)$ is uniformly continuous on $[0,L] \times [0,L] \times [-r_0, r_0]$, we obtain $\zeta_L(\Gamma, \varepsilon) \to 0$ as $n \to \infty$. Now for an arbitrarily fixed $t \in [0, L]$ and taking into account (11), we get

$$|(Tu)(t) - (Tv)(t)| \le \zeta(\varepsilon)\int_0^L \zeta_L(g, \varepsilon)ds = \phi(\varepsilon) + L\zeta_L(g, \varepsilon). \tag{14}$$

Finally, by combining (13) and (14) on the base of the above established fact concerning the quantity $\zeta_L(\Gamma, \varepsilon)$, we conclude that the operator T is continuous on the ball B_{r_0}.

Step (3): Fix numbers $\varepsilon > 0$ and $L > 0$. We choose arbitrarily nonempty subset U of the ball B_{r_0} and $t, s \in [0, L]$ such that $|t - s| \le \varepsilon$. Without loss of generality, assume that $s < t$. Then

$$\begin{aligned} |(Tu)(t) - (Tu)(s)| &\le |\Delta(t,u(t)) - \Delta(s,u(s))| + |\int_0^t \Gamma(t,\alpha,u(\alpha))d\alpha - \int_0^s \Gamma(s,\alpha,u(\alpha))d\alpha| \\ &\le |\Delta(t,u(t)) - \Delta(s,u(t)) + \Delta(s,u(t)) - \Delta(s,u(s))| \\ &+ |\int_0^t \Gamma(t,\alpha,u(\alpha))d\alpha - \int_0^t \Gamma(s,\alpha,u(\alpha))d\alpha| \\ &+ |\int_0^t \Gamma(s,\alpha,u(\alpha))d\alpha - \int_0^s \Gamma(s,\alpha,u(\alpha))d\alpha| \\ &\le \zeta_L^1(\Delta,\varepsilon) + \phi(|u(t) - u(s)|) + \int_0^t |\Gamma(t,\alpha,u(\alpha))d\alpha - \Gamma(s,\alpha,u(\alpha))d\alpha| \\ &+ \int_s^t |\Gamma(s,\alpha,u(\alpha))|d\alpha \\ &\le \zeta_L^1(\Delta,\varepsilon) + \phi(\zeta_L(u,\varepsilon)) + \int_0^t \zeta^1(\Gamma,\varepsilon)d\alpha + m(s)\int_s^t n(\alpha)d\alpha \\ &\le \zeta_L^1(\Delta,\varepsilon) + \phi(\phi_L(u,\varepsilon)) + L\zeta_L^1(\Gamma,\varepsilon) + \varepsilon\sup\{m(s)n(t) : t, s \in [0, L]\}, \end{aligned} \tag{15}$$

for all $u \in U$, where

$$\zeta_L^1(\Delta, \varepsilon) = \sup\{|\Delta(t,u) - \Delta(s,u)| : t, s \in [0, L], u \in [-r_0, r_0], |t - s| \le \varepsilon\},$$

and

$$\zeta_L^1(\Gamma, \varepsilon) = \sup\{|\Gamma(t,\alpha,u) - \Gamma(s,\alpha,u)| : t, s, \alpha \in [0, L], u \in [-r_0, r_0], |t - s| \le \varepsilon\}.$$

By uniform continuity of the function Δ and Γ, we deduce that $\zeta_1^L(\Delta, \varepsilon) \to 0$ and $\zeta_L^1(\Gamma, \varepsilon) \to 0$ as $\varepsilon \to 0$. Also, since the functions $m = m(t)$ and $n = n(t)$ are continuous on $\mathbf{R}^+$, we obtain

$$\sup\{m(s)n(t) : t, s \in [0, L]\} < \infty,$$

and by (15), we get

$$\zeta_L^0(TU) \leq \lim_{\varepsilon \to 0} \phi(\zeta_L(U, \varepsilon)).$$

By right-continuity of the ϕ, observe that

$$\zeta_L^0(TU) \leq \zeta(\zeta_L^0(U)),$$

and, finally

$$\zeta^0(TU) \leq \phi(\zeta^0(U)). \tag{16}$$

Step (4): Let $u, v \in U$ be two arbitrary functions. Then for $t \in \mathbf{R}$, we get

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| &\leq |\Delta(t, u(t)) - \Delta(t, v(t))| + |\int_0^t |\Gamma(t, s, u(s))|ds + \int_0^s |\Gamma(t, s, v(s))|ds \\ &\leq \phi(|u(t) - v(t)|) + 2m(t) \int_0^t n(s)ds \\ &= \phi(|u(t) - v(t)|) + 2c(t). \end{aligned} \tag{17}$$

By (16), we derive that

$$diam(TU)(t) \leq \phi(diam U(t)) + 2c(t).$$

By right-continuity of the ϕ, observe that

$$\limsup_{t \to \infty} diam(TU)(t) \leq \phi(\limsup_{t \to \infty} diam(U)(t)). \tag{18}$$

By combining (16),(17),(18) and assumption (*iii*), we have

$$\zeta^0(TU) + \limsup_{t \to \infty} diam(TU)(t) \leq \phi(\zeta^0(U) + \limsup_{t \to \infty} diam(U)(t)),$$

from which,

$$\phi(\zeta^0(U) + \limsup_{t \to \infty} diam(U)(t)) - \zeta^0(TU) + \limsup_{t \to \infty} diam(TU)(t) \geq 0.$$

So,

$$\phi(\mu(U)) - \mu(TU) \geq 0, \tag{19}$$

Since $\varrho(t, s) = \phi(s) - t$, by (19), we have

$$\varrho(\mu(TU), \mu(U)) \geq 0. \tag{20}$$

Finally, by (20) and applying Theorem 2.3, one can conclude desired result.

□

The following lemma plays a crucial role on finding a solution for an important functional integral equation which appears in

Lemma 4.2. *Let $\varphi : [0, +\infty) \to [0, +\infty)$ be a function such that*

$$\varphi(t) = \begin{cases} 0 & if\ t = 0 \\ \frac{3n+4}{3n+6}x + \frac{1}{3(n+1)(n+2)} & if\ t \in (0, 1]\ such\ that\ \frac{1}{n+1} \leq t < \frac{1}{n}\ for\ some\ n \in \mathbb{N} \\ \frac{5}{6}t & if\ t > 1. \end{cases}$$

Then

(p_1) *φ is strictly increasing and continuous on* $[0,+\infty)$,

(p_2) *φ is a Meir-Keeler function so it is an $R-$function. Thus, it is a $SR-$function.*

Proof. (p_1) : Clearly, $\varphi(0) = 0$ and $\varphi'(s) > 0$ for $s \in (0,1)$. Indeed, on each interval $\left(\frac{1}{n+1}, \frac{1}{n}\right)$ we have that $\phi'(x) = (3n+4)/(3n+6) > 0$, so φ is strictly increasing on $\left(\frac{1}{n+1}, \frac{1}{n}\right)$, and as $\varphi'(x) = 5/6 > 0$ on $(1,+\infty)$, then φ is strictly increasing on $(1,+\infty)$. Let show that φ is continuous in $[0,+\infty)$. On each point $x \in (0,+\infty)$, φ is continuous because it is derivable at x. Moreover, if $x = 1/n$ for some $n \in \mathbb{N}$, then

$$\begin{aligned}\varphi(\tfrac{1}{n}) &= \lim_{x\to(\frac{1}{n})^-} \varphi(x) = \lim_{x\to(\frac{1}{n})^-} \left[\tfrac{3n+4}{3n+6}x + \tfrac{1}{3(n+1)(n+2)}\right] \\ &= \tfrac{3n+4}{3n+6}.\tfrac{1}{n} + \tfrac{1}{3(n+1)(n+2)} = \tfrac{3n+2}{3n(n+1)}\end{aligned}$$

and on the other hand,

$$\begin{aligned}\varphi(\tfrac{1}{n}) &= \lim_{x\to(\frac{1}{n})^+} \varphi(x) = \lim_{x\to(\frac{1}{n})^+} \left[\tfrac{3(n-1)+4}{3(n-1)+6}x + \tfrac{1}{3n(n+1)}\right] \\ &= \tfrac{3(n-1)+4}{3(n-1)+6}.\tfrac{1}{n} + \tfrac{1}{3n(n+1)} = \tfrac{3n+2}{3n(n+1)}.\end{aligned}$$

This is also valid for $n = 1$ because the limit from the right takes the value $5/6$. As a consequence,

$$\lim_{x\to(\frac{1}{n})^-} \varphi(x) = \varphi(\frac{1}{n}) = \lim_{x\to(\frac{1}{n})^+} \varphi(x)$$

and φ is continuous at $1/n$. The continuity at $x = 0$ follows from the fact that $\varphi(x) < x$ for all $x \in (0,+\infty)$. So φ is strictly increasing on $[0,+\infty)$.

(p_2) : We claim that φ satisfies in Meir-Keeler condition:

(MK) For every $\epsilon > (0,+\infty)$ there exists $\delta > 0$ such that $\varphi(x) \leq \epsilon$ for all $x \in [\epsilon, \epsilon+\delta)$.

Let $\epsilon(0,1]$ be arbitrary (the case $\epsilon > 1$ is obvious because $\varphi'(x) = \frac{5}{6}x$ if $x \in (1+\infty)$. Then, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0+1} < \epsilon \leq \frac{1}{n_0}$. We distinguish two cases:

– If $\frac{1}{n_0+1} < \epsilon < \frac{1}{n_0}$, then φ is a continuous, derivable function at $t = \epsilon$ such that $\varphi'(\epsilon) < \epsilon$ and $0 < \varphi'(\epsilon) < 1$. In fact, φ is affine on the interval $(\frac{1}{n_0+1}, \frac{1}{n_0})$. Let $\delta > 0$ be such that

$$\delta < \min\left\{\frac{\epsilon - \varphi(\epsilon)}{\varphi'(\epsilon)}, \frac{1}{n_0} - \epsilon\right\}.$$

Thus

$$[\epsilon, \epsilon+\delta] \subset \left[\epsilon, \epsilon + \frac{1}{n_0} - \epsilon\right] = \left[\epsilon, \frac{1}{n_0}\right] \subset \left(\frac{1}{n_0+1}, \frac{1}{n_0}\right]$$

As φ is affine on the interval, its derivation is constant. Therefore, for all $x \in [\epsilon, \epsilon+\delta]$,

$$\varphi'(\epsilon) = \frac{\varphi(x) - \varphi(\epsilon)}{x - \epsilon}.$$

So, it means that,

$$\begin{aligned}\varphi(x) &= \varphi(\epsilon) + (x-\epsilon)\varphi'(\epsilon) \\ &\leq \varphi(\epsilon) + \delta\varphi'(\epsilon) \\ &\leq \varphi(\epsilon) + \tfrac{\epsilon-\varphi(\epsilon)}{\varphi'(\epsilon)}.\varphi'(\epsilon) \\ &= \epsilon.\end{aligned}$$

– Assume that $\epsilon = \frac{1}{n_0}$.
In this case

$$\begin{array}{rcl} \epsilon - \varphi(\epsilon) & = & \frac{1}{n_0} - \left(\frac{3n_0+4}{3n_0+6}x + \frac{1}{3(n_0+1)(n_0+2)}\right) \\ & = & \frac{1}{3n_0(n_0+1)} \\ & = & \frac{\epsilon^2}{3(\epsilon+1)} > 0 \end{array}$$

Since φ is continuous and strictly increasing at ϵ, let $\delta > 0$ be such that

$$\varphi(x) - \varphi(\epsilon) = |\varphi(x) - \varphi(\epsilon)| < \frac{\epsilon^2}{3(\epsilon+1)} \quad for\ all\ x \in [\epsilon, \epsilon+\delta].$$

Henceforth, for all $x \in [\epsilon, \epsilon + \delta]$,

$$\varphi(x) < \varphi(\epsilon) + \frac{\epsilon^2}{3(\epsilon+1)} = \varphi(\epsilon) + \epsilon - \varphi(\epsilon) = \epsilon.$$

Thus, in any case, φ satisfies in Meir-Keeler condition.

□

Example 4.3. *Consider the following functional integral equation*

$$u(t) = \varphi(|u(t)|) + \int_0^t \frac{se^{-2t}|u(s)|}{u(s)+5} ds \tag{21}$$

where

$$\varphi(t) = \begin{cases} 0 & if\ t = 0 \\ \frac{3n+4}{3n+6}x + \frac{1}{3(n+1)(n+2)} & if\ t \in (0,1]\ such\ that\ \frac{1}{n+1} \le t < \frac{1}{n}\ for\ some\ n \in \mathbb{N} \\ \frac{5}{6}t & if\ t > 1. \end{cases}$$

We put

$$\Delta(t,u) = \varphi(u(t)) \quad and \quad \Gamma(t,s,u) = \frac{se^{-2t}|u|}{u+5}$$

We choose $\varrho(t,s) = \varphi(s) - t$ for all $t, s > 0$. By Lemma 4.2 φ is an $L-$function. Also, for arbitrarily fixed $u, v \in \mathbf{R}^+$ and for $t > 0$, we have

$$\begin{array}{rcl} |\Delta(t,u) - \Delta(t,v)| & = & |\varphi(u(t)) - \varphi(v(t))| \\ & \le & \varphi(|u(t) - v(t)|) \end{array}$$

So Δ satisfies (ii). Obviously Δ satisfies (i). Moreover, we get

$$|\Gamma(t,s,u)| \le se^{-2t} \quad for\ \ all \quad t,s \in \mathbf{R}^+ \quad and \quad u \in \mathbf{R}$$

If we put $m(t) = e^{-2t}$,$n(s) = s$, then assumption (iii) is satisfied. Also we have

$$\lim_{t\to\infty} m(t) \int_0^t n(s)ds = \lim_{t\to\infty} e^{-2t} \int_0^t sds = 0.$$

Now, let us calculate the constant Z appearing in assumption (iv). We have

$$Z = \sup\{|\Delta(t,0)| + m(t) \int_0^t n(s)ds : t \ge 0\} = \sup\{\frac{1}{2}t^2 e^{-2t} : t \ge 0\} = 2e^{-4} = 0.036631$$

Now, consider the inequality from (*iv*)*, having now the form*

$$\varphi(r) + Z \leq r$$

that is to say that $\varrho(r, r) + Z \leq 0$*. If* $r \geq 1$ *then we should have* $r \geq 6Z$*, so* $r_0 = 2$ *is sufficient. If* $0 < r \leq 1$*, then we can choose*

$$r_0 \geq \max\{r, \frac{1}{2n+2} + \frac{q(n+2)}{2}\}$$

as a sufficient number. The case $r = 0$ *is trivial. So, all conditions of Theorem 4.1 are satisfied. Hence, (21) has at least one solution in* $BC(\mathbf{R}^+)$.

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