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Biharmonic Pseudo-Riemannian Submersions from 3-Manifolds

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Abstract. We classify the pseudo-Riemannian biharmonic submersion from a 3-dimensional space form onto a surface.

1. Introduction

The theory of Riemannian submersions was initiated by O'Neill [14] and Gray [11]. One of the well known example of a Riemannian submersion is the projection of a Riemannian product manifold on one of its factors. Presently, there is an extensive literature on the Riemannian submersions with different conditions imposed on the total space and on the fibres. A systematic exposition could be found in A. Besse's book [4]. Pseudo-Riemannian submersions were introduced by O'Neill [15]. Magid classified pseudo-Riemannian submersions with totally geodesic fibres from an anti-de Sitter space onto a Riemannian manifold [13]. Then Băditou gave the classification of the pseudo-Riemannian submersions with (para) complex connected totally geodesic fibres from a (para) complex pseudo-hyperbolic space onto a pseudo Riemannian manifold [1, 3].

A map between Riemannian manifolds is harmonic if the divergence of its differential vanishes. The first major study of harmonic maps has been begun by J. Eells and J. H. Sampson [9]. In [9], Eells and Sampson defined biharmonic maps between Riemannian manifolds as an extension of harmonic maps and Jiang obtained their first and second variational formulas [12].

During the last decade important progress has been made in the study of both the geometry and the analytic properties of biharmonic maps. A fundamental problem in the study of biharmonic maps is to classify all proper biharmonic maps between certain model spaces. An example of this was proved independently by Chen-Ishikawa [7] and Jiang [12] that every biharmonic surface in a Euclidean 3-space E^3 is a minimal surface. Later, Caddeo et al. showed that the theorem remains true if the target Euclidean space is replaced by 3-dimensional hyperbolic space form [5]. Chen and Ishikawa also proved that biharmonic Riemannian surface in E^3_1 is a harmonic surface [6]. For Riemannian submersions, Wang and Ou stated that Riemannian submersion from a 3-dimensional space form into a surface is biharmonic if and only if it is harmonic [19].

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The above results give us the motivation for preparing this study. In this paper, we study the biharmonic pseudo-Riemannian submersions from 3-manifolds.

The main purpose of section §2 is to give a brief information about pseudo-Riemannian submersions, biharmonic maps and space forms. In this section, we also give some properties of fundamental tensors and fundamental equations which we will use them to obtain our results. In section §3, we investigate the biharmonicity of a pseudo-Riemannian submersion from a 3-manifold by using the integrability data of a special orthonormal frame adapted to a pseudo-Riemannian submersion. Finally, we give a complete classification of biharmonic pseudo-Riemannian submersions from a 3-dimensional pseudo-Riemannian space form.

2. PRELIMINARIES

2.1. *Pseudo-Riemannian submersions with totally geodesic fibre*

In this subsection we recall several notions and results which will be needed throughout the paper.

Let (M, g) be an m-dimensional connected pseudo-Riemannian manifold of index s $(0 \leq s \leq m)$, let (B, g') be an n-dimensional connected pseudo-Riemannian manifold of index $r \leq s, (0 \leq r \leq n)$. In case of Riemannian submersion, the fibers are always Riemannian manifolds.

A pseudo-Riemannian submersion is a smooth map $\pi : M \to B$ which is onto and satisfies the following three axioms:

$S1$. $\pi_* |_p$ is onto for all $p \in M$,

$S2$. the restriction of the metric to the fibres $\pi^{-1}(b)$, $b \in B$ are non degenerate ,

$S3$. π_* preserves scalar products of vectors normal to fibres.

We shall always assume that the dimension of the fibres dim$M$ - dim$B$ is positive and the fibres are connected. By S2, one can observe fibres as spacelike and timelike cases.

The vectors tangent to fibres are called vertical and those normal to fibres are called horizontal. We denote by V the vertical distribution and by H the horizontal distribution. The fundamental tensors of a submersion were defined by O'Neill ([14], [15]). They are $(1, 2)$-tensors on M, given by the formulas:

$$\begin{aligned} T(E, F) &= T_E F = h\nabla_{\nu E} \nu F + \nu \nabla_{\nu E} hF, \\ A(E, F) &= A_E F = \nu \nabla_{hE} hF + h\nabla_{hE} \nu F, \end{aligned} \tag{1}$$

for any $E, F \in \chi(M)$. Here ∇ denotes the Levi-Civita connection of (M, g). These tensors are called integrability tensors for the pseudo-Riemannian submersions. We use the h and ν letters to denote the orthogonal projections on the vertical and horizontal distributions respectively. A vector field X on M is called basic if X is horizontal and π-related to a vector field X_* on B, i.e. $\pi_*(X_p) = X_{*\pi(p)}$ for all $p \in M$. The following lemmas are well known (see [14], [15]).

Lemma 2.1. *Let $\pi : (M, g) \to (B, g')$ be a pseudo-Riemannian submersion. If X, Y are basic vector fields on M, then*

i) $g(X, Y) = g'(X_*, Y_*) \circ \pi$,

ii) $h[X, Y]$ is basic and π-related to $[X_*, Y_*]$,

iii) $h(\nabla_X Y)$ is a basic vector field corresponding to $\overset{B}{\nabla}_{X_*} Y_*$ where ∇^B is the connection on B.

iv) for any vertical vector field V, the bracket $[X, V]$ is vertical.

Lemma 2.2. *For any vertical U, W and horizontal X, Y vector fields, the tensor fields T and A satisfy*

i)$T_U W = T_W U$,

ii)$A_X Y = -A_Y X = \frac{1}{2}\nu [X, Y]$.

Moreover, if X is basic and U is vertical then $h(\nabla_U X) = h(\nabla_X U) = A_X U$. It is not difficult to observe that T acts on the fibers as the second fundamental form and reverse the vertical distributions. It is easy to see that a Riemannian submersion $\pi : M \to B$ has totally geodesic fibers if and only if T vanishes identically.

We define the curvature tensor R of M by $R(E,F) = \nabla_E\nabla_F - \nabla_F\nabla_E - \nabla_{[E,F]}$ for any vector fields E, F on M. The pseudo-Riemannian curvature $(0,4)$-tensor is defined by

$$R(E,F,G,H) = g(R(E,F)G,H).$$

Let us recall the sectional curvature of pseudo-Riemannian manifolds for nondegenerate planes. Let M be a pseudo-Riemannian manifold and P be a non-degenerate tangent plane to M at p. The number

$$K_{X\wedge Y} = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}$$

is independent on the choice of basis X, Y for P and is called the sectional curvature. We use notation $R_{ijkl} = g(R(e_i,e_j)e_k,e_l)$. Next, we can give the following lemma:

Lemma 2.3 ([15]). *Let $\pi : (M,g) \to (B,g')$ be a pseudo-Riemannian submersion. K and K^B denote the sectional curvatures in M and B, respectively. If X, Y are basic vector fields on M, then*

$$K^B_{X_*\wedge Y_*} = K_{X\wedge Y} + \frac{3g(A_XY, A_XY)}{g(X,X)g(Y,Y) - g(X,Y)^2}. \tag{2}$$

In [17], Escobales gave a classification of Riemannian submersions with connected totally geodesic fibres from a sphere to a Riemannian manifold and then Ranjan [16] dropped Escobales's classification into three categories: (a) $S^{2n+1} \to CP^n, n \geq 1$, with the fibres S^1; (b) $S^{4n+3} \to HP^n, n \geq 1$, with the fibres S^3; (c) $S^{8n+7} \to CaP^n$, $n = 1,2$ with the fibres S^7, where CP^n, HP^n and CaP^n are complex projective, quaternionic projective and Cayley projective space, respectively.

In the Lorentz space case, Magid [13] proved that if $\pi : H^{2n+1}_1(c) \to B^{2n}$ be a pseudo-Riemannian submersion with totally geodesic fibres onto a Riemannian manifold then, B^{2n} is a Kaehler manifold holomorphically isometric to complex hyperpolic space $CH^n(4c)$.

In [2] Baditou and Ianuş generalized Magid's result and classified the pseudo-Riemannian submersions with connected complex totally geodesic fibres from a complex pseudo hyperbolic space onto a Riemannian manifold. These pseudo-Riemannian submersions are observed as mainly three categories : (1) $H^{2m+1}_1 \to \mathbb{C}H^m$, (2) $H^{4m+3}_3 \to H(H^m)$ or (3) $H^{15}_7 \to H^8(-4)$, where $\mathbb{C}H^m$and $H(H^m)$ are complex hyperbolic space and quaternionic hyperbolic space, respectively. Then Baditoiu [1] improved these results under the assumption that the dimension of the fibres is less than or equal to three.

Recently, Baditoiu [3] generalized previous results without any assumption for dimension of the fibres and proved that any pseudo-Riemannian submersions with connected, totally geodesic fibres from a real pseudo hyperbolic space onto a pseudo-Riemannian manifold is equivalent to one of the (para) Hopf pseudo-Riemannian submersions: (i) $H^{2m+1}_{2t+1} \to \mathbb{C}H^m_t, 0 \leq t \leq m$, (ii) $H^{2m+1}_m \to AP^m$, (iii) $H^{4m+3}_{4t+3} \to H(H^m_t), 0 \leq t \leq m$, (iv) $H^{4m+3}_{2m+1} \to BP^m$, (v) $H^{15}_{15} \to H^8_8(-4)$, (vi) $H^{15}_7 \to H^8_4(-4)$ or (vii) $H^{15}_7 \to H^8_4(-4)$, where $\mathbb{C}H^m_t$ and $H(H^m_t)$ are the indefinite complex and quaternionic pseudo-hyperbolic spaces of holomorphic, respectively, quaternionic curvature -4; AP^m is the para-complex projective space of real dimension $2m$, signature (m,m) and para-holomorphic curvature -4; BP^m is the para-quaternionic projective space of real dimension $4m$, signature $(2m,2m)$ and para-quaternionic curvature -4.

In summary, for three dimensional, these (para) pseudo-Riemannian submersions with connected, totally geodesic fibres fall into one of the following cases:

(a_1) $\pi : S^3(1) \to CP^1$, (a_2) $\pi : H^3_1(-1) \to H^2(-4) = CH^1$, (a_3) $\pi : H^3_1(-1) \to H^2_1(-4) = AH^1$, (a_4) $\pi : H^3_3(-1) \to H^2_2(-4) = CH^1_1$

We will finish this subsection by the following Theorem of Uniqueness:

Theorem 2.4 ([3]). *Let $\pi_1, \pi_2 : H^a_l \to B$ be two pseudo-Riemannian submersions with connected, totally geodesic fibres from a pseudo-hyperbolic space onto a pseudo-Riemannian manifold. Then there exists an isometry $f : H^a_l \to H^a_l$ such that $\pi_2 \circ f = \pi_1$. In particular, π_1 and π_2 are equivalent.*

2.2. Biharmonic maps

Let M^m and B^n be pseudo-Riemannian manifolds of dimensions m and n, respectively, and $\varphi : M^m \to B^n$ a smooth map. We denote by ∇^M and ∇^B the Levi-Civita connections on M^m and B^n, respectively. Then the tension field $\tau(\varphi)$ is a section of the vector bundle $\varphi^* TB^n$ defined by

$$\tau(\varphi) = \mathrm{trace}(\nabla^\varphi d\varphi) = \sum_{i=1}^m g(e_i, e_i)(\nabla^\varphi_{e_i} d\varphi(e_i) - d\varphi(\nabla_{e_i} e_i)).$$

Here ∇^φ and $\{e_i\}$ denote the induced connection by φ on the bundle $\varphi^* TB^n$, which is the pull-back of ∇^B, and a local orthonormal frame field of M^m, respectively. A smooth map φ is called a harmonic map if its tension field vanishes. A map φ is called biharmonic if it is a critical point of the energy

$$E_2(\varphi) = \frac{1}{2} \int_\Omega g(\tau(\varphi), \tau(\varphi) dv_g$$

for every compact domains Ω of M^m, where dv_g is the volume form of M^m. Using same argument in Riemannian case, the bitension field can be defined by

$$\tau_2(\varphi) = \sum_{i=1}^m g(e_i, e_i)((\nabla^\varphi_{e_i} \nabla^\varphi_{e_i} - \nabla^\varphi_{\nabla_{e_i} e_i})\tau(\varphi) - R^B(d\varphi(e_i), \tau(\varphi))d\varphi(e_i)), \tag{3}$$

where R^B is the curvature tensor of B^n (see [8], [12], [18]). A smooth map φ is a biharmonic map (or 2-harmonic map) if its bitension field vanishes (see [12], [18]). By definition, a harmonic map is clearly biharmonic map. Non harmonic biharmonic maps are called proper biharmonic maps.

3. THE THEOREMS AND PROOFS

In this section, we will prove our classification Theorem and corollaries. Firstly, we will recall well known theorems:

Theorem 3.1 ([10]). *A pseudo-Riemannian submersion $\pi : (M, g) \to (B, g')$ is a harmonic map if and only if each fibre is a minimal submanifold.*

Theorem 3.2 ([1],[13],[16],[17]). *Let $\pi : (M^3_r(c), g) \to (B^2_s, g')$ be a (para) pseudo-Riemannian submersion with connected totally geodesic fibres, where $0 \le r \le 3$, $0 \le s \le 2$ and $c \neq 0$.In summary, for three dimensional, these (para) pseudo-Riemannian submersions with connected, totally geodesic fibres. Then π is one of the following types:*

Timelike Fiber	*Spacelike Fiber*
$H^3_3(-1) \xrightarrow{\pi} H^2_2(-4) = CH^1_1$;[1]	$H^3_1(-1) \xrightarrow{\pi} H^2_1(-4) = AH^1$;[1]
$H^3_1(-1) \xrightarrow{\pi} H^2(-4) = CH^1$;[13]	$S^3(1) \xrightarrow{\pi} S^2\left(\frac{1}{2}\right) = CP^1$;[16],[17].

We will report following theorems which give us the motivation to study on this paper.

Theorem 3.3 ([6]). *Let $x : M \to E^3_s$ $(s = 0, 1)$ be a biharmonic isometric immersion of a Riemannian surface M into E^3_s .Then x is harmonic.*

Theorem 3.4 ([20]). *If M is a complete biharmonic space-like surface in S^3_1 or R^3_1, then it must be totally geodesic, i.e. S^2 or R^2.*

Theorem 3.5 ([19]). *Let $\pi : (M^3(c), g) \to (B^2, g')$ be a Riemannian submersion from a space form of constant sectional curvature c. Then, π is biharmonic if and only if it is harmonic, and this holds if and only if it is a harmonic morphism.*

Let $\pi : (M_r^3, g) \to (B_s^2, g^{'})$ be a pseudo-Riemannian submersion where $0 \le r \le 3$, $0 \le s \le 2$. Let us consider a local pseudo orthonormal frame $\{e_1, e_2, e_3\}$ such that e_1, e_2 are basic and e_3 is vertical . Then, it is well known (see [14]) that $[e_1, e_3]$ and $[e_2, e_3]$ are vertical and $[e_1, e_2]$ is π-related to $[\varepsilon_1, \varepsilon_2]$, where $\{\varepsilon_1, \varepsilon_2\}$ is a pseudo orthonormal frame in the base manifold.

Let $\{e_1, e_2, e_3\}$ be an orthonormal frame adapted to with e_3 being vertical where $g(e_i, e_i) = \delta_i = \mp 1$. If we assume that

$$[\varepsilon_1, \varepsilon_2] = L_1\varepsilon_1 + L_2\varepsilon_2, \tag{4}$$

for $L_1, L_2 \in C^\infty(B)$ and use the notations $l_i = L_i \circ \pi$, $i = 1, 2$. Then, we have

$$\begin{aligned} [e_1, e_3] &= \lambda e_{3,} \\ [e_2, e_3] &= \mu e_{3,} \\ [e_1, e_2] &= l_1e_1 + l_2e_2 - 2\sigma e_{3.} \end{aligned} \tag{5}$$

where λ, μ and $\sigma \in C^\infty(M)$. Here l_1, l_2, λ, μ and σ are the integrability functions of the adapted frame of the pseudo-Riemannian submersion π.

Proposition 3.6. *Let $\pi : (M_r^3, g) \to (B_s^2, g^{'})$ be a pseudo-Riemannian submersion with the adapted frame $\{e_1, e_2, e_3\}$ and the integrability functions l_1, l_2, λ, μ and σ. Then, the pseudo-Riemannian submersion π is biharmonic if and only if*

$$\begin{aligned} \Delta^M\lambda &= -\delta_2l_1e_1(\mu) - \delta_2e_1(\mu l_1) - \delta_2l_2e_2(\mu) - \delta_2e_2(\mu l_2) \\ &\quad +\delta_2\lambda\mu l_1 + \delta_2\mu^2l_2 + \lambda\left\{\delta_2l_1^2 + \delta_1l_2^2 - \delta_1\delta_2K^B\right\}, \\ \Delta^M\mu &= \delta_1l_1e_1(\lambda) + \delta_1e_1(\lambda l_1) + \delta_1l_2e_2(\lambda) + \delta_1e_2(\lambda l_2) \\ &\quad -\delta_1\lambda\mu l_2 - \delta_1\lambda^2l_1 + \mu\left\{\delta_2l_1^2 + \delta_1l_2^2 - \delta_1\delta_2K^B\right\}, \end{aligned} \tag{6}$$

where $K^B = R^B_{1221} \circ \pi = \delta_2e_1(l_2) - \delta_1e_2(l_1) - \delta_1l_1^2 - \delta_2l_2^2$ is the sectional curvature of pseudo-Riemannian manifold $(B_s^2, g^{'})$.

Proof. Let ∇ denote the Levi-Civita connection of the pseudo-Riemannian manifold (M_r^3, g). Using (5), Koszul formula and after a straightforward computation, we have

$$\begin{aligned} \nabla_{e_1}e_1 &= -\delta_1\delta_2l_1e_{2,} \quad \nabla_{e_1}e_2 = l_1e_1 - \sigma e_3, \\ \nabla_{e_1}e_3 &= \delta_2\delta_3\sigma e_2, \quad \nabla_{e_2}e_1 = -l_2e_2 + \sigma e_3, \\ \nabla_{e_2}e_2 &= \delta_1\delta_2l_2e_1, \quad \nabla_{e_2}e_3 = -\delta_1\delta_3\sigma e_1, \\ \nabla_{e_3}e_1 &= \delta_2\delta_3\sigma e_2 - \lambda e_3, \quad \nabla_{e_3}e_2 = -\delta_1\delta_3\sigma e_1 - \mu e_3, \\ \nabla_{e_3}e_3 &= \delta_1\delta_3\lambda e_1 + \delta_2\delta_3\mu e_2. \end{aligned} \tag{7}$$

The tension of the pseudo-Riemannian submersion τ is given by

$$\tau(\pi) = \sum_{i=1}^{3} g(e_i, e_i)\left[\nabla^\pi_{e_i}d\pi(e_i) - d\pi(\nabla^M_{e_i}e_i)\right] = -\delta_3d\pi(\nabla^M_{e_3}e_3) = -\delta_1\lambda\varepsilon_1 - \delta_2\mu\varepsilon_2. \tag{8}$$

After some calculation by using (7) we get

$$\begin{aligned} \tau^2(\pi) &= \sum_{i=1}^{3} g(e_i,e_i)\left\{\nabla^{\pi}_{e_i}\nabla^{\pi}_{e_i}\tau(\pi) - \nabla^{\pi}_{\nabla^M_{e_i}e_i}\tau(\pi) - R^B(d\pi(e_i),\tau(\pi))d\pi(e_i)\right\} \\ &= \delta_1\left[\begin{array}{c} \nabla^{\pi}_{e_1}(-\delta_1 e_1(\lambda)\varepsilon_1 - \delta_1\lambda\nabla^{\pi}_{e_1}\varepsilon_1) + \nabla^{\pi}_{e_1}(-\delta_2 e_1(\mu)\varepsilon_2 - \delta_2\mu\nabla^{\pi}_{e_1}\varepsilon_2) \\ +\delta_1\delta_2 l_1\nabla^{\pi}_{e_2}(-\delta_1\lambda\varepsilon_1 - \delta_2\mu\varepsilon_2) + \delta_2\mu R^B(\varepsilon_1,\varepsilon_2)\varepsilon_1 \end{array}\right] \\ &\quad +\delta_2\left[\begin{array}{c} \nabla^{\pi}_{e_2}(-\delta_1 e_2(\lambda)\varepsilon_1 - \delta_1\lambda\nabla^{\pi}_{e_2}\varepsilon_1) + \nabla^{\pi}_{e_2}(-\delta_2 e_2(\mu)\varepsilon_2 - \delta_2\mu\nabla^{\pi}_{e_2}\varepsilon_2) \\ -\delta_1\delta_2 l_2\nabla^{\pi}_{e_1}(-\delta_1\lambda\varepsilon_1 - \delta_2\mu\varepsilon_2) + \delta_1\lambda R^B(\varepsilon_2,\varepsilon_1)\varepsilon_2 \end{array}\right] \\ &\quad \delta_3\left[\begin{array}{c} \nabla^{\pi}_{e_3}(-\delta_1 e_3(\lambda)\varepsilon_1 - \delta_1\lambda\nabla^{\pi}_{e_3}\varepsilon_1) + \nabla^{\pi}_{e_3}(-\delta_2 e_3(\mu)\varepsilon_2 - \delta_2\mu\nabla^{\pi}_{e_3}\varepsilon_2) \\ -\delta_1\delta_3\lambda\nabla^{\pi}_{e_1}(-\delta_1\lambda\varepsilon_1 - \delta_2\mu\varepsilon_2) - \delta_2\delta_3\mu\nabla^{\pi}_{e_2}(-\delta_1\lambda\varepsilon_1 - \delta_2\mu\varepsilon_2) \end{array}\right]. \end{aligned}$$

Now we calculate Laplace of λ and μ. Since $grad\lambda = \delta_1 e_1(\lambda)e_1 + \delta_2 e_2(\lambda)e_2 + \delta_3 e_3(\lambda)e_3$, we obtain

$$\begin{aligned} \Delta^m\lambda &= \sum_{i=1}^{3} g(e_i,e_i)g(\nabla_{e_i}grad\lambda, e_i) \\ &= \delta_1 e_1(e_1(\lambda)) + \delta_2 e_2(e_2(\lambda)) + \delta_3 e_3(e_3(\lambda)) + \delta_2 e_2(\lambda)l_1 - \delta_1 e_1(\lambda)l_2 \\ &\quad -\delta_1 e_1(\lambda)\lambda - \delta_2 e_2(\lambda)\mu. \end{aligned}$$

Using same calculations for μ we get

$$\begin{aligned} \Delta^m\mu &= \delta_1 e_1(e_1(\mu)) + \delta_2 e_2(e_2(\mu)) + \delta_3 e_3(e_3(\mu)) + \delta_2 e_2(\mu)l_1 - \delta_1 e_1(\mu)l_2 \\ &\quad -\delta_1 e_1(\mu)\lambda - \delta_2 e_2(\mu)\mu. \end{aligned}$$

$$\begin{aligned} \tau^2(\pi) &= \delta_1\left[\begin{array}{c} -\Delta^M\lambda - \delta_2 l_1 e_1(\mu) - \delta_2 e_1(\mu l_1) - \delta_2 l_2 e_2(\mu) - \delta_2 e_2(\mu l_2) \\ +\delta_2\lambda\mu l_1 + \delta_2\mu^2 l_2 + \lambda\left\{\delta_2 l_1^2 + \delta_1 l_2^2 - \delta_1\delta_2 K^B\right\} \end{array}\right]\varepsilon_1 \\ &\quad +\delta_2\left[\begin{array}{c} -\Delta^M\mu + \delta_1 l_1 e_1(\lambda) + \delta_1 e_1(\lambda l_1) + \delta_1 l_2 e_2(\lambda) + \delta_1 e_2(\lambda l_2) \\ -\delta_1\lambda\mu l_2 - \delta_1\lambda^2 l_1 + \mu\left\{\delta_2 l_1^2 + \delta_1 l_2^2 - \delta_1\delta_2 K^B\right\} \end{array}\right]\varepsilon_2, \end{aligned}$$

which completes the proof.

When the integrability function $\mu = 0$ we have the following corollary.

Corollary 3.7. *Let $\pi : (M^3_r, g) \to (B^2_s, g')$ be a pseudo-Riemannian submersion with an adapted frame $\{e_1, e_2, e_3\}$ and the integrability functions l_1, l_2, λ, μ and σ with $\mu = 0$. Then, the pseudo-Riemannian submersion π is biharmonic if and only if*

$$\begin{aligned} -\delta_1\Delta^M\lambda + \lambda\left\{\delta_1\delta_2 l_1^2 + l_2^2 - \delta_2 K^B\right\} &= 0, \qquad (9) \\ \delta_1\delta_2 l_1 e_1(\lambda) + \delta_1\delta_2 e_1(\lambda l_1) + \delta_1\delta_2 l_2 e_2(\lambda) + \delta_1\delta_2 e_2(\lambda l_2) - \delta_1\delta_2\lambda^2 l_1 &= 0. \end{aligned}$$

□

The following lemmas will be used to prove classification Theorem.

Lemma 3.8. *Let $\pi : M^3_r(c) \to (B^2_s, g')$ be a pseudo-Riemannian submersion from a space form of constant sectional curvature c. Then, for any orthonormal frame $\{e_1, e_2, e_3\}$ on $M^3_r(c)$ adapted to the pseudo-Riemannian submersion with e_3 being vertical, all the integrability functions l_1, l_2, λ, μ and σ are constant along fibers of π, i.e.,*

$$e_3(l_1) = e_3(l_2) = e_3(\mu) = e_3(\lambda) = e_3(\sigma) = 0 \qquad (10)$$

Proof. From definition, $l_i = F_i \circ \pi$ for $i = 1, 2$ we can conclude that l_1 and l_2 are constant along the fibers. It remains to show that

$$e_3(\mu) = e_3(\lambda) = e_3(\sigma) = 0. \tag{11}$$

Using the Jacobi identity to the frame $\{e_1, e_2, e_3\}$, we have

$$2e_3(\sigma) + \lambda l_1 + \mu l_2 + e_2(\lambda) - e_1(\mu) = 0. \tag{12}$$

By using (12) and the fact that $M_1^3(c)$ has constant sectional curvature c, calculating R^M_{1312}, R^M_{1313}, R^M_{1323}, R^M_{1212}, R^M_{1223}, R^M_{2313}, R^M_{2323} respectively, we get

$$\begin{aligned} i) e_1(\sigma) - 2\lambda\sigma &= 0, \\ ii)\ \delta_1 e_1(\lambda) + \delta_1\delta_2\delta_3\sigma^2 - \delta_1\lambda^2 + \delta_2\mu l_1 &= c, \\ iii) - e_1(\mu) + e_3(\sigma) + \lambda l_1 + \lambda\mu &= 0, \\ iv) - \delta_2 e_2(l_1) + \delta_1 e_1(l_2) - \delta_2 l_1^2 - \delta_1 l_2^2 - 3\delta_1\delta_2\delta_3\sigma^2 &= c, \\ v) e_2(\sigma) - 2\mu\sigma &= 0, \\ vi) - e_2(\lambda) - e_3(\sigma) - \mu l_2 + \lambda\mu &= 0, \\ vii)\ \delta_1\delta_2\delta_3\sigma^2 + \delta_2 e_2(\mu) - \delta_1\lambda l_2 - \delta_2\mu^2 &= c. \end{aligned} \tag{13}$$

Applying e_3 to both sides of the equation *iv*) of (13) and using $e_3e_1 = [e_3, e_1] + e_1e_3$ and $e_3e_2 = [e_3, e_2] + e_2e_3$, we obtain

$$\sigma e_3(\sigma) = 0,$$

which implies

$$e_3(\sigma) = 0.$$

Using the last equation and applying e_3 to both sides of the equations *i*) and *v*) of (13) respectively, we get

$$e_3(\lambda) = 0, \quad e_3(\mu) = 0.$$

□

Case 1. Spacelike Fiber

Submersion Signature of g Signature of g'	New Orthonormal frame of Base Manifold
$\pi : (M_1^3, g) \to (B_1^2, g')$ $(e_1, e_2, e_3; +, -, +)$ $(\varepsilon_1, \varepsilon_2; +, -)$	$\varepsilon_1' = -\frac{\bar\lambda}{\sqrt{\bar\lambda^2-\bar\mu^2}}\varepsilon_1 + \frac{\bar\mu}{\sqrt{\bar\lambda^2-\bar\mu^2}}\varepsilon_2,\ \varepsilon_2' = -\frac{\bar\mu}{\sqrt{\bar\lambda^2-\bar\mu^2}}\varepsilon_1 + \frac{\bar\lambda}{\sqrt{\bar\lambda^2-\bar\mu^2}}\varepsilon_2$;if $\bar\lambda^2 - \bar\mu^2 > 0$ $\varepsilon_1' = -\frac{\bar\mu}{\sqrt{\bar\mu^2-\bar\lambda^2}}\varepsilon_1 + \frac{\bar\lambda}{\sqrt{\bar\mu^2-\bar\lambda^2}}\varepsilon_2,\ \varepsilon_2' = -\frac{\bar\lambda}{\sqrt{\bar\mu^2-\bar\lambda^2}}\varepsilon_1 + \frac{\bar\mu}{\sqrt{\bar\mu^2-\bar\lambda^2}}\varepsilon_2$;if $\bar\mu^2 - \bar\lambda^2 > 0$
$\pi : (M_2^3, g) \to (B_2^2, g')$ $(e_1, e_2, e_3; -, -, +)$ $(\varepsilon_1, \varepsilon_2; -, -)$	$\varepsilon_1' = \frac{\bar\lambda}{\sqrt{\bar\lambda^2+\bar\mu^2}}\varepsilon_1 + \frac{\bar\mu}{\sqrt{\bar\lambda^2+\bar\mu^2}}\varepsilon_2,\ \varepsilon_2' = \frac{\bar\mu}{\sqrt{\bar\lambda^2+\bar\mu^2}}\varepsilon_1 - \frac{\bar\lambda}{\sqrt{\bar\lambda^2+\bar\mu^2}}\varepsilon_2$
$\pi : (M^3, g) \to (B^2, g')$ $(e_1, e_2, e_3; +, +, +)$ $(\varepsilon_1, \varepsilon_2; +, +)$	$\varepsilon_1' = \frac{\bar\lambda}{\sqrt{\bar\lambda^2+\bar\mu^2}}\varepsilon_1 + \frac{\bar\mu}{\sqrt{\bar\lambda^2+\bar\mu^2}}\varepsilon_2,\ \varepsilon_2' = -\frac{\bar\mu}{\sqrt{\bar\lambda^2+\bar\mu^2}}\varepsilon_1 + \frac{\bar\lambda}{\sqrt{\bar\lambda^2+\bar\mu^2}}\varepsilon_2$

Table 1

Case 2. Timelike Fiber

Submersion Signature of g Signature of g'	New Orthonormal frame of Base Manifold
$\pi : (M_1^3, g) \to (B^2, g')$ $(e_1, e_2, e_3; +, +, -)$ $(\varepsilon_1, \varepsilon_2; +, +)$	$\varepsilon_1' = \frac{\bar\lambda}{\sqrt{\bar\lambda^2+\bar\mu^2}}\varepsilon_1 + \frac{\bar\mu}{\sqrt{\bar\lambda^2+\bar\mu^2}}\varepsilon_2, \varepsilon_2' = \frac{\bar\mu}{\sqrt{\bar\lambda^2+\bar\mu^2}}\varepsilon_1 - \frac{\bar\lambda}{\sqrt{\bar\lambda^2+\bar\mu^2}}\varepsilon_2$
$\pi : (M_2^3, g) \to (B_1^2, g')$ $(e_1, e_2, e_3; +-, -)$ $(\varepsilon_1, \varepsilon_2 : +, -)$	$\varepsilon_1' = -\frac{\bar\lambda}{\sqrt{\bar\lambda^2-\bar\mu^2}}\varepsilon_1 + \frac{\bar\mu}{\sqrt{\bar\lambda^2-\bar\mu^2}}\varepsilon_2, \varepsilon_2' = -\frac{\bar\mu}{\sqrt{\bar\lambda^2-\bar\mu^2}}\varepsilon_1 + \frac{\bar\lambda}{\sqrt{\bar\lambda^2-\bar\mu^2}}\varepsilon_2$;if $\bar\lambda^2 - \bar\mu^2 > 0$ $\varepsilon_1' = -\frac{\bar\mu}{\sqrt{\bar\mu^2-\bar\lambda^2}}\varepsilon_1 + \frac{\bar\lambda}{\sqrt{\bar\mu^2-\bar\lambda^2}}\varepsilon_2, \varepsilon_2' = -\frac{\bar\lambda}{\sqrt{\bar\mu^2-\bar\lambda^2}}\varepsilon_1 + \frac{\bar\mu}{\sqrt{\bar\mu^2-\bar\lambda^2}}\varepsilon_2$;if $\bar\mu^2 - \bar\lambda^2 > 0$
$\pi : (M_3^3, g) \to (B_2^2, g')$ $(e_1, e_2, e_3; -, -, -)$ $(\varepsilon_1, \varepsilon_2 : -, -)$	$\varepsilon_1' = \frac{\bar\lambda}{\sqrt{\bar\lambda^2+\bar\mu^2}}\varepsilon_1 + \frac{\bar\mu}{\sqrt{\bar\lambda^2+\bar\mu^2}}\varepsilon_2, \varepsilon_2' = \frac{\bar\mu}{\sqrt{\bar\lambda^2+\bar\mu^2}}\varepsilon_1 - \frac{\bar\lambda}{\sqrt{\bar\lambda^2+\bar\mu^2}}\varepsilon_2$

Table 2

Lemma 3.9. *Let $\pi : (M_r^3(c), g) \to (B_s^2, g')$ be a pseudo-Riemannian submersion with an adapted frame $\{e_1, e_2, e_3\}$ and the integrability functions l_1, l_2, λ, μ and σ . Then, there exists another adapted orthonormal frame $\left\{e_1', e_2', e_3' = e_3\right\}$ on $M_r^3(c)$ with integrability functions $\mu' = 0$, and $\sigma' = \sigma$.*

Proof. Applying the same method in ([19], Lemma 3.2) and using Lemma 3.8 , Table 1 and Table 2, one can complete the proof of the lemma. □

Now we will give a classification of biharmonic pseudo-Riemannian submersions.

Classification Theorem: *Let $\pi : M_r^3(c) \to B_s^2$ be a pseudo-Riemannian submersion from a space form of constant sectional curvature c. Then, π is biharmonic if and only if it is equivalent to one of the following submersions:*

Timelike Fiber	Spacelike Fiber
$\pi_1 : H_3^3(-1) \to H_2^2(-4) = CH_1^1$;	$\pi_6 : E_2^3 \to E_2^2$;
$\pi_2 : E_3^3 \to E_2^2$;	$\pi_7 : H_1^3(-1) \to H_1^2(-4) = AH^1$;
$\pi_3 : H_1^3(-1) \to H^2(-4) = CH^1$;	$\pi_8 : E_1^3 \to E_1^2$;
$\pi_4 : E_1^3 \to E^2$;	$\pi_9 : S^3(1) \to S^2\left(\frac{1}{2}\right) = CP^1$;is proved by [19]
$\pi_5 : E_2^3 \to E_1^2$;	$\pi_{10} : E^3 \to E^2$,is proved by [19]

Table 3

Proof. By Lemma 3.9, we can choose an orthonormal frame $\{e_1, e_2, e_3\}$ adapted to the pseudo-Riemannian submersion with integrability functions l_1, l_2, λ, μ and σ with $\mu = 0$. According to this frame (13) reduces to

$$\begin{aligned} a_1) e_1(\sigma) - 2\lambda\sigma &= 0, \\ a_2) \delta_1 e_1(\lambda) + \delta_1\delta_2\delta_3\sigma^2 - \delta_1\lambda^2 &= c, \\ a_3) \lambda l_1 &= 0, \\ a_4) - \delta_2 e_2(l_1) + \delta_1 e_1(l_2) - \delta_2 l_1^2 - \delta_1 l_2^2 - 3\delta_1\delta_2\delta_3\sigma^2 &= c, \\ a_5) e_2(\sigma) &= 0, \\ a_6) e_2(\lambda) &= 0, \\ a_7) \delta_1\delta_2\delta_3\sigma^2 - \delta_1\lambda l_2 &= c. \end{aligned} \tag{14}$$

From a_3) of (14), we have either $\lambda = 0$ or $l_1 = 0$. If $\lambda = 0$, from (8) the tension field of π vanishes. This means that pseudo-Riemannian submersion is harmonic. If $l_1 = 0$ and $\lambda \neq 0$, this case can not happen. We will prove this by a contradiction.

Case I: $\lambda \neq 0$, $l_1 = 0$ and $l_2 = 0$. So, from a_4), a_7) in (14), we have $\sigma = c = 0$. If we put $l_1 = l_2 = \sigma = 0$ and $\mu = 0$ into (9) we obtain

$\Delta^M \lambda = 0,$

which, one can easily get by using a_2), a_6) of (14) ,

$$\lambda^3 = 0.$$

It follows that $\lambda = 0$ which is a contradiction.

Case II: $\lambda \neq 0$, $l_1 = 0$ and $l_2 \neq 0$. In this case, by using $l_1 = 0$ and a_5), a_6) and a_7) of (14), (9) reduces to

$$-\delta_1 \Delta^M \lambda + \lambda \left[-\delta_2 c - 3\delta_1\delta_3\sigma^2 + l_2^2\right] = 0, \tag{15}$$

where $K^B = c + 3\delta_1\delta_2\delta_3\sigma^2$ obtained from curvature formula for a pseudo-Riemannian submersion. Using a_1), a_2) of (14) and after a straightforward calculation yields

$$\begin{aligned}\Delta^M \lambda &= \delta_1 e_1(e_1(\lambda)) - \delta_1 e_1(\lambda) l_2 - \delta_1 e_1(\lambda)\lambda \\ \Delta^M \lambda &= -5\delta_1\delta_2\delta_3\lambda\sigma^2 + \delta_1\lambda^3 + \lambda c + l_2(-c + \delta_1\delta_2\delta_3\sigma^2 - \delta_1\lambda^2).\end{aligned}$$

Substituting this into (15) and using a_7) we obtain

$$\lambda\left[\delta_3(6\delta_2 - 3\delta_1)\sigma^2 - \lambda^2 - (2\delta_1 + \delta_2)c\right] = 0. \tag{16}$$

We accept $\lambda \neq 0$, so (16) is equivalent to

$$\lambda^2 = \delta_3(6\delta_2 - 3\delta_1)\sigma^2 - (2\delta_1 + \delta_2)c. \tag{17}$$

After applying e_1 to both sides of (17), we get

$$\lambda e_1(\lambda) = \delta_3(6\delta_2 - 3\delta_1)\sigma e_1(\sigma).$$

Combining this and a_1) , a_2) in (14), we have

$$\lambda(\lambda^2 - \delta_2\delta_3\sigma^2 + \delta_1 c) = 2\delta_3(6\delta_2 - 3\delta_1)\lambda\sigma^2.$$

By assumption $\lambda \neq 0$, this turned into

$$\lambda^2 + \delta_1 c = \delta_3(13\delta_2 - 6\delta_1)\sigma^2,$$

or

$$\lambda^2 = \delta_3(13\delta_2 - 6\delta_1)\sigma^2 - \delta_1 c. \tag{18}$$

Applying e_1 to both sides of (18) and again using a_1), a_2) in (14) we get

$$\lambda^2 = \delta_3(27\delta_2 - 12\delta_1)\sigma^2 - \delta_1 c. \tag{19}$$

Combining (17), (18) with (19) we have $\lambda = \sigma = c = 0$. This implies there is a contradiction. Because our assumption is $\lambda \neq 0$.So we have $\lambda = \mu = 0$. If we use (7) in the first equation of (1) we get $T(e_i, e_j) = 0$, $1 \leq i, j \leq 3$. It means that fiber is totally geodesic. By (a_2)of (14), we have

$$\delta_1\delta_2\delta_3\sigma^2 = c. \tag{20}$$

Using the last equation and Theorem 3.2 , we get our classification. □

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