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EP elements and $*-$Strongly Regular Rings

Hua Yao[a], Junchao Wei[a]

[a] *School of Mathematics, Yangzhou University, Yangzhou, 225002, P. R. China*

Abstract. Let R be a ring with involution $*$. An element $a \in R$ is called $*-$strongly regular if there exists a projection p of R such that $p \in comm^2(a)$, $ap = 0$ and $a + p$ is invertible, and R is said to be $*-$strongly regular if every element of R is $*-$strongly regular. We discuss the relations among strongly regular rings, $*-$strongly regular rings, regular rings and $*-$regular rings. Also, we show that an element a of a $*-$ring R is $*-$strongly regular if and only if a is EP. We finally give some characterizations of EP elements.

1. Introduction

In this article, all rings are associative with identity unless otherwise stated, and modules will be unitary modules. Let R be a ring, write $E(R)$, $N(R)$, $U(R)$, $J(R)$ and $Z(R)$ to denote the set of all idempotents, the set of all nilpotents, the set of units, the Jacobson radical and the center of R, respectively.

Rings in which every element is the product of a unit and an idempotent which commute are said to be strongly regular, and have been studied by many authors. According to Koliha and Patricio [11], the commutant and double commutant of an element $a \in R$ are defined by $comm(a) = \{x \in R | xa = ax\}$ and $comm^2(a) = \{x \in R | xy = yx \text{ for all } y \in comm(a)\}$. It is known that a ring R is strongly regular if and only if for each $a \in R$, there exists an idempotent $p \in comm^2(a)$ such that $a + p \in U(R)$ and $ap = 0$.

Let R be a ring and write $R^{qnil} = \{a \in R | 1 + ax \in U(R) \text{ for every } x \in comm(a)\}$. Recall that an element $a \in R$ is called polar (quasipolar) provided that there exists an idempotent $p \in R$ such that $p \in comm^2(a)$, $a + p \in U(R)$ and $ap \in N(R)$ ($ap \in R^{qnil}$), the idempotent p is unique, we denote it by a^{π}, which is called a spectral idempotent of a. A ring R is polar [7] (quasipolar [18]) in the case that every element in R is polar (quasipolar). [5, Theorem 2.4] shows that a ring R is strongly regular if and only if R is a quasipolar ring and $R^{qnil} = \{0\}$.

Following [3], an element a of a ring R is called group invertible if there is $a^{\sharp} \in R$ such that

$$aa^{\sharp}a = a, \; a^{\sharp}aa^{\sharp} = a^{\sharp}, \; aa^{\sharp} = a^{\sharp}a.$$

Denote by $R^{\sharp}$ the set of all group invertible elements of R. Clearly, a ring R is strongly regular if and only if $R = R^{\sharp}$.

An involution $a \longmapsto a^*$ in a ring R is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \; (a + b)^* = a^* + b^*, \; (ab)^* = b^*a^*.$$

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Email addresses: dalarston@126.com (Hua Yao), jcweiyz@126.com (Junchao Wei)

A ring R with an involution $*$ is called $*-$ring. An element $a^{\dagger}$ in a $*$-ring R is called the Moore-Penrose inverse (or MP-inverse)[**?**]f a, if Penr

$$aa^{\dagger}a = a,\ a^{\dagger}aa^{\dagger} = a^{\dagger},\ aa^{\dagger} = (aa^{\dagger})^*,\ a^{\dagger}a = (a^{\dagger}a)^*.$$

In this case, we call a is MP-invertible in R. The set of all MP-invertible elements of R is denoted by $R^{\dagger}$.

An involution $*$ of R is called proper if $x^*x = 0$ implies $x = 0$ for all $x \in R$. Following [1], a $*-$ring R is $*-$regular if and only if R is regular and the involution is proper.

An idempotent p of a $*-$ring R is called projection if $p = p^*$. Denote by $PE(R)$ the set of all projection elements of R. Clearly, $PE(R) \subseteq E(R)$. It is known that an idempotent e in a $*-$ring R is projection if and only if $e = e^*e$ if and only if $Re = Re^*$. [6, Lemma 2.1] shows that a $*-$ring R is $*-$regular if and only if for each $a \in R$, there exists $p \in PE(R)$ such that $aR = pR$.

Following [11], a $*-$ring R is $*-$regular if and only if $R = R^{\dagger}$. Due to [9], a $*-$ring R is said to satisfy the $k-$term star-cancellation law (or SC_k) if

$$a_1^*a_1 + \cdots + a_k^*a_k = 0 \Longrightarrow a_1 = \cdots = a_k = 0.$$

[10] shows that the 2×2 matrix ring $M_2(R)$ over a $*-$ring R is $*-$regular if and only if R is regular and satisfies SC_2.

Duo to [8], an element a of a $*-$ring R is said to be *EP* if $a \in R^{\#} \cap R^{\dagger}$ and $a^{\#} = a^{\dagger}$. In [14], many characterizations of *EP* elements are given.

The *EP* matrices and *EP* linear operators on Banach or Hilbert spaces have been investigated by many authors. This article is motivated by the papers [6, 14]. In this paper, we shall first give some new characterizations of *EP* elements. Next, we introduce $*-$strongly regular elements and $*-$strongly regular rings. We investigate the characterizations of $*-$strongly regular rings. Finally, we discuss $*-$exchange rings. With the help of $*-$exchange rings, we give some characterizations of $*-$strongly regular rings.

2. Some Characterizations of *EP* elements

Let R be a $*-$ring and $a \in R^{\dagger}$. Then by [14, Theorem 1.1], one knows that $a^* = a^*aa^{\dagger} = a^{\dagger}aa^*$. Hence we have the following proposition.

Proposition 2.1. *Let R be a $*-$ring and $a \in R$. Then a is an EP element if and only if $a \in R^{\dagger}$ and $Ra = Ra^{\dagger}$.*

Proof. Suppose that a is *EP*. Then $a \in R^{\dagger} \cap R^{\#}$ and $a^{\dagger} = a^{\#}$, it follows that $Ra = Ra^{\dagger}a = Ra^{\#}a = Raa^{\#} = Ra^{\#} = Ra^{\dagger}$.

Conversely, assume that $a \in R^{\dagger}$ and $Ra = Ra^{\dagger}$. Then $Ra = Raa^{\dagger} = R(aa^{\dagger})^* = R(a^{\dagger})^*a^* \subseteq Ra^* = Ra^*aa^{\dagger} \subseteq Ra^{\dagger} = Ra$, it follows that $Ra = Ra^*$. By [13, Theorem 3.1], one knows that a is *EP*. □

Similar to the proof of Proposition 2.1, we have the following corollary.

Corollary 2.2. *Let R be a $*-$ring and $a \in R$. Then a is an EP element if and only if $a \in R^{\dagger}$ and $aR = a^{\dagger}R$.*

It is known that for a $*-$ring R, $a \in R$ is *EP* if and only if $a^{\dagger}$ is *EP*. Hence we can obtain the following corollary.

Corollary 2.3. *Let R be a $*-$ring and $a \in R$. Then a is an EP element if and only if $a \in R^{\dagger}$ and $Ra^* = R(a^{\dagger})^*$.*

Proof. Suppose that a is *EP*. Then Proposition 2.1 and [13, Theorem 3.1] imply $Ra^* = Ra = Ra^{\dagger}$. Note that $a^{\dagger}$ is *EP*. Then [13, Theorem 3.1] gives $Ra^{\dagger} = R(a^{\dagger})^*$. Hence $Ra^* = R(a^{\dagger})^*$.

Conversely, assume that $Ra^* = R(a^{\dagger})^*$. Then $aR = a^{\dagger}R$, by Corollary 2.2, one gets a is *EP*. □

Theorem 2.4. *Let R be a $*-$ring and $a \in R$. Then the following conditions are equivalent:*
(1) a is EP;
(2) $a \in R^{\dagger}$ and $Ra = R(a^{\dagger})^n$ for each $n \geq 2$;
(3) $a \in R^{\dagger}$ and $Ra = R(a^{\dagger})^n$ for some $n \geq 2$.

Proof. (1) $\Longrightarrow$ (2) Since a is EP, by Proposition 2.1, we have $a \in R^{\dagger}$ and $Ra = Ra^{\dagger}$. Noting that $Ra^{\dagger} = Raa^{\dagger}$. Hence $Ra = Ra^{\dagger} = Raa^{\dagger} = Ra^{\dagger}a^{\dagger} = R(a^{\dagger})^2$, repeating the process, one obtains that $Ra = R(a^{\dagger})^n$ for each $n \geq 2$.

(2) $\Longrightarrow$ (3) It is trivial.

(3) $\Longrightarrow$ (1) Since $Ra = R(a^{\dagger})^n$ for some $n \geq 2$, $Ra \subseteq Ra^{\dagger} = Ra^*$. Note that $Ra^{\dagger} = Raa^{\dagger}$. Then $Ra^{\dagger} = R(a^{\dagger})^{n+1} \subseteq R(a^{\dagger})^n = Ra$, it follows that $Ra \subseteq Ra^* = Ra^{\dagger} \subseteq Ra$. Hence $Ra = Ra^* = Ra^{\dagger}$, this implies that a is EP. □

Let R be a $*-$ring and $a \in R$. Then it is easy to show that $a \in R^{\dagger}$ and $aa^* = 0$ imply $a = 0$. Also, $a \in R^{\dagger} \cap R^{\#}$ is EP if and only if $aa^{\#} = a^{\dagger}a$. Hence we have the following theorem.

Theorem 2.5. *Let R be a $*-$ring and $a \in R$. Then the following conditions are equivalent:*

(1) a is EP;

(2) $a \in R^{\dagger} \cap R^{\#}$ and $a^{\dagger}a^2a^* = a^2a^{\dagger}a^*$;

(3) $a \in R^{\dagger} \cap R^{\#}$ and $a^{\dagger}a^2a^* = aa^*aa^{\#}$;

(4) $a \in R^{\dagger} \cap R^{\#}$ and $a^{\dagger}a^na^* = a^{n-1}a^*a^{\dagger}a$ for some $n \geq 2$.

Proof. (1) $\Longrightarrow$ (2) It is trivial.

(2) $\Longrightarrow$ (3) Suppose that $a^{\dagger}a^2a^* = a^2a^{\dagger}a^*$. Then $aa^* = a^{\#}a^2a^* = a^{\#}a(a^{\dagger}a^2a^*) = a^{\#}a(a^2a^{\dagger}a^*) = a^2a^{\dagger}a^* = a^{\dagger}a^2a^*$, it follows that $aa^* = (aa^*)^* = aa^*a^{\dagger}a$, one obtains $a^* = a^{\dagger}aa^* = a^{\dagger}aa^*a^{\dagger}a = a^*a^{\dagger}a$, so $a = (a^*a^{\dagger}a)^* = a^{\dagger}a^2$. Hence $aa^*aa^{\#} = aa^*(a^{\dagger}a^2)a^{\#} = a(a^*a^{\dagger}a) = aa^* = a^{\dagger}a^2a^*$.

(3) $\Longrightarrow$ (4) Suppose that $a^{\dagger}a^2a^* = aa^*aa^{\#}$. Then similar to (2) $\Longrightarrow$ (3), one can show that $a = a^{\dagger}a^2$ and $a^* = a^*a^{\dagger}a$. Hence $a^{\dagger}a^na^* = (a^{\dagger}a^2)a^{n-2}a^* = a^{n-1}a^* = a^{n-1}a^*a^{\dagger}a$.

(4) $\Longrightarrow$ (1) Assume that $a^{\dagger}a^na^* = a^{n-1}a^*a^{\dagger}a$. Then $a^{n-1}a^* = a^{\#}a^na^* = a^{\#}a(a^{\dagger}a^na^*) = a^{\#}a(a^{n-1}a^*a^{\dagger}a) = a^{n-1}a^*a^{\dagger}a$, it follows that $aa^* = (a^{\#})^{n-2}a^{n-1}a^* = (a^{\#})^{n-2}a^{n-1}a^*a^{\dagger}a = aa^*a^{\dagger}a$, so $a^* = a^{\dagger}aa^* = a^{\dagger}(aa^*a^{\dagger}a) = a^*a^{\dagger}a$, this gives $(a^{\#})^*a^* = (a^{\#})^*a^*a^{\dagger}a$, so $aa^{\#} = a^{\dagger}a$. Hence a is EP. □

Remark: The condition (4) of Theorem 2.5 exists in [12, Theorem 2.1(xii)] for $m = n - 1$ and $n = 1$.

Theorem 2.6. *Let R be a $*-$ring and $a \in R$. Then the following conditions are equivalent:*

(1) a is EP;

(2) $a \in R^{\dagger} \cap R^{\#}$ and $a^2a^{\dagger} + a^{\#}aa^{\dagger} = a + a^{\dagger}$;

(3) $a \in R^{\dagger} \cap R^{\#}$ and $a^2a^{\dagger} + a^{\#} = a + a^{\dagger}$;

(4) $a \in R^{\dagger} \cap R^{\#}$ and $a^{\#}aa^{\dagger} + a^{\dagger}aa^{\#} = 2a^{\dagger}$;

(5) $a \in R^{\dagger} \cap R^{\#}$ and $a^{\dagger} + a^{\#} = 2a^{\dagger}aa^{\#}$;

(6) $a \in R^{\dagger} \cap R^{\#}$ and $a^{\dagger} + a^{\#} = 2a^{\dagger}a^{\dagger}a$.

Proof. (1) $\Longrightarrow$ $(i), i = 2, 3, 4, 5, 6$ They are trivial.

(2) $\Longrightarrow$ (1) From the assumption $a^2a^{\dagger} + a^{\#}aa^{\dagger} = a + a^{\dagger}$, we get $a^2a^{\dagger}a + a^{\#}aa^{\dagger}a = a^2 + a^{\dagger}a$. So, $a^{\#}a = a^{\dagger}a$, it follows that a is EP.

(3) $\Longrightarrow$ (1) By the equality $a^2a^{\dagger} + a^{\#} = a + a^{\dagger}$, we get $a^2a^{\dagger}a + a^{\#}a = a^2 + a^{\dagger}a$, this gives $a^{\#}a = a^{\dagger}a$. Hence a is EP.

(4) $\Longrightarrow$ (1) Using the equality $a^{\#}aa^{\dagger} + a^{\dagger}aa^{\#} = 2a^{\dagger}$, we have $2aa^{\dagger} = aa^{\#}aa^{\dagger} + aa^{\dagger}aa^{\#} = aa^{\dagger} + aa^{\#}$, it follows that $aa^{\dagger} = aa^{\#}$. Hence a is EP.

(5) $\Longrightarrow$ (1) The equality $a^{\dagger} + a^{\#} = 2a^{\dagger}aa^{\#}$ gives $aa^{\dagger} + aa^{\#} = 2aa^{\dagger}aa^{\#} = 2aa^{\#}$, again we have $aa^{\dagger} = aa^{\#}$. Hence a is EP.

(6) $\Longrightarrow$ (1) If $a^{\dagger} + a^{\#} = 2a^{\dagger}a^{\dagger}a$, then $a^{\dagger} + a^{\#} = 2a^{\dagger}a^{\dagger}a = a^{\dagger}a(2a^{\dagger}a^{\dagger}a) = a^{\dagger}a(a^{\dagger} + a^{\#}) = a^{\dagger} + a^{\dagger}aa^{\#}$, one obtains that $a^{\#} = a^{\dagger}aa^{\#}$. Hence $a^{\dagger}a = a^{\#}a$ and so a is EP. □

Remark: The condition (4) of Theorem 2.6 exists in [12, Theorem 2.1(xv)] for $n = 1$.

Theorem 2.7. *Let R be a $*-$ring. Then $E(R) = PE(R)$ if and only if every element of $E(R)$ is EP.*

Proof. Let $e \in E(R)$. If $E(R) = PE(R)$, then $e = e^*$. It is not difficult to verify that e is EP with $e^{\#} = e^{\dagger} = e$. Conversely, we assume that e is EP. Then $e^{\#} = e^{\dagger}$, it follows that $e = ee^{\#}e = ee^{\#}$ and so $e^{\dagger} = e^{\#} = (ee^{\#})e^{\#} = ee^{\#} = e$. Hence $e \in PE(R)$. □

Recall that a ring R is directly finite if $ab = 1$ implies $ba = 1$ for any $a, b \in R$. Clearly, a ring R is directly finite if and only if right invertible element of R is invertible.

Theorem 2.8. *Let R be a $*-$ring. Then the following conditions are equivalent:*
(1) R is a directly finite ring;
(2) Every right invertible element of R is group invertible;
(3) Every right invertible element of R is EP.

Proof. (1) $\Longrightarrow$ (3) It is trivial because every invertible element is *EP*.
(3) $\Longrightarrow$ (2) It is evident.
(2) $\Longrightarrow$ (1) Suppose that $a, b \in R$ with $ab = 1$. By hypothesis, $a \in R^{\#}$, so $1 = ab = (aa^{\#})(ab) = aa^{\#} = a^{\#}a$, one obtains that a is invertible. Hence R is directly finite. □

Recall that a ring R is reduced if $N(R) = \{0\}$. Using the *EP* elements, we can characterize reduced rings as follows.

Theorem 2.9. *Let R be a $*-$ring. Then the following conditions are equivalent:*
(1) R is a reduced ring;
(2) Every element of $N(R)$ is group invertible;
(3) Every element of $N(R)$ is EP.

Proof. (1) $\Longrightarrow$ (3) $\Longrightarrow$ (2) They are trivial.
(2) $\Longrightarrow$ (1) Suppose that the condition (2) holds. If R is not reduced, then there exists $b \in R\backslash\{0\}$, let n be the positive integer such that $b^n = 0$ and $b^{n-1} \neq 0$. Choose $a = b^{n-1}$. Then $a \in R\backslash\{0\}$ with $a^2 = 0$. Since $a \in R^{\#}$, $a = a^2a^{\#} = 0$, which is a contradiction. Hence R is reduced. □

Theorem 2.10. *Let R be a $*-$ring and $a \in R$. Then a is EP if and only if there exists (unique) $p \in PE(R)$ such that $pa = ap = 0$ and $a + p \in U(R)$.*

Proof. It is similar to the proof of [2, Theorem 2.1]. □

Also, similar to the proof of [2, Theorem 2.1], we have the following corollary.

Corollary 2.11. *Let R be a $*-$ring and $a \in R$. Then a is EP if and only if there exists unique $p \in PE(R)$ such that $pa = ap = 0$ and $a - p \in U(R)$.*

Corollary 2.12. *Let R be a $*-$ring and $a \in R$. Then a is EP if and only if there exists $p \in PE(R)$ such that $p \in comm^2(a)$, $ap = 0$ and $a + p \in U(R)$.*

Proof. The sufficiency follows from Theorem 2.10.
The necessity: Noting that $p = 1 - a^{\#}a$ in Theorem 2.10. Then, for any $x \in comm(a)$, we have $(1 - p)xp = a^{\#}axp = a^{\#}xap = 0$ and $px(1 - p) = pxaa^{\#} = paxa^{\#} = 0$, this implies that $px = pxp = xp$. Hence $p \in comm^2(a)$, we are done. □

Similarly, we have the following corollary.

Corollary 2.13. *Let R be a $*-$ring and $a \in R$. Then a is EP if and only if there exists unique $p \in PE(R)$ such that $p \in comm^2(a)$, $ap = 0$ and $a - p \in U(R)$.*

Theorem 2.14. *Let R be a $*-$ring and $a \in R$. Then a is EP if and only if there exists $b \in comm^2(a)$, $ab = ba \in PE(R)$, $a = a^2b$ and $b = ab^2$.*

Proof. Suppose that a is *EP*. Then by Corollary 2.12, there exists $p \in PE(R)$ such that $p \in comm^2(a)$, $ap = 0$ and $a + p \in U(R)$. Choose $b = (a + p)^{-1}(1 - p)$. Then clearly, $b \in comm^2(a)$ and $ab = ba = 1 - p \in PE(R)$. By a simple computation, we have $a = a^2b$ and $b = ab^2$.
Conversely, assume that there exists $b \in comm^2(a)$, $ab = ba \in PE(R)$, $a = a^2b$ and $b = ab^2$. Choose $p = 1 - ab$. Then $p \in PE(R)$, $ap = a - a^2b = 0 = pa$ and $pb = b - ab^2 = 0 = bp$. Note that $(a + p)(b + p) = ab + p = 1$. Then $a + p \in U(R)$, by Theorem 2.10, a is *EP*. □

3. $*-$Strongly Regular Rings

Recall that an element a of a ring R is strongly regular if $a \in a^2R \cap Ra^2$. It is well known that $a \in R$ is strongly regular if and only if there exist $e \in E(R)$ and $u \in U(R)$ such that $a = eu = ue$.

Let R be a $*-$ring. An element $a \in R$ is called $*-$strongly regular if there exist $p \in PE(R)$ and $u \in U(R)$ such that $a = pu = up$. A ring R is called $*-$strongly regular if every element of R is $*-$strongly regular.

Clearly, $*-$strongly regular elements are strongly regular, and so $*-$strongly regular rings are strongly regular. However, the converse is not true by the following example.

Example 3.1. *Let D be a division ring and $R = D \oplus D$. Set $*$ be an involution of R defined by $*((a, b)) = (b, a)$. Evidently, R is a strongly regular ring, but R is not $*-$strongly regular. In fact $(1, 0)$ is not a $*-$strongly regular element.*

Theorem 3.2. *Let R be a $*-$ring. Then R is a $*-$strongly regular ring if and only if R is a strongly regular ring with $E(R) = PE(R)$.*

Proof. Suppose that R is a $*-$strongly regular ring and $e \in E(R)$. Then there exist $p \in PE(R)$ and $u \in U(R)$ such that $e = pu = up$, this gives $e = pe = ep$. Note that $p = eu^{-1}$. Then $p = ep = e$, so $E(R) \subseteq PE(R)$, this shows that $E(R) = PE(R)$.

The converse is trivial. □

Theorem 3.3. *Let R be a $*-$ring and $a \in R$. Then a is EP if and only if a is $*-$strongly regular.*

Proof. Suppose that a is EP. Then, by Theorem 2.10, there exists $p \in PE(R)$ such that $a + p \in U(R)$ and $ap = pa = 0$. Write $a + p = u \in U(R)$. Then $a = a(1 - p) = u(1 - p) = (1 - p)u$. Since $1 - p \in PE(R)$, a is $*-$strongly regular.

Conversely, assume that a is $*-$strongly regular. Then there exist $p \in PE(R)$ and $u \in U(R)$ such that $a = pu = up$. Since $(a + 1 - p)(u^{-1}p + 1 - p) = (u^{-1}p + 1 - p)(a + 1 - p) = 1$, $a + 1 - p \in U(R)$. Noting that $a(1 - p) = (1 - p)a = 0$ and $1 - p \in PE(R)$. Hence a is EP by Theorem 2.10. □

Theorem 3.4. *Let R be a $*-$ring. Then R is $*-$strongly regular if and only if R is Abel and for each $a \in R$, $Ra = Ra^*a$.*

Proof. Suppose that R is $*-$strongly regular. Note that $*-$strongly regular rings are strongly regular. Then R is also Abel. Now let $a \in R$. Then a is $*-$strongly regular, so there exist $p \in PE(R)$ and $u \in U(R)$ such that $a = pu = up$. Hence $a^*a = u^*up$, one obtains that $Ra^*a = Rp = Ra$.

Conversely, assume that R is Abel and for each $a \in R$, $Ra = Ra^*a$. Write that $a = da^*a$ for some $d \in R$. Then $(ad^*)^2 = ad^*ad^* = (da^*a)d^*ad^* = d(a^*ad^*)ad^* = da^*ad^* = ad^*$. Noting that R is Abel, ad^* is a central idempotent of R, so da^* is a central idempotent of R, this gives that $a = (da^*)a = a(da^*)$. Hence $Ra \subseteq Ra^*$. By [4, Proposition 2.7], R is a $*$-regular ring, so $a \in R^{\dagger}$. Thus by [13, Theorem 3.1], one knows that a is EP, by Theorem 3.3, a is $*-$strongly regular. Hence R is $*-$strongly regular. □

Corollary 3.5. *A $*-$ring R is a $*-$strongly regular ring if and only if R is an Abel ring and $*-$regular ring.*

Let R be a ring and write $ZE(R) = \{x \in R | ex = xe$ for each $e \in E(R)\}$. It is easy to show that $ZE(R)$ is a subring of R and $Z(R)$, the center, of R is contained in $ZE(R)$.

Let R be a $*-$ring. Choose $a \in ZE(R)$ and $e \in E(R)$. Since $e^* \in E(R)$, $ae^* = e^*a$, it follows that $ea^* = a^*e$. Hence $a^* \in ZE(R)$, so $ZE(R)$ becomes a $*-$ring.

Theorem 3.6. *Let R be a $*-$regular ring. Then $ZE(R)$ is a $*-$strongly regular ring.*

Proof. Let $a \in ZE(R)$. Since R is a $*-$regular ring, by [6, Lemma 2.1], there exists $p \in PE(R)$ such that $aR = pR$. Write $p = ab$ for some $b \in R$. Then $a = pa = aba$. Choose $e \in E(R)$. Then $ae = ea$, it follows that $(1 - p)epa = (1 - p)ea = (1 - p)ae = 0$, this gives $(1 - p)ep = 0$, that is, $ep = pep$. Since $e^* \in E(R)$, $e^*p = pe^*p$, one obtains $pe = pep$. Hence $ep = pe$, this implies $p \in ZE(R)$. Note that $ba \in E(R)$. Then

$ba^2 = (ba)a = a(ba) = a = pa = ap = a^2b$, it follows that $b^3a^2 = a^2b^3$. Since $ba^2e = ae = ea = ea^2b = a^2eb$, $b^3a^2e = a^2eb^3 = ea^2b^3$, this implies that $a^2b^3 \in ZE(R)$. Choose $c = a^2b^3 \in ZE(R)$. Then $ac = a^3b^3 = a^2(ab)b^2 = a^2pb^2 = pa^2b^2 = a^2b^2 = a(ab)b = apb = ab = p$. Hence $aZE(R) = paZE(R) \subseteq pZE(R) = acZE(R) \subseteq aZE(R)$, by [6, Lemma 2.1], $ZE(R)$ is a $*-$regular ring. Note that $ZE(R)$ is Abel. Then by Corollary 3.5, we have $ZE(R)$ is $*-$strongly regular. □

Clearly, if R is an Abel ring, then $ZE(R) = R$. Hence Corollary 3.5 and Theorem 3.6 give the following corollary.

Corollary 3.7. *Let R be a $*-$ring. Then R is a $*-$strongly regular ring if and only if R is an Abel ring and $ZE(R)$ is a $*-$strongly regular ring.*

Due to [16], a $*-$ring is $*-$Abel if every projection is central. Clearly, Abel $*-$rings are $*-$Abel. A $*-$ring R is called $*-$quasi-normal if $pR(1-p)Rp = 0$ for each $p \in PE(R)$. Clearly, $*-$Abel rings are $*-$quasi-normal.

Corollary 3.8. *Let R be a $*-$ring. Then R is a $*-$strongly regular ring if and only if R is a $*-$quasi-normal $*-$regular ring.*

Proof. The necessity follows from Corollary 3.5.

Conversely, assume that R is a $*-$quasi-normal $*-$regular ring. Then R is a semiprime ring and $pR(1-p)Rp = 0$ for each $p \in PE(R)$, this implies $pR(1-p) = 0 = (1-p)Rp$. Hence R is $*-$Abel, by Corollary 3.5, R is $*-$strongly regular. □

Corollary 3.9. *If R is a $*-$strongly regular ring, then so is pRp for any $p \in PE(R)$.*

Proof. It follows from Corollary 3.5 and [6, Proposition 2.8]. □

4. $*-$Exchange Rings

Definition 4.1. *Let R be a $*-$ring and $a \in R$. If there exists $p \in PE(R)$ such that $p \in aR$ and $1-p \in (1-a)R$, then a is called $*-$exchange element of R. And a $*-$ring R is said to be $*-$exchange if every element of R is $*-$exchange.*

Clearly, any $*-$exchange element of a $*-$ring R is exchange and the converse is true whenever $PE(R) = E(R)$.

Lemma 4.2. *Let R be a $*-$ring and $x \in R$. If x is $*-$strongly regular, then x is $*-$exchange.*

Proof. Suppose that x is $*-$strongly regular. Then there exist $u \in U(R)$ and $p \in PE(R)$ such that $x = pu = up$, and hence $x(1-p) = 0$. Note that $p = xu^{-1}$ and $(1-x)(1-p) = 1-p$. Hence x is $*-$exchange. □

Lemma 4.3. *Let R be a $*-$ring and $x \in R$. Then the following conditions are equivalent:*

(1) x is $*-$exchange;

(2) There exists $p \in PE(R)$ such that $p - x \in (x - x^2)R$.

Proof. (1) $\Longrightarrow$ (2) Assume that x is $*-$exchange. Then there exists $p \in PE(R)$ such that $p \in xR$ and $1-p \in (1-x)R$, this gives $p - x = (1-x)p - x(1-p) \in (x-x^2)R$.

(2) $\Longrightarrow$ (1) Let $p \in PE(R)$ satisfy $p - x \in (x-x^2)R$. Write $p - x = (x-x^2)c$ for some $c \in R$. It follows that $p = x(1 + (1-x)c) \in xR$ and $1 - p = (1-x)(1-xc) \in (1-x)R$. Hence x is $*-$exchange. □

Let R be a $*-$ring and I be an (one-sided) ideal of R. I is called $*-$(one-sided) ideal of R if $a^* \in I$ for each $a \in I$. Clearly, the Jacobson radical $J(R)$ of a $*-$ring R is $*-$ideal.

Lemma 4.4. *Let R be a $*-$exchange ring and I a $*-$right ideal of R. Then the projection elements can be lifted modulo I.*

Proof. Let $x \in R$ satisfy $x - x^2 \in I$. Since R is $*-$exchange, there exists $p \in PE(R)$ such that $p - x \in (x - x^2)R$ by Lemma 4.3. Note that I is a $*-$right ideal of R. Hence $p - x \in I$, we are done. □

Lemma 4.5. *If R is a $*-$exchange ring, then $E(R) = PE(R)$.*

Proof. Let $e \in E(R)$. Then by the hypothesis, there exists $p \in PE(R)$ such that $p \in eR$ and $1 - p \in (1 - e)R$. It follows that $p = ep = e$. Hence $e \in PE(R)$, this gives $E(R) \subseteq PE(R)$. Therefore $E(R) = PE(R)$. □

Let R be a $*-$ring and I a $*-$ideal of R. For each $\bar{a} = a + I$ in $\bar{R} = R/I$, we define $\bar{a}^* = a^* + I$. Then R/I becomes a $*-$ring.

Theorem 4.6. *Let R be a $*-$ring. Then R is a $*-$exchange ring if and only if*
(1) $R/J(R)$ is $*-$exchange ring;
(2) Projection elements can be lifted modulo $J(R)$;
(3) $E(R) = PE(R)$.

Proof. Suppose that R is $*-$exchange. Then the projection elements can be lifted modulo $J(R)$ by Lemma 4.4 and $E(R) = PE(R)$ by Lemma 4.5. Note that R is exchange. Then $R/J(R)$ is exchange, it follows that $R/J(R)$ is $*-$exchange because $E(R) = PE(R)$.

Conversely, let $a \in R$. Since $\bar{R} = R/J(R)$ is $*-$exchange, there exists $p \in R$ such that $\bar{p} \in PE(\bar{R}) \cap \bar{a}\bar{R}$ and $\bar{1} - \bar{p} \in (\bar{1} - \bar{a})\bar{R}$. Note that the projection elements can be lifted modulo $J(R)$. Then we can assume that $p \in PE(R)$. Let $b, c \in R$ satisfy $p - ab \in J(R)$ and $1 - p - (1 - a)c \in J(R)$. Write $u = 1 - p + ab$. Then $u \in U(R)$. Let $e = upu^{-1}$. Then we have $e^2 = e = abpu^{-1} \in aR$. Note that $E(R) = PE(R)$. Then $e \in PE(R)$. Since $p - ab \in J(R)$, $\bar{a}\bar{b} = \bar{p}$, it follows that $\bar{u} = \bar{1} - \bar{p} + \bar{a}\bar{b} = \bar{1}$, so $\bar{e} = \bar{a}\bar{b}\bar{p}\bar{u}^{-1} = \bar{p}$, $e - p \in J(R)$, it follows that $1 - e - (1 - a)c = 1 - p - (1 - a)c + p - e \in J(R)$. Write $1 - e - (1 - a)c = d \in J(R)$. Then $1 = e(1 - d)^{-1} + (1 - a)c(1 - d)^{-1}$. Choose $f = e + e(1-d)^{-1}(1-e)$. Then $f \in PE(R) \cap aR$ and $1 - f = (1 - e(1-d)^{-1})(1-e) = (1-a)c(1-d)^{-1}(1-e) \in (1-a)R$. Therefore a is $*-$exchange and so R is $*-$exchange. □

Theorem 4.6 implies the following corollary.

Corollary 4.7. *A $*-$ring R is $*-$exchange if and only if R is exchange and $PE(R) = E(R)$.*

Lemma 4.8. *Let R be a $*-$ring. Then $E(R) = PE(R)$ if and only if for each $e, g \in E(R)$, $e^*e = ee^*$ and $g^*g = 0$ implies $g = 0$.*

Proof. Suppose that $E(R) = PE(R)$ and $e \in E(R)$. We claim that $eR(1 - e) = 0$. If not, then there exists $a \in R$ such that $ea(1-e) \neq 0$. Note that $g = e + ea(1-e) \in E(R) = PE(R)$. Then $e + ea(1-e) = g = g^* = e^* + (1-e^*)a^*e^* = e + (1-e)a^*e$, it follows that $ea(1-e) = (1-e)a^*e$, so $ea(1-e) = 0$, which is a contradiction. Hence $eR(1-e) = 0$. Similarly, we can show that $(1 - e)Re = 0$. Hence $e^*e = ee^*e = ee^*$.

Now assume that $g \in E(R)$ and $g^*g = 0$. Noting that $E(R) = PE(R)$. Then $g^* = g$, so $g = 0$.

Conversely, let $e \in E(R)$. Then by hypothesis, one has $e^*e = ee^*$. Since $e - e^*e \in E(R)$ and $(e - e^*e)^*(e - e^*e) = 0$, again by hypothesis, one obtains that $e - e^*e = 0$, this implies $e \in PE(R)$. Hence $E(R) = PE(R)$. □

By the proof of Lemma 4.8, we have the following corollary.

Corollary 4.9. *Let R be a $*-$ring and $E(R) = PE(R)$. Then R is an Abel ring.*

It is known that Abel exchange rings are clean. Hence Theorem 4.6 and Corollary 4.9 imply the following corollary.

Corollary 4.10. *$*-$exchange rings are clean.*

Since clean rings are always exchange, hence Theorem 4.6 and Corollary 4.10 give the following corollary.

Corollary 4.11. *Let R be a $*-$ring. Then the following conditions are equivalent:*
(1) R is a $*-$exchange ring;
(2) R is an exchange ring and $E(R) = PE(R)$;
(3) R is a clean ring and $E(R) = PE(R)$.

The following corollary follows from [17, Theorem 3.3, Corollary 3.4, Theorem 3.12, Corollary 4.9], Corollary 4.7 and Corollary 4.9.

Corollary 4.12. *Let R be a $*-$exchange ring and P is an ideal of R.*
(1) If P is a prime ideal of R, then R/P is a local ring;
(2) If P is a left (right) primitive ideal of R, then R/P is a division ring;
(3) R is a left and right quasi-duo ring;
(4) R has stable range one.

Theorem 4.13. *The following conditions are equivalent for a $*-$ring R:*
(1) R is a $*-$strongly regular ring;
(2) R is a semiprime $*-$exchange ring and every prime ideal of R is maximal;
(3) R is a semiprime $*-$exchange ring and every prime ideal of R is left (right) primitive.

Proof. (1) $\Longrightarrow$ (2) Suppose that R is $*-$strongly regular. Then, by Lemma 4.2, R is $*-$exchange, this implies R is left and right quasi-duo by Corollary 4.12. Note that R is strongly regular. Hence, by [19, Theorem 2.6], R is a semiprime and every prime ideal of R is maximal.

(2) $\Longrightarrow$ (3) It is trivial.

(3) $\Longrightarrow$ (1) Suppose that R is a semiprime $*-$exchange ring and every prime ideal of R is left (right) primitive. Then R is left and right quasi-duo by Corollary 4.12 and $PE(R) = E(R)$ by Theorem 4.6. Note that R is strongly regular by [19, Theorem 2.6]. Hence R is $*-$strongly regular by Theorem 3.2. □

Corollary 4.14. *Let R be a $*-$exchange semiprimitive ring such that every left R-module has a maximal submodule, then R is $*-$strongly regular.*

Proof. Note that R is left and right quasi-duo and $PE(R) = E(R)$ by Corollary 4.7 and Corollary 4.12. Then, by [19, Lemma 3.2], R is von neumann regular, it follows that R is $*-$strongly regular by Theorem 3.2. □

Corollary 4.15. *Let R be a $*-$exchange ring. If every prime ideal of R is left (right) primitive, then $R/J(R)$ is $*-$strongly regular.*

Proof. Since R is a $*-$exchange ring, by Theorem 4.6, $R/J(R)$ is $*-$exchange. Note that $R/J(R)$ is semiprime and every prime ideal of $R/J(R)$ is left (right) primitive. Then, by Theorem 4.13, one obtains that $R/J(R)$ is $*-$strongly regular. □

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