Filomat 32:2 (2018), 517–530
https://doi.org/10.2298/FIL1802517A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Variational Approach for Fractional Boundary Value Systems Depending on Two Parameters

Ghasem A. Afrouzi[a], Samad Mohseni Kolagar[a], Armin Hadjian[b], Jiafa Xu[c]

[a]*Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran*
[b]*Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, P.O. Box 1339, Bojnord 94531, Iran*
[c]*School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China*

Abstract. In this paper, we prove the existence of infinitely many solutions to nonlinear fractional boundary value systems, depending on two real parameters. The approach is based on critical point theory and variational methods. We also give an example to illustrate the obtained results.

1. Introduction

In this paper, we are interested in ensuring the existence of infinitely many solutions for the following fractional boundary value system

$$\begin{cases} {}_tD_T^{\alpha_i}\Big(a_i(t)\,{}_0D_t^{\alpha_i}u_i(t)\Big) &= \lambda F_{u_i}(t,u_1(t),\dots,u_n(t)) + \mu G_{u_i}(t,u_1(t),\dots,u_n(t)) \\ & \quad +h_i(u_i(t)), \qquad \text{a.e. } t\in[0,T], \\ u_i(0)=u_i(T)=0, \end{cases} \tag{1}$$

for $1\le i\le n$, where $\alpha_i\in(0,1]$, ${}_0D_t^{\alpha_i}$ and ${}_tD_T^{\alpha_i}$ are the left and right Riemann-Liouville fractional derivatives of order α_i respectively, $a_i\in L^\infty([0,T])$ with $a_{i0}:=\operatorname{ess\,inf}_{[0,T]}a_i>0$ for $1\le i\le n$, λ is a positive parameter, μ is a nonnegative parameter, $F,G:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ are continuous functions with respect to $t\in[0,T]$ for every $(x_1,\dots,x_n)\in\mathbb{R}^n$ and are C^1 with respect to $(x_1,\dots,x_n)\in\mathbb{R}^n$ for a.e. $t\in[0,T]$, $F(t,0,\dots,0)=0$ and $G(t,0,\dots,0)=0$ for a.e. $t\in[0,T]$, F_{u_i} and G_{u_i} denotes the partial derivative of F and G with respect to u_i, respectively, and $h_i:\mathbb{R}\to\mathbb{R}$ are Lipschitz continuous functions with the Lipschitz constants $L_i>0$ for $1\le i\le n$, i.e.,

$$|h_i(x_1)-h_i(x_2)|\le L_i|x_1-x_2|, \tag{2}$$

for every $x_1,x_2\in\mathbb{R}$, and $h_i(0)=0$ for $1\le i\le n$.

Fractional differential equations has proved to be an important tool in the modeling of dynamical systems associated with phenomena such as fractals and chaos. In fact, this branch of calculus has found

2010 *Mathematics Subject Classification*. Primary 34A08; Secondary 26A33, 35A15
Keywords. Fractional differential equations; Riemann-Liouville fractional derivatives; Variational methods.
Received: 10 March 2017; Accepted: 12 November 2017
Communicated by Jelena Manojlović
Corresponding author: Jiafa Xu
Email addresses: afrouzi@umz.ac.ir (Ghasem A. Afrouzi), mohseni.samad@gmail.com (Samad Mohseni Kolagar), a.hadjian@ub.ac.ir, hadjian83@gmail.com (Armin Hadjian), xujiafa292@sina.com (Jiafa Xu)

its applications in various disciplines of science and engineering such as mechanics, electricity, chemistry, biology, economics, control theory, signal and image processing, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electrodynamics of complex medium, viscoelasticity and damping, control theory, wave propagation, percolation, identification, and fitting of experimental data. Fractional differential equations serve as an excellent tool for the description of hereditary properties of various materials and processes.The interest in the study of fractional order differential equations lies in the fact that fractional order models are found to be more accurate than the classical integer-order models; that is, there are more degrees of freedom in the fractional order models. In consequence, the subject of fractional differential equations is gaining more and more attention; see for instance the monographs of Miller and Ross [17], Samko et al [12], Podlubny [18], Hilfer [9], Kilbas et al [13] and the papers [1–4, 6, 7]. See also [14, 15, 22, 24–30] and references therein.

Critical point theory has been very useful in determining the existence of solutions for integer order differential equations with some boundary conditions; see for instance, in the vast literature on the subject, the classical books [16, 19, 21, 23] and references therein. But until now, there are a few results for fractional boundary value problems (briefly BVP) which were established exploiting this approach, since it is often very difficult to establish a suitable space and variational functional for fractional problems.

In [5], Bai established the existence of infinitely many solutions for the following perturbed nonlinear fractional boundary value problem

$$\begin{cases} {}_tD_T^{\alpha}\Big({}_0D_t^{\alpha}u(t)\Big) = \lambda a(t)f(u(t)) + \mu g(t,u(t)), & \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where $\alpha \in (0,1]$, λ and μ are non-negative parameters, $a : [0,T] \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ and $g : [0,T] \times \mathbb{R}$ are three given continuous functions.

Also, by applying the critical point theorem due to Bonanno and Marano [8], the authors in [25] provided a new approach to study the existence of at least three weak solutions for the coupled system

$$\begin{cases} {}_tD_T^{\alpha}\Big(a(t){}_0D_t^{\alpha}u(t)\Big) = \lambda F_u(t,u(t),v(t)), & 0 < t < T, \\ {}_tD_T^{\beta}\Big(b(t){}_0D_t^{\beta}v(t)\Big) = \lambda F_v(t,u(t),v(t)), & 0 < t < T, \\ u(0) = u(T) = 0, \ \ v(0) = v(T) = 0, \end{cases}$$

where λ is a positive real parameter, $0 < \alpha, \beta \le 1$, $a, b \in L^{\infty}[0,T]$ with $a_0 := \operatorname{ess\,inf}_{[0,T]} a(t) > 0$ and $b_0 := \operatorname{ess\,inf}_{[0,T]} b(t) > 0$, and $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is a function such that $F(\cdot, x, y)$ is continuous in $[0,T]$ for every $(x,y) \in \mathbb{R}^2$ and $F(t,\cdot,\cdot)$ is a C^1 function for any $t \in [0,T]$.

Further, with the same assumptions as above, the existence of infinitely many weak solutions for the following fractional differential system has been achieved in [26] via critical point theory

$$\begin{cases} {}_tD_T^{\alpha}\Big(a(t){}_0D_t^{\alpha}u(t)\Big) = \lambda F_u(t,u(t),v(t)) + h_1(u(t)), & 0 < t < T, \\ {}_tD_T^{\beta}\Big(b(t){}_0D_t^{\beta}v(t)\Big) = \lambda F_v(t,u(t),v(t)) + h_2(v(t)), & 0 < t < T, \\ u(0) = u(T) = 0, \ \ v(0) = v(T) = 0, \end{cases}$$

where $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ are two Lipschitz continuous functions with the Lipschitz constants $L_1, L_2 \ge 0$.

In the present paper, motivated by the above works and using Ricceri's variational principle (see [20]) we ensure the existence of infinitely many weak solutions for system (1). More precisely, starting from the results obtained in [26] and with the same method, we are interested in looking for a class of perturbations, namely μG_{u_i}, for which (1) still preserves multiple solutions.

This paper is organized as follows. In Section 2, we present some necessary preliminary facts that will be needed in the paper. In Section 3 our main result and some significative consequences and an example is presented.

2. Preliminaries

In this section, we first introduce some necessary definitions and properties of the fractional calculus which are used in this paper.

Definition 2.1. Let u be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order $\alpha > 0$ for a function u are defined by

$$_aD_t^\alpha u(t) := \frac{d^n}{dt^n}\, {}_aD_t^{\alpha-n}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1}u(s)ds,$$

and

$$_tD_b^\alpha u(t) := (-1)^n\frac{d^n}{dt^n}\, {}_tD_b^{\alpha-n}u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b (t-s)^{n-\alpha-1}u(s)ds,$$

for every $t \in [a,b]$, provided the right-hand sides are pointwise defined on $[a,b]$, where $n-1 \le \alpha < n$ and $n \in \mathbb{N}$.
Here, $\Gamma(\alpha)$ is the standard gamma function given by

$$\Gamma(\alpha) := \int_0^{+\infty} z^{\alpha-1}e^{-z}dz.$$

Set $AC^n([a,b],\mathbb{R})$ the space of functions $u : [a,b] \to \mathbb{R}$ such that $u \in C^{n-1}([a,b],\mathbb{R})$ and $u^{(n-1)} \in AC([a,b],\mathbb{R})$. Here, as usual, $C^{n-1}([a,b],\mathbb{R})$ denotes the set of mappings having $(n-1)$ times continuously differentiable on $[a,b]$. In particular, we denote $AC([a,b],\mathbb{R}) := AC^1([a,b],\mathbb{R})$.

Proposition 2.2 ([12, 13]). *We have the following property of fractional integration*

$$\int_a^b [{}_aD_t^{-\alpha}u(t)]v(t)dt = \int_a^b [{}_tD_b^{-\alpha}v(t)]u(t)dt, \quad \alpha > 0,$$

provided that $u \in L^p([a,b],\mathbb{R})$, $v \in L^q([a,b],\mathbb{R})$ and $p \ge 1$, $q \ge 1$, $1/p + 1/q \le 1+\alpha$ or $p \ne 1$, $q \ne 1$, $1/p+1/q = 1+\alpha$.

Proposition 2.3 ([11]). *If $u(a) = u(b) = 0$, $u \in L^\infty([a,b],\mathbb{R}^N)$, $v \in L^1([a,b],\mathbb{R}^N)$, or $v(a) = v(b) = 0$, $v \in L^\infty([a,b],\mathbb{R}^N)$, $u \in L^1([a,b],\mathbb{R}^N)$, then*

$$\int_a^b [{}_aD_t^{\alpha}u(t)]v(t)dt = \int_a^b [{}_tD_b^{\alpha}v(t)]u(t)dt, \quad 0 < \alpha \le 1.$$

To establish a variational structure for the main problem, it is necessary to construct appropriate function spaces. Following [10], denote by $C_0^\infty([0,T],\mathbb{R})$ the set of all functions $g \in C^\infty([0,T],\mathbb{R})$ with $g(0) = g(T) = 0$.

Definition 2.4. *Let $0 < \alpha_i \le 1$ for $1 \le i \le n$. The fractional derivative space $E_0^{\alpha_i}$ is defined by the closure with respect to the weighted norm*

$$\|u_i\|_{\alpha_i} := \left(\int_0^T a_i(t)|{}_0D_t^{\alpha_i}u_i(t)|^2dt + \int_0^T |u_i(t)|^2dt\right)^{1/2}, \quad \forall u_i \in E_0^{\alpha_i}. \tag{3}$$

Clearly, the fractional derivative space $E_0^{\alpha_i}$ is the space of functions $u_i \in L^2[0,T]$ having an α_i-order fractional derivative ${}_0D_t^{\alpha}u_i \in L^2[0,T]$ and $u_i(0) = u_i(T) = 0$ for $1 \le i \le n$. Based on [10, Proposition 3.1], we know for $0 < \alpha_i \le 1$, the space $E_0^{\alpha_i}$ is a reflexive and separable Banach space.

For every $u_i \in E_0^{\alpha^i}$, set

$$\|u_i\|_{L^s} := \left(\int_0^T |u_i(t)|^sdt\right)^{1/s}, \qquad s \ge 1,$$

and

$$\|u_i\|_\infty := \max_{t\in[0,T]} |u_i(t)|.$$

Lemma 2.5 ([25]). *Let $\alpha_i \in (1/2, 1]$ for $1 \le i \le n$. For all $u_i \in E_0^{\alpha_i}$, we have*

$$\|u_i\|_{L^2} \le \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)\sqrt{a_{i0}}}\Big(\int_0^T a_i(t)|_0D_t^{\alpha_i}u_i(t)|^2dt\Big)^{1/2}, \tag{4}$$

$$\|u_i\|_\infty \le \frac{T^{\alpha_i-1/2}}{\Gamma(\alpha_i)\sqrt{a_{i0}(2\alpha_i-1)}}\Big(\int_0^T a_i(t)|_0D_t^{\alpha_i}u_i(t)|^2dt\Big)^{1/2}. \tag{5}$$

Hence, we can consider $E_0^{\alpha_i}$ with respect to the norm

$$\|u_i\|_{\alpha_i} := \left(\int_0^T a_i(t)|_0D_t^{\alpha_i}u_i(t)|^2dt\right)^{1/2}, \quad \forall\ u_i \in E_0^{\alpha_i}, \tag{6}$$

for $1 \le i \le n$, which is equivalent to (3).
Similarly to [10, Proposition 3.3], we have the following property of the fractional derivative space $E_0^{\alpha_i}$ for $1 \le \alpha_i \le n$.

Lemma 2.6 ([10]). *Assume that $\frac{1}{2} < \alpha_i \le 1$ for $1 \le i \le n$, and the sequence $\{u_n\}$ converges weakly to u in $E_0^{\alpha_i}$, i.e., $u_n \rightharpoonup u$. Then $\{u_n\}$ converges strongly to u in $C([0,T]),\mathbb{R})$, i.e., $\|u_n - u\|_\infty \to 0$, as $n \to \infty$.*

Throughout this paper, we let X be the Cartesian product of the n spaces $E_0^{\alpha_i}$ for $1 \le i \le n$, i.e., $X = E_0^{\alpha_1} \times E_0^{\alpha_2} \times \cdots \times E_0^{\alpha_n}$ equipped with the norm

$$\|u\| := \sum_{i=1}^n \|u_i\|_{\alpha_i}, \quad u = (u_1, u_2, \dots, u_n),$$

where $\|u_i\|_{\alpha_i}$ is defined in (6). Obviously, X is compactly embedded in $(C([0,T]),\mathbb{R}))^n$.

We mean by a *(weak) solution* of system (1), any $u = (u_1, u_2, \dots, u_n) \in X$ such that

$$\int_0^T \sum_{i=1}^n a_i(t)_0D_t^{\alpha_i}u_i(t)_0D_t^{\alpha_i}v_i(t)dt - \lambda\int_0^T \sum_{i=1}^n F_{u_i}(t, u_1(t), \dots, u_n(t))v_i(t)dt$$
$$-\mu\int_0^T \sum_{i=1}^n G_{u_i}(t, u_1(t), \dots, u_n(t))v_i(t)dt - \int_0^T \sum_{i=1}^n h_i(u_i(t))v_i(t)dt = 0$$

for all $v = (v_1, v_2, \dots, v_n) \in X$.

Below we recall Theorem 2.5 of [20] which is essential tool in the our paper.

Theorem 2.7. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly continuous. For every $r > \inf_X \Phi$, put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\left(\sup_{v\in\Phi^{-1}(-\infty,r)}\Psi(v)\right) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r\to+\infty}\varphi(r), \qquad \text{and} \qquad \delta := \liminf_{r\to(\inf_X\Phi)^+}\varphi(r).$$

Then, the following properties hold:

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional*

$$I_\lambda := \Phi - \lambda\Psi$$

to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) *If $\gamma < +\infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*
   (b$_1$) *I_λ possesses a global minimum, or*
   (b$_2$) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$, then for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either*
   (c$_1$) *there is a global minimum of Φ which is a local minimum of I_λ, or*
   (c$_2$) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ, with $\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi$.*

3. Main Results

In the present section we discuss the existence of infinitely many solutions for system (1). For $u = (u_1, \ldots, u_n) \in X$, we define

$$\Upsilon(u) := \sum_{i=1}^{n} \Upsilon_i(u_i),$$

where

$$\Upsilon_i(x) := \int_0^T H_i(x(s))ds \qquad \text{and} \qquad H_i(x) := \int_0^x h_i(z)dz, \quad 1 \le i \le n,$$

for every $s \in [0, T]$ and $x \in \mathbb{R}$.
Let

$$k := \max_{1\le i\le n}\left\{\frac{T^{\alpha_i-\frac{1}{2}}}{\Gamma(\alpha_i)\sqrt{a_{i0}(2\alpha_i-1)}}\right\},$$

$$k' := \min_{1\le i\le n}\left\{1-\frac{L_iT^{2\alpha_i}}{(\Gamma(\alpha_i+1))^2a_{i0}}\right\},$$

$$\rho := \max_{1\le i\le n}\left\{1+\frac{L_iT^{2\alpha_i}}{(\Gamma(\alpha_i+1))^2a_{i0}}\right\}.$$

For a given constant $\theta \in (0, \frac{1}{2})$ and for all $1 \le i \le n$, set

$$\begin{aligned}
P_i(\alpha_i, \theta) \quad = \quad & \frac{1}{2\theta^2T^2}\left\{\int_0^T a_i(t)t^{2(1-\alpha_i)}dt + \int_{\theta T}^T a_i(t)(t-\theta T)^{2(1-\alpha_i)}dt\right.\\
& + \int_{(1-\theta)T}^T a_i(t)(t-(1-\theta)T)^{2(1-\alpha_i)}dt - 2\int_{(1-\theta)T}^T a_i(t)(t^2-(1-\theta)Tt)^{1-\alpha_i}dt\\
& \left. -2\int_{\theta T}^T a_i(t)(t^2-\theta Tt)^{1-\alpha_i}dt + 2\int_{(1-\theta)T}^T a_i(t)(t^2-\theta Tt+\theta(1-\theta)T^2)^{1-\alpha_i}dt\right\},
\end{aligned}$$

$$\Delta := \min_{1\le i\le n}\{P_i(\alpha_i,\theta)\},$$

$$\Delta' := \max_{1\le i\le n}\{P_i(\alpha_i,\theta)\}.$$

For all $\gamma > 0$ we set

$$Q(\gamma) := \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n |x_i| \le \gamma\right\}.$$

Theorem 3.1. *Let $\frac{1}{2} < \alpha_i \le 1$ for $1 \le i \le n$. Assume that there exists $\theta \in (0, \frac{1}{2})$ such that*

(A_1) $F(t, x_1, \dots, x_n) \geq 0$ *for each* $(t, x_1, \dots, x_n) \in ([0, \theta T] \cup [(1-\theta)T, T]) \times \mathbb{R}^n$*;*

(A_2) $\liminf_{\xi\to+\infty} \frac{\int_0^T \sup_{(x_1,\dots,x_n)\in Q(\xi)} F(t,x_1,\dots,x_n)dt}{\xi^2} < \frac{k'}{2k^2n^2\rho\Delta'} \limsup_{\xi\to+\infty} \frac{\int_{\theta T}^{(1-\theta)T} F(t,\Gamma(2-\alpha_1)\xi,\dots,\Gamma(2-\alpha_n)\xi)dt}{\xi^2}$.

Then, for each $\lambda \in \Lambda :=]\lambda_1, \lambda_2[$ *where*

$$\lambda_1 := \frac{\rho\Delta'}{\limsup_{\xi\to+\infty} \frac{\int_{\theta T}^{(1-\theta)T} F(t,\Gamma(2-\alpha_1)\xi,\dots,\Gamma(2-\alpha_n)\xi)dt}{\xi^2}},$$

$$\lambda_2 := \frac{\frac{k'}{2k^2n^2}}{\liminf_{\xi\to+\infty} \frac{\int_0^T \sup_{(x_1,\dots,x_n)\in Q(\xi)} F(t,x_1,\dots,x_n)dt}{\xi^2}},$$

for every non-negative function $G : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ *satisfying the condition*

$$G_\infty := \limsup_{\xi\to+\infty} \frac{\int_0^T \sup_{(x_1,\dots,x_n)\in Q(\xi)} G(t, x_1, \dots, x_n)dt}{\xi^2} < +\infty, \tag{7}$$

and for every $\mu \in [0, \mu_{G,\lambda}[$ *where*

$$\mu_{G,\lambda} := \frac{k'}{2k^2n^2G_\infty}\Big(1 - \lambda\frac{2k^2n^2}{k'} \liminf_{\xi\to+\infty} \frac{\int_0^T \sup_{(x_1,\dots,x_n)\in Q(\xi)} F(t, x_1, \dots, x_n)dt}{\xi^2}\Big),$$

system (1) *has an unbounded sequence of weak solutions in* X*.*

Proof. Our aim is to apply Theorem 2.7(b) to system (1). To this end, fix $\bar{\lambda} \in \Lambda$ and let G be a function satisfying our assumptions. Since $\bar{\lambda} < \lambda_2$, we have

$$\mu_{G,\bar{\lambda}} = \frac{k'}{2k^2n^2G_\infty}\Big(1 - \bar{\lambda}\frac{2k^2n^2}{k'} \liminf_{\xi\to+\infty} \frac{\int_0^T \sup_{(x_1,\dots,x_n)\in Q(\xi)} F(t, x_1, \dots, x_n)dt}{\xi^2}\Big) > 0.$$

Now fix $\bar{\mu} \in]0, \mu_{G,\bar{\lambda}}[$. Set

$$J(t, \xi_1, \dots, \xi_n) := F(t, \xi_1, \dots, \xi_n) + \frac{\bar{\mu}}{\bar{\lambda}}G(t, \xi_1, \dots, \xi_n)$$

for every $t \in [0, T]$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. We define the mappings $\Phi, \Psi : X \to \mathbb{R}$ by

$$\Phi(u) := \sum_{i=1}^n \frac{\|u_i\|_{\alpha_i}^2}{2} - \Upsilon(u),$$

$$\Psi(u) := \int_0^T J(t, u_1(t), \dots, u_n(t))dt,$$

for each $u = (u_1, \dots, u_n) \in X$ and put

$$I_{\bar{\lambda},\bar{\mu}}(u) := \Phi(u) - \bar{\lambda}\Psi(u), \qquad u \in X.$$

Let us prove that Φ and Ψ satisfy the required conditions. Since X is compactly embedded in $(C([0, T], \mathbb{R}))^n$, it is well known that Ψ is well-defined Gâteaux differentiable functional whose Gâteaux derivative at $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)(v) = \int_0^T \sum_{i=1}^n J_{u_i}(t, u_1(t), \dots, u_n(t))v_i(t)dt$$

for every $v = (v_1, \dots, v_n) \in X$. Moreover, Ψ is sequentially weakly continuous.

The functional Φ is a Gâteaux differentiable functional with the differential at $u \in X$

$$\Phi'(u)(v) = \int_0^T \sum_{i=1}^n a_i(t) {}_0D_t^{\alpha_i} u_i(t) {}_0D_t^{\alpha_i} v_i(t)dt - \int_0^T \sum_{i=1}^n h_i(u_i(t))v_i(t)dt,$$

for every $v \in X$. Further, Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive functional on X.

Clearly, the weak solutions of system (1) are exactly the critical points of the functional $I_{\bar{\lambda},\bar{\mu}}$. Moreover, since (2) holds for every $x_1, \dots, x_n \in \mathbb{R}$ and $h_1(0) = \cdots = h_n(0) = 0$, one has $|h_i(x)| \le L_i|x|$, $1 \le i \le n$ for all $x \in \mathbb{R}$. It follows from (4) and (5) that

$$\begin{aligned}
\Phi(u) &\ge \frac{\sum_{i=1}^n \|u_i\|_{\alpha_i}^2}{2} - \Big|\int_0^T \sum_{i=1}^n H_i(u_i(t))dt\Big| \\
&\ge \frac{\sum_{i=1}^n \|u_i\|_{\alpha_i}^2}{2} - \sum_{i=1}^n \frac{L_i}{2}\int_0^T |u_i(t)|^2 dt \\
&\ge \Big(\frac{1}{2} - \frac{L_i T^{2\alpha_i}}{2(\Gamma(\alpha_i+1))^2 a_{i0}}\Big)\|u_i\|_{\alpha_i}^2 \\
&\ge \frac{k'}{2}\sum_{i=1}^n \|u_i\|_{\alpha_i}^2,
\end{aligned} \tag{8}$$

for all $u \in X$, and so Φ is coercive.

Now, let us verify that $\bar{\lambda} < \frac{1}{\gamma}$. Let $\{\xi_k\}$ be a sequence of positive numbers such that $\xi_k \to \infty$ as $k \to \infty$ and

$$\lim_{k\to+\infty} \frac{\int_0^T \sup_{(x_1,\dots,x_n)\in Q(\xi_k)} F(t, x_1, \dots, x_n)dt}{\xi_k^2} = \liminf_{\xi\to+\infty} \frac{\int_0^T \sup_{(x_1,\dots,x_n)\in Q(\xi)} F(t, x_1, \dots, x_n)dt}{\xi^2}.$$

Put $r_k := \frac{k'\xi_k^2}{2k^2n^2}$ for all $k \in \mathbb{N}$. Since $\max_{t\in[0,T]} |u_i(t)| \le k\|u_i\|_{\alpha_i}$ for all $u_i \in E_0^{\alpha_i}([0, T])$ and $1 \le i \le n$, we have

$$\sup_{t\in[0,T]} \sum_{i=1}^n |u_i(t)|^2 \le k^2 \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 \tag{9}$$

for each $u = (u_1, \dots, u_n) \in X$. So, from (8) and (9) we have

$$\begin{aligned}
\Phi^{-1}(]-\infty, r_k[) \quad &:= \quad \left\{u \in X : \frac{k'}{2}\Big(\sum_{i=1}^n \|u_i\|_{\alpha_i}^2\Big) < r_k\right\} \\
&\subseteq \quad \left\{u \in X : \sum_{i=1}^n |u_i(x)|^2 \le \frac{2k^2}{k'} r_k \quad \text{for each t} \in [0, \mathrm{T}]\right\} \\
&\subseteq \quad \left\{u \in X : \sum_{i=1}^n |u_i(t)| \le \xi_k \quad \text{for each t} \in [0, \mathrm{T}]\right\}.
\end{aligned}$$

Hence, taking into account that $\Phi(0,\dots,0)=\Psi(0,\dots,0)=0$, for every k large enough, one has

$$\begin{aligned}
\varphi(r_k) &= \inf_{u\in\Phi^{-1}((-\infty,r_k[)} \frac{\sup_{u\in\Phi^{-1}((-\infty,r_k[)}\Psi(v)-\Psi(u)}{r_k-\Phi(u)} \\
&\le \frac{\sup_{u\in\Phi^{-1}((-\infty,r_k[)}\Psi(v)}{r_k} \le \frac{\int_0^T \sup_{(x_1,\dots,x_n)\in Q(\xi_k)} J(t,x_1,\dots,x_n)dt}{\frac{k'\xi_k^2}{2k^2n^2}} \\
&= \frac{\int_0^T \sup_{(x_1,\dots,x_n)\in Q(\xi_k)}[F(t,x_1,\dots,x_n)+\frac{\bar{\mu}}{\bar{\lambda}}G(t,x_1,\dots,x_n)]dt}{\frac{k'\xi_k^2}{2k^2n^2}} \\
&\le \frac{\int_0^T \sup_{(x_1,\dots,x_n)\in Q(\xi_k)} F(t,x_1,\dots,x_n)dt}{\frac{k'}{2k^2n^2}\xi_k^2} + \frac{\bar{\mu}}{\bar{\lambda}}\frac{\int_0^T \sup_{(x_1,\dots,x_n)\in \Phi(\xi_k)} G(t,x_1,\dots,x_n)dt}{\frac{k'\xi_k^2}{2k^2n^2}}.
\end{aligned}$$

Moreover, from assumption (A_2) and (7) one has

$$\liminf_{k\to\infty}\frac{\int_0^T \sup_{(x_1,\dots,x_n)\in Q(\xi_k)} F(t,x_1,\dots,x_n)dt}{\frac{k'\xi_k^2}{2k^2n^2}} + \lim_{k\to\infty}\frac{\bar{\mu}}{\bar{\lambda}}\frac{\int_0^T \sup_{(x_1,\dots,x_n)\in Q(\xi_k)} G(t,x_1,\dots,x_n)dt}{\frac{k'\xi_k^2}{2k^2n^2}} < +\infty,$$

which implies

$$\liminf_{k\to\infty}\frac{\int_0^T \sup_{(x_1,\dots,x_n)\in Q(\xi_k)} J(t,x_1,\dots,x_n)dt}{\xi_k^2} < +\infty.$$

Therefore,

$$\gamma \le \liminf_{k\to\infty}\varphi(r_k) \le \frac{2k^2n^2}{k'}\liminf_{k\to\infty}\frac{\int_0^T \sup_{(x_1,\dots,x_n)\in Q(\xi_k)} J(t,x_1,\dots,x_n)dt}{\xi_k^2} < +\infty. \tag{10}$$

The assumption $\bar{\mu}\in]0,\mu_{G,\bar{\lambda}}[$ immediately yields $\bar{\lambda}<\frac{1}{\gamma}$.

The next step is to show that for fixed $\bar{\lambda}$ the functional $I_{\bar{\lambda},\bar{\mu}}$ has no global minimum. Let us verify that $I_{\bar{\lambda},\bar{\mu}}$ is unbounded from below. Since

$$\begin{aligned}
\frac{1}{\bar{\lambda}} &< \frac{1}{\rho\Delta'}\limsup_{\xi\to+\infty}\frac{\int_{\theta T}^{(1-\theta)T} F(t,\Gamma(2-\alpha_1)\xi,\dots,\Gamma(2-\alpha_n)\xi)dt}{\xi^2} \\
&\le \limsup_{\xi\to+\infty}\frac{\int_{\theta T}^{(1-\theta)T} J(t,\Gamma(2-\alpha_1)\xi,\dots,\Gamma(2-\alpha_n)\xi)dt}{\xi^2},
\end{aligned}$$

we can consider a real sequence $\{d_k\}$ and a positive constant τ such that $d_k\to\infty$ as $k\to\infty$ and

$$\frac{1}{\bar{\lambda}} < \tau < \frac{1}{\rho\Delta'}\frac{\int_{\theta T}^{(1-\theta)T} F(t,\Gamma(2-\alpha_1)d_k,\dots,\Gamma(2-\alpha_n)d_k)dt}{d_k^2} \tag{11}$$

for each $k\in\mathbb{N}$ large enough. For all $k\in\mathbb{N}$, and $\theta\in(0,\frac{1}{2})$ define $\{w_k=(w_{1k},\dots,w_{nk})\}$ by setting

$$\omega_{ik}(t) := \begin{cases} \frac{\Gamma(2-\alpha_i)d_k}{hT}t, & t\in[0,\theta T[, \\ \Gamma(2-\alpha_i)d_k, & t\in[\theta T,(1-\theta)T], \\ \frac{\Gamma(2-\alpha_i)d_k}{\theta T}(T-t), & t\in](1-\theta)T,T], \end{cases} \tag{12}$$

for $1 \le i \le n$. Clearly $\omega_{ik}(0) = \omega_{ik}(T) = 0$ and $\omega_{ik} \in L^2[0,T]$ for $1 \le i \le n$. A direct calculation shows that

$$
{}_0D_t^{\alpha_i}\omega_{ik}(t) = \begin{cases} \frac{d_k}{\theta T} t^{1-\alpha_i}, & t \in [0, \theta T[, \\ \frac{d_k}{\theta T}(t^{1-\alpha_i} - (t-\theta T)^{1-\alpha_i}), & t \in [\theta T, (1-\theta)T], \\ \frac{d_k}{\theta T}(t^{1-\alpha_i} - (t-\theta T)^{1-\alpha_i} - (t-(1-\theta)T)^{1-\alpha_i}), & t \in](1-\theta)T, T], \end{cases}
$$

for $1 \le i \le n$. Furthermore,

$$
\begin{aligned}
\int_0^T a_i(t)|{}_0D_t^{\alpha_i}\omega_{ik}(t)|^2 dt &= \int_0^{\theta T} + \int_{\theta T}^{(1-\theta)T} + \int_{(1-\theta)T}^{T} a_i(t)|{}_0D_t^{\alpha_i}\omega_{ik}(t)|^2 dt \\
&= \frac{d_k^2}{\theta^2 T^2}\Bigg\{\int_0^T a_i(t) t^{2(1-\alpha_i)} dt + \int_{\theta T}^T a_i(t)(t-\theta T)^{2(1-\alpha_i)} dt \\
&\quad + \int_{(1-\theta)T}^T a_i(t)(t-(1-\theta)T)^{2(1-\alpha_i)} dt - 2\int_{\theta T}^T a_i(t)(t^2 - \theta T t)^{1-\alpha_i} dt \\
&\quad -2\int_{(1-\theta)T}^T a_i(t)(t^2 - (1-\theta)Tt)^{1-\alpha} dt \\
&\quad +2\int_{(1-\theta)T}^T a_i(t)(t^2 - \theta T t + \theta(1-\theta)T^2)^{1-\alpha_i} dt\Bigg\} \\
&= 2P_i(\alpha_i, \theta) d_k^2,
\end{aligned}
$$

for $1 \le i \le n$. Thus, $\omega_k \in X$, and in particular,

$$
\|\omega_{ik}\|_{\alpha_i}^2 = \int_0^T a_i(t)|{}_0D_t^{\alpha_i}\omega_{ik}(t)|^2 dt = 2P(\alpha_i, \theta) d_k^2,
$$

for $1 \le i \le n$. So,

$$
\Phi(\omega_k) = P_i(\alpha_i, \theta) d_k^2.
$$

On the other hand, similar to (8), we have

$$
\begin{aligned}
\Phi(\omega_k) &= \sum_{i=1}^n \frac{\|\omega_{ik}\|_{\alpha_i}^2}{2} - \Upsilon(\omega) \\
&\le \frac{\rho}{2}\Big(\sum_{i=1}^n \|\omega_{ik}\|_{\alpha_i}^2\Big) \\
&= \rho\Big(\sum_{i=1}^n P(\alpha_i, h)\Big) d_k^2 \\
&\le \rho\Delta' d_k^2.
\end{aligned} \tag{13}
$$

Bearing in mind assumption (A_1) and since G is nonnegative, from the definition of Ψ we infer

$$
\Psi(w_k) \ge \int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha_1)d_k, \dots, \Gamma(2-\alpha_n)d_k) dt. \tag{14}
$$

So, according to (11), (13) and (14),

$$
\begin{aligned}
I_{\bar\lambda,\bar\mu}(w_k) &\le \rho\Delta' d_k^2 - \bar\lambda \int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha_1)d_k, \dots, \Gamma(2-\alpha_n)d_k) dt \\
&< \rho\Delta'(1 - \bar\lambda\tau) d_k^2
\end{aligned} \tag{15}
$$

for every $k \in \mathbb{N}$ large enough. Hence, $I_{\bar{\lambda},\bar{\mu}}$ is unbounded from below, and so has no global minimum. Therefore, applying Theorem 2.7(b) we deduce that there is a sequence $\{u_k = (u_{1k}, \dots, u_{nk})\} \subset X$ of critical points of $I_{\bar{\lambda},\bar{\mu}}$ such that $\lim_{k\to\infty} \|(u_{1k}, \dots, u_{nk})\| = +\infty$. Hence, the conclusion is achieved. □

Remark 3.2. Under the conditions

$$\liminf_{\xi\to+\infty} \frac{\int_0^T \sup_{(x_1,\dots,x_n)\in Q(\xi)} F(t, x_1, \dots, x_n)dt}{\xi^2} = 0,$$

and

$$\limsup_{\xi\to+\infty} \frac{\int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha_1)\xi, \dots, \Gamma(2-\alpha_n)\xi)dt}{\xi^2} = +\infty,$$

from Theorem 3.1 we see for every $\lambda > 0$ and $\mu \in [0, \frac{k'}{2k^2n^2G_\infty}[$ system (1) admits infinitely many weak solutions in X. Moreover, if $G_\infty = 0$, the result holds for every $\lambda > 0$ and $\mu \geq 0$.

Here we point out the following consequence of Theorem 3.1 with $\mu = 0$.

Corollary 3.3. *Assume that there exists $\theta \in (0, \frac{1}{2})$ such that assumption* (A$_1$) *holds. Suppose that*

(B$_1$) $\liminf_{\xi\to+\infty} \frac{\int_0^T \sup_{(x_1,\dots,x_n)\in Q(\xi)} F(t,x_1,\dots,x_n)dt}{\xi^2} < \frac{k'}{2k^2n^2}$;

(B$_2$) $\limsup_{\xi\to+\infty} \frac{\int_{\theta T}^{(1-\theta)T} F(t,\Gamma(2-\alpha_1)\xi,\dots,\Gamma(2-\alpha_n)\xi)dt}{\xi^2} > \rho\Delta'$.

Then, the system

$$\begin{cases} {}_tD_T^{\alpha_i}\Big(a_i(t)\,{}_0D_t^{\alpha_i}u_i(t)\Big) = F_{u_i}(t, u_1(t), \dots, u_n(t)) + h_i(u_i) \quad a.e.\ t \in [0, T] \\ u_i(0) = u_i(T) = 0, \end{cases}$$

for $1 \leq i \leq n$, has an unbounded sequence of weak solutions in X.

In the same way as in the proof of Theorem 3.1 but using conclusion (c) of Theorem 2.7 instead of (b), we will obtain the following result.

Theorem 3.4. *Assume that all the hypotheses of Theorem* 3.1 *hold except for assumption* (A$_2$). *Suppose that*

(A$_3$) $\liminf_{\xi\to0^+} \frac{\int_0^T \sup_{(x_1,\dots,x_n)\in Q(\xi)} F(t,x_1,\dots,x_n)dt}{\xi^2} < \frac{k'}{2k^2n^2\rho\Delta'} \limsup_{\xi\to0^+} \frac{\int_{\theta T}^{(1-\theta)T} F(t,\Gamma(2-\alpha_1)\xi,\dots,\Gamma(2-\alpha_n)\xi)dt}{\xi^2}$.

Then, for each $\lambda \in]\lambda_3, \lambda_4[$ where

$$\lambda_3 := \frac{\rho\Delta'}{\limsup_{\xi\to0^+} \frac{\int_{\theta T}^{(1-\theta)T} F(t,\Gamma(2-\alpha_1)\xi,\dots,\Gamma(2-\alpha_n)\xi)dt}{\xi^2}},$$

$$\lambda_4 := \frac{\frac{k'}{2k^2n^2}}{\liminf_{\xi\to0^+} \frac{\int_0^T \sup_{(x_1,\dots,x_n)\in Q(\xi)} F(t,x_1,\dots,x_n)dt}{\xi^2}},$$

for every non-negative function $G : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ satisfying the condition

$$G_0 := \limsup_{\xi\to0^+} \frac{\int_0^T \sup_{(x_1,\dots,x_n)\in Q(\xi)} G(t, x_1, \dots, x_n)dt}{\xi^2} < +\infty, \tag{16}$$

and for every $\mu \in [0, \mu'_{G,\lambda}[$ *where*

$$\mu'_{G,\lambda} := \frac{k'}{2k^2n^2G_0}\Big(1 - \lambda\frac{2k^2n^2}{k'}\liminf_{\xi\to 0^+}\frac{\int_0^T \sup_{(x_1,\ldots,x_n)\in Q(\xi)} F(t,x_1,\ldots,x_n)dt}{\xi^2}\Big),$$

system (1) *has a sequence of weak solutions, which strongly converges to zero in* X.

Proof. Fix $\bar\lambda \in]\lambda_3, \lambda_4[$ and let G be a function satisfying (16). Since $\bar\lambda < \lambda_4$, one has

$$\mu'_{G,\bar\lambda} = \frac{k'}{2k^2n^2G_0}\Big(1 - \bar\lambda\frac{2k^2n^2}{k'}\liminf_{\xi\to 0^+}\frac{\int_0^T \sup_{(x_1,\ldots,x_n)\in Q(\xi)} F(t,x_1,\ldots,x_n)dt}{\xi^2}\Big) > 0.$$

Fix $\bar\mu \in]0, \mu'_{G,\bar\lambda}[$ and put

$$J(t,\xi_1,\ldots,\xi_n) := F(t,\xi_1,\ldots,\xi_n) + \frac{\bar\mu}{\bar\lambda}G(t,\xi_1,\ldots,\xi_n)$$

for every $t \in [0,T]$ and $\xi = (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n$. We take Φ, Ψ and $I_{\bar\lambda,\bar\mu}$ as in the proof of Theorem 3.1. We verify that $\bar\lambda < \frac{1}{\gamma}$. For this, let $\{\xi_k\}$ be a sequence of positive number such that $\xi_k \to 0^+$ as $k \to \infty$ and

$$\lim_{k\to+\infty}\frac{\int_0^T \sup_{(x_1,\ldots,x_n)\in Q(\xi_k)} F(t,x_1,\ldots,x_n)dt}{\xi_k^2} = \liminf_{\xi\to 0^+}\frac{\int_0^T \sup_{(x_1,\ldots,x_n)\in Q(\xi)} F(t,x_1,\ldots,x_n)dt}{\xi 2}.$$

By the fact that $\inf_X \Phi = 0$ and the definition of δ, we have $\delta = \liminf_{r\to 0^+}\varphi(r)$. Then, as in showing (10) in the proof of Theorem 3.1, we can prove that $\delta < +\infty$, and hence $\bar\lambda < \frac{1}{\delta}$.

Let $\bar\lambda$ be fixed. We claim that the functional $I_{\bar\lambda,\bar\mu}$ does not have a local minimum at zero. For this, let $\{d_k\}$ be a sequence of positive numbers such that $d_k \to 0^+$ as $k \to \infty$ and pick $\tau > 0$ such that

$$\frac{1}{\bar\lambda} < \tau < \frac{1}{\rho\Delta'}\frac{\int_{\theta T}^{(1-\theta)T} F(t,\Gamma(2-\alpha_1)d_k,\ldots,\Gamma(2-\alpha_n)d_k)dt}{d_k^2}$$

for each $k \in \mathbb{N}$ large enough. Let $\{w_k = (w_{1k},\ldots,w_{nk})\}$ be a sequence in X with w_{ik} defined in (12). Note that $\bar\lambda\tau > 1$. Then, as in showing (15), we can obtain that

$$\begin{aligned} I_{\bar\lambda,\bar\mu}(\omega_k) &\le \rho\Delta' d_k^2 - \bar\lambda\int_{\theta T}^{(1-\theta)T} F(t,\Gamma(2-\alpha_1)d_k,\ldots,\Gamma(2-\alpha_n)d_k)dt \\ &< (1-\bar\lambda\tau)d_k^2\rho\Delta' < 0 \end{aligned}$$

for every $k \in \mathbb{N}$ large enough. Since $I_{\bar\lambda,\bar\mu}(0) = 0$, this implies that the functional $I_{\bar\lambda,\bar\mu}$ does not have a local minimum at zero.

Hence, part (c) of Theorem 2.7 ensures that there exists a sequence $\{u_k = (u_{1k},\ldots,u_{nk})\}$ in X of critical points of $I_{\bar\lambda,\bar\mu}$ which weakly converges to zero. In view of the fact that the embedding $X \hookrightarrow (C([0,T],\mathbb{R}))^n$ is compact, we know that the critical points converge strongly to zero, and the proof is completed. □

Remark 3.5. Under the conditions

$$\liminf_{\xi\to 0^+}\frac{\int_0^T \sup_{(x_1,\ldots,x_n)\in Q(\xi)} F(t,x_1,\ldots,x_n)dt}{\xi^2} = 0,$$

and

$$\limsup_{\xi\to 0^+}\frac{\int_{\theta T}^{(1-\theta)T} F(t,\Gamma(2-\alpha_1)\xi,\ldots,\Gamma(2-\alpha_n)\xi)dt}{\xi^2} = +\infty,$$

Theorem 3.4 ensures that for every $\lambda > 0$ and $\mu \in [0, \frac{k'}{2k^2n^2G_0}[$ system (1) admits infinitely many weak solutions in X. Moreover, if $G_0 = 0$, the result holds for every $\lambda > 0$ and $\mu \ge 0$.

Now we present the following example to illustrate the above result.

Example 3.6. Consider the system

$$\begin{cases} {}_tD_1^{0.75}\Big((2+t^2)\,{}_0D_t^{0.75}x_1(t)\Big) = \lambda F_{x_1}(t,x_1(t),x_2(t)) + \mu G_{x_1}(t,x_1(t),x_2(t)) \\ \qquad\qquad\qquad\qquad\qquad +h_1(x_1(t)) \quad \text{a.e. } t\in[0,1], \\ {}_tD_1^{0.8}\Big((1+t)\,{}_0D_t^{0.8}x_2(t)\Big) = \lambda F_{x_2}(t,x_1(t),x_2(t)) + \mu G_{x_2}(t,x_1(t),x_2(t)) \\ \qquad\qquad\qquad\qquad\qquad +h_2(x_2(t)) \quad \text{a.e. } t\in[0,1], \\ x_1(0)=x_2(0)=x_1(1)=x_2(1)=0, \end{cases} \tag{17}$$

where $h_1(x_1) = \frac{1}{4}\sin x_1$ and $h_2(x_2) = \frac{1}{9}x_2$. Moreover, for all $(t,x_1,x_2)\in[0,1]\times\mathbb{R}^2$, let $F:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ be defined as

$$F(t,x_1,x_2) := \begin{cases} 0 & \text{for all } (t,x_1,x_2)\in[0,1]\times\{0\}^2, \\ g(t)x_1^2(1-\sin(\ln(|x_1|))) + k(t)x_2^2(1-\cos(\ln(|x_2|))) & \text{for all } (t,x_1,x_2)\in[0,1]\times(\mathbb{R}-\{0\})^2, \end{cases}$$

where $g,k:[0,1]\to\mathbb{R}$ are non-negative continuous functions. Let $\theta=\frac{1}{4}$. We observe that

$$\liminf_{\xi\to0^+}\frac{\int_0^1 \sup_{|x_1|+|x_2|\le\xi} F(t,x_1,x_2)dt}{\xi^2} = 0$$

and

$$\limsup_{\xi\to0^+}\frac{\int_{1/4}^{3/4} F(t,\Gamma(1.25)\xi,\Gamma(0.2)\xi)dt}{\xi^2} = +\infty.$$

Now, let $G:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ be a function defined by

$$G(t,x_1,x_2) = 1-\cos(x_1x_2).$$

By definition, $G\in C^1(\mathbb{R}^2)$ and

$$\limsup_{\xi\to0^+}\frac{\int_0^1 \sup_{|x_1|+|x_2|\le\xi} G(t,x_1,x_2)dt}{\xi^2} = 0 < \infty.$$

All hypotheses of Remark 3.5 are satisfied. Then for all $(\lambda,\mu)\in]0,+\infty[\times[0,+\infty[$ the system 17 admits a sequence of weak solutions which strongly converges to 0 in $E_0^{0.75}\times E_0^{0.8}$.

As an application of Theorem 3.1 we consider the case $n=1$.

Corollary 3.7. *Let $\frac{1}{2}<\alpha\le1$, $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be an L^1-Carathéodory function and $h:\mathbb{R}\to\mathbb{R}$ be a Lipschitz continuous function with the Lipschitz constant $L>0$. Put $F(t,x):=\int_0^x f(t,\xi)d\xi$ for each $(t,x)\in[0,T]\times\mathbb{R}$. Assume that there exist two constants $\theta\in(0,\frac{1}{2})$ and $\eta>0$ such that*

(C_1) *$F(t,x)\ge0$ for each $(t,x)\in[0,\theta T]\cup[(1-\theta)T,T]\times\mathbb{R}$;*

(C_2) $\liminf_{\xi\to+\infty}\frac{\int_0^T\sup_{|x|\le\xi}F(t,x)dt}{\xi^2} < \frac{k'}{2k^2n^2\rho\Delta'}\limsup_{\xi\to+\infty}\frac{\int_{\theta T}^{(1-\theta)T}F(t,\Gamma(2-\alpha)\xi)dt}{\xi^2}$.

Then, for each $\lambda\in]\lambda_5,\lambda_6[$ where

$$\lambda_5 := \frac{\rho\Delta'}{\limsup_{\xi\to+\infty}\frac{\int_{\theta T}^{(1-\theta)T}F(t,\Gamma(2-\alpha)\eta)dt}{\xi^2}},$$

$$\lambda_6 := \frac{\frac{k'}{2k^2n^2}}{\liminf_{\xi\to+\infty} \frac{\int_0^T \sup_{|x|\le\xi} F(t,x)dt}{\xi^2}},$$

for every L^1-Carathéodory function $g : [0,T]\times\mathbb{R}\to\mathbb{R}$ whose potential $G(t,x) := \int_0^x g(t,\xi)d\xi$ for $(t,x)\in[0,T]\times\mathbb{R}$ is a non-negative function satisfying the condition

$$G_\infty := \lim_{\xi\to+\infty} \frac{\int_0^T \sup_{|x|\le\xi} G(t,x)dt}{\xi^2} < +\infty,$$

and for every $\mu\in[0,\mu_{G,\lambda}[$ where

$$\mu_{G,\lambda} := \frac{k'}{2k^2n^2G_\infty}\Big(1-\lambda\frac{2k^2n^2}{k'}\liminf_{\xi\to+\infty}\frac{\int_0^T \sup_{|x|\le\xi} F(t,x)dt}{\xi^2}\Big),$$

the system

$$\begin{cases} {}_tD_T^\alpha\Big(a(t)_0D_t^\alpha u(t)\Big) & = \lambda f(t,u(t)) + \mu g(t,u(t)) \\ & \quad +h(u(t)) \quad a.e.\ t\in[0,T] \\ u(0)=u(T)=0, & \end{cases}$$

has an unbounded sequence of weak solutions in X.

Acknowledgement

This work was supported by the National Natural Science Foundation of China(Grant No. 11601048), Natural Science Foundation of Chongqing (Grant No. cstc2016jcyjA0181), the Science and Technology Research Program of Chongqing Municipal Education Commission(Grant No. KJ1703050), Natural Science Foundation of Chongqing Normal University (Grant No. 16XYY24, 15XLB011).

References

[1] R.P. Agarwal, M. Benchohra, S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, Acta Applicandae Mathematicae 109 (2010) 973–1033.
[2] B. Ahmad, J.J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions, Computers & Mathematics with Applications 58 (2009) 1838–1843.
[3] C. Bai, Impulsive periodic boundary value problems for fractional differential equation involving Riemann-Liouville sequential fractional derivative, Journal of Mathematical Analysis and Applications 384 (2011) 211–231.
[4] C. Bai, Solvability of multi-point boundary value problem of nonlinear impulsive fractional differential equation at resonance, Electronic Journal of Qualitative Theory of Differential Equations 89 (2011) 1–19.
[5] C. Bai, Infinitely many solutions for a perturbed nonlinear fractional boundary-value problem, Electronic Journal of Differential Equations 136 (2013) 1–12.
[6] Z. Bai, H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, Journal of Mathematical Analysis and Applications 311 (2005) 495–505.
[7] M. Benchohra, S. Hamani, S.K. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions, Nonlinear Analysis 71 (2009) 2391–2396.
[8] G. Bonanno, S.A. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, Applied Analysis 89 (2010) 1–10.
[9] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
[10] F. Jiao, Y. Zhou, Existence of solutions for a class of fractional boundary value problems via critical point theory, Computers & Mathematics with Applications 62 (2011) 1181–1199.
[11] F. Jiao, Y. Zhou, Existence results for fractional boundary value problem via critical point theory, International Journal of Bifurcation and Chaos 22 (2012) 1250086 (17 pages).
[12] A.A. Kilbas, O.I. Marichev, S.G. Samko, Fractional Integral and Derivatives (Theory and Applications), Gordon and Breach, Switzerland, 1993.
[13] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[14] N. Kosmatov, Integral equations and initial value problems for nonlinear differential equations of fractional order, Nonlinear Analysis 70 (2009) 2521–2529.
[15] V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, Nonlinear Analysis 69 (2008) 2677–2682.
[16] J. Mawhin, M. Willem, Critical Point Theorey and Hamiltonian Systems, Springer, New York, 1989.
[17] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.
[18] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[19] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, in: CBMS, vol. 65, Amer. Math. Soc. 1986.
[20] B. Ricceri, A general variational principle and some of its applications, Journal of Computational and Applied Mathematics 113 (2000) 401–410.
[21] M. Struwe, Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3, Springer-Verlag, Berlin-Heidelberg, 1990.
[22] Z. Wei, W. Dong, J. Che, Periodic boundary value problems for fractional differential equations involving a Riemann-Liouville fractional derivative, Nonlinear Analysis 73 (2010) 3232–3238.
[23] M. Willem, Minimax Theorems, Birkhäuser, 1996.
[24] S. Zhang, Positive solutions to singular boundary value problem for nonlinear fractional differential equation, Computers & Mathematics with Applications 59 (2010) 1300–1309.
[25] Y. Zhao, H. Chen, B. Qin, Multiple solutions for a coupled system of nonlinear fractional differential equations via variational methods, Applied Mathematics and Computation 257 (2015) 417–427.
[26] Y. Zhao, H. Chen, Q. Zhang, Infinitely many solutions for fractional differential system via variational method, Journal of Applied Mathematics and Computing 50 (2016) 589–609.
[27] Y. Cui, Uniqueness of solution for boundary value problems for fractional differential equations, Applied Mathematics Letters 51 (2016) 48–54.
[28] J. Jiang, L. Liu, Y. Wu, Positive solutions to singular fractional differential system with coupled boundary conditions, Communications in Nonlinear Science and Numerical Simulation 18 (2013) 3061–3074.
[29] J. Wu, X. Zhang, L. Liu, Y. Wu, Y. Cui, The convergence analysis and error estimation for unique solution of a p-Laplacian fractional differential equation with singular decreasing nonlinearity, Boundary Value Problems 82 (2018) 1–15.
[30] X. Hao, H. Wang, L. Liu, Y. Cui, Positive solutions for a system of nonlinear fractional nonlocal boundary value problems with parameters and p-Laplacian operator, Boundary Value Problems 182 (2017) 1–18.