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Complete Heyting Algebra-Valued Convergence Semigroups

T.M.G. Ahsanullah[a], Fawzi Al-Thukair[a]

[a]*Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia*

Abstract. Considering a complete Heyting algebra $\mathbb{H}$, we introduce a notion of stratified $\mathbb{H}$-convergence semigroup. We develop some basic facts on the subject, besides obtaining conditions under which a stratified $\mathbb{H}$-convergence semigroup is a stratified $\mathbb{H}$-convergence group. We supply a variety of natural examples; and show that every stratified $\mathbb{H}$-convergence semigroup with identity is a stratified $\mathbb{H}$-quasi-uniform convergence space. We also show that given a commutative cancellative semigroup equipped with a stratified $\mathbb{H}$-quasi-unifom structure satisfying a certain property gives rise to a stratified $\mathbb{H}$-convergence semigroup via a stratified $\mathbb{H}$-quasi-uniform convergence structure.

1. Introduction

Inspired on the one hand by the enormous work done on *classical convergence groups* mostly within the framework of sequential convergence lead by J. Novák [42] and on the other, by the *classical convergence group* and *its uniformization* initiated by D. C. Kent and G. D. Richardson within the framework of filter-theoretic convergence (cf. [32–35, 45]), we generalized the notion of convergence groups into the context of *frame-valued convergence groups* [2] and later, *lattice-valued convergence ring and its uniform convergence structures* [1]. Furthermore, we studied *enriched lattice-valued convergence groups* in [3]. All of these works are based on lattice-valued filter-theoretic convergence structures attributed to G. Jäger undertaken since 2001(cf. [21]), which originally initiated by U. Höhle and A. Šostak [19], and independently, by U. Höhle in [20].

One of the motivations of this work is stemmed from the fact that the category of classical convergence spaces [6–8, 10, 13, 16, 32, 33, 37, 38, 42, 46] is a better behaved category than the category of topological spaces; there are enormous quantity of research papers appeared over the years on topological semigroups alone, but unfortunately, we were able to find a very few papers devoted to convergence semigroups within the scope of filter-theoretic treatments (cf. [17, 18]); this is, however, in contrast to the category of convergence groups (cf. [32–35, 45]). This has stimulated our interest further to look for lattice-valued convergence semigroups based on whatever we found from existing papers on classical convergence semigroups [17, 18]. In this paper, we intend to develop a basic theory on lattice-valued convergence

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Email addresses: tmga1@ksu.edu.sa (T.M.G. Ahsanullah), thukair@ksu.edu.sa (Fawzi Al-Thukair)

semigroups; unlike previous findings [1–3], here classical results on semigroup theory are particularly used like their classical counterpart [9, 11, 14, 40, 41, 44, 49].

We organize our work as follows. In Section 2, we present various facts from existing articles which will be used throughout the text. Section 3 deals with, the main notion of stratified $\mathbb{H}$-convergence semigroups along with various results. Here we show among others that a compact Hausdorff-separated stratified $\mathbb{H}$-convergence semigroup contains an idempotent element; also, we show that the set of idempotent elements in a compact Hausdorff-separated stratified $\mathbb{H}$-convergence semigroup is $\top$-closed [30], and hence compact [27]. We give conditions for which a Hausdorff-separated stratified $\mathbb{H}$-Choquet convergence semigroup produces a stratified $\mathbb{H}$-convergence group. In section 4, we give results on ideals in stratified $\mathbb{H}$-convergence semigroups. We provide in Section 5, a wide variety of examples including natural examples; one of the examples here states that a commutative cancellative semigroup equipped with a stratified $\mathbb{H}$-quasi-uniform structure satisfying a certain property, produces a stratified $\mathbb{H}$-convergence semigroup via stratified $\mathbb{H}$-neighborhood system [19, 24]. A similar example is given in Section 6 but this time via stratified $\mathbb{H}$-quasi-uniform convergence structure. In this section, the main result is to show that every stratified $\mathbb{H}$-convergence semigroup with identity element is a stratified $\mathbb{H}$-quasi-uniform convergence space.

2. Preliminaries

Throughout the text we consider $\mathbb{H} = (\mathbb{H}, \leq, \wedge)$, a complete Heyting algebra. This means that the lattice $\mathbb{H}$ is a complete lattice, where finite meets are distributive over arbitrary joins; that is, for all $\alpha, \beta_j \in \mathbb{H}$, $\alpha \wedge \bigvee_{j\in J} \beta_j = \bigvee_{j\in J} \left(\alpha \wedge \beta_j\right)$. The set of all $\mathbb{H}$-sets is denoted by $\mathbb{H}^X$ (= $\{\nu \colon X \longrightarrow \mathbb{H}\}$). If $A \subseteq X$, then a constant $\mathbb{H}$-set with value $\alpha \in \mathbb{H}$ on A, is denoted by α_A, and is defined as $\alpha_A(x) = \alpha$, if $x \in A$ and $\alpha_A(x) = \bot$, elsewhere. In particular, $\top_X(x) = \top$, the characteristic function of X and $\bot_X(x) = \bot$, the zero function. The residuated implication operation $\rightarrow \colon \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{H}$ is defined by: $\alpha \rightarrow \beta = \bigvee\{\gamma \in \mathbb{H} | \alpha \wedge \gamma \leq \beta\}$. Then $\alpha \leq \beta \rightarrow \gamma \Leftrightarrow \alpha \wedge \beta \leq \gamma$.

Lemma 2.1. ([20]) *Let $\mathbb{H} = (\mathbb{H}, \leq, \wedge)$ be a complete Heyting algebra. Then the following are satisfied for all $\alpha, \beta, \gamma, \delta \in \mathbb{H}$:*
(i) $\alpha \leq \beta \Rightarrow \alpha \rightarrow \gamma \geq \beta \rightarrow \gamma$ and $\gamma \rightarrow \alpha \leq \gamma \rightarrow \beta$;
(ii) $\alpha \rightarrow (\beta \wedge \gamma) = (\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \gamma)$;
(iii) $\left(\bigvee_{j\in J} \alpha_j\right) \rightarrow \beta = \bigwedge_{j\in J}(\alpha_j \rightarrow \beta)$;
(iv) $(\alpha \rightarrow \beta) \wedge (\gamma \rightarrow \delta) \leq (\alpha \wedge \gamma) \rightarrow (\beta \wedge \delta)$.

Definition 2.2. ([19]) A map $\mathcal{F} \colon \mathbb{H}^X \longrightarrow \mathbb{H}$ is called an $\mathbb{H}$-*filter* on X if the conditions below are satisfied:
(HF1) $\mathcal{F}(\top_X) = \top, \mathcal{F}(\bot_X) = \bot$;
(HF2) if $\nu_1, \nu_2 \in \mathbb{H}^X$ with $\nu_1 \leq \nu_2$, then $\mathcal{F}(\nu_1) \leq \mathcal{F}(\nu_2)$;
(HF3) $\mathcal{F}(\nu_1) \wedge \mathcal{F}(\nu_2) \leq \mathcal{F}(\nu_1 \wedge \nu_2)$, $\forall \nu_1, \nu_2 \in \mathbb{H}^X$;
(S$\mathbb{H}$) An $\mathbb{H}$-filter $\mathcal{F}$ is called a *stratified* $\mathbb{H}$-*filter* if $\forall \alpha \in \mathbb{H}$, $\alpha \leq \mathcal{F}(\alpha_X)$.

The set of all stratified $\mathbb{H}$-filters on X is denoted by $\mathcal{F}^s_{\mathbb{H}}(X)$. On $\mathcal{F}^s_{\mathbb{H}}(X)$, partial ordering $\leq$ is defined by: if $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_{\mathbb{H}}(X)$, then $\mathcal{F} \leq \mathcal{G} \Leftrightarrow \mathcal{F}(\nu) \leq \mathcal{G}(\nu)$, $\forall \nu \in \mathbb{H}^X$. A maximal element in the partially ordered set $\left(\mathcal{F}^s_{\mathbb{H}}(X), \leq\right)$ is called a *stratified* $\mathbb{H}$-*ultrafilter*. We denote $\mathcal{F}^{su}_{\mathbb{H}}(X)$ as the set of all stratified $\mathbb{H}$-ultrafilters [19]. If $x \in X$, then $[x] \in \mathcal{F}^s_{\mathbb{H}}(X)$, called *point stratified* $\mathbb{H}$-*filter* on X, and is defined as $[x](\nu) = \nu(x)$, for all $\nu \in \mathbb{H}^X$.

If $f \colon X \longrightarrow Y$ is a function, then $f^{\leftarrow} \colon \mathbb{H}^Y \longrightarrow \mathbb{H}^X$ is defined for any $\mu \in \mathbb{H}^Y$ by $f^{\leftarrow}(\mu) = \mu \circ f$; and $f^{\rightarrow} \colon \mathbb{H}^X \longrightarrow \mathbb{H}^Y$ is defined by: $f^{\rightarrow}(\nu)(y) = \bigvee\{\nu(x) | f(x) = y\}$, $\forall \nu \in \mathbb{H}^X, y \in Y$. Moreover, if $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$, then the stratified L-filter $f^{\Rightarrow}(\mathcal{F}) \colon \mathbb{H}^Y \longrightarrow \mathbb{H}$ on Y is defined for any $\mu \in \mathbb{H}^Y$ by: $f^{\Rightarrow}(\mathcal{F})(\mu) = \mathcal{F}\left(f^{\leftarrow}(\mu)\right) = \mathcal{F}(\mu \circ f)$. This is also true for $\mathcal{F} \in \mathcal{F}^{su}_{\mathbb{H}}(X)$, that is, in which case $f^{\Rightarrow}(\mathcal{F}) \in \mathcal{F}^{su}_{\mathbb{H}}(X)$.

If $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(Y)$, then $f^{\Leftarrow}(\mathcal{F}) \colon \mathbb{H}^X \longrightarrow \mathbb{H}$ defined by: $[f^{\Leftarrow}(\mathcal{F})](\nu) = \bigvee\{\mathcal{F}(\mu) | \mu \in \mathbb{H}^Y, f^{\leftarrow}(\mu) \leq \nu\}$, for all $\nu \in \mathbb{H}^X$, is a stratified $\mathbb{H}$-filter on X if and only if for all $\mu \in \mathbb{H}^Y, f^{\leftarrow}(\mu) = \bot_X \Rightarrow \mathcal{F}(\mu) = \bot$. If, however,

$f\colon X \longrightarrow$ is a surjective map, and $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$, then $f^{\Leftarrow}(\mathcal{F}) \in \mathcal{F}^s_{\mathbb{H}}(X)$, and the relation $f^{\Rightarrow}(f^{\Leftarrow}(\mathcal{F})) = \mathcal{F}$ holds [19].

If $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_{\mathbb{H}}(X)$, then $\mathcal{F} \vee \mathcal{G} \in \mathcal{F}^s_{\mathbb{H}}(X)$ if and only if $\nu \wedge \mu = \bot_X \implies \mathcal{F}(\nu) \wedge \mathcal{G}(\mu) = \bot$. In this case $\mathcal{F} \vee \mathcal{G}$ is defined for any $\xi \in \mathbb{H}^X$ by: $\mathcal{F} \vee \mathcal{G}(\xi) = \bigvee\{\mathcal{F}(\nu) \wedge \mathcal{G}(\mu)\colon \nu \wedge \mu \le \xi\}$.

Let $\nu \in \mathbb{H}^X$ and $\mu \in \mathbb{H}^Y$, then their *product* $\nu \times \mu \in \mathbb{H}^{X\times Y}$ is defined by $\nu \times \mu = \nu \circ pr_X \wedge \mu \circ pr_Y$, where $pr_X\colon X \times Y \longrightarrow X, (x, y) \mapsto x$ and $pr_Y\colon X \times Y \longrightarrow Y, (x, y) \mapsto y$, are projections on X and Y, respectively.

If $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$ and $\mathcal{G} \in \mathcal{F}^s_{\mathbb{H}}(Y)$, then the *product of* $\mathcal{F}$ and $\mathcal{G}$ is a stratified $\mathbb{H}$-filter, defined for any $\nu \in \mathbb{H}^{X\times Y}$ by: $\mathcal{F} \times \mathcal{G}(\nu) = \bigvee\{\mathcal{F}(\nu_1) \wedge \mathcal{G}(\nu_2)\colon \nu_1 \in \mathbb{H}^X, \nu_2 \in \mathbb{H}^Y, \nu_1 \times \nu_2 \le \nu\}$ (cf. [21]).

If $A \subseteq X$, and $\imath_A\colon A \hookrightarrow X$ is the inclusion map, then for any $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$, $\mathcal{F}_A(\nu) = \imath_A^{\leftarrow}(\mathcal{F})(\nu) = \bigvee\{\mathcal{F}(\mu)\colon \mu|_A \le \nu\} \in \mathcal{F}^s_{\mathbb{H}}(A)$ if and only if $\mu|_A = \bot_A$ implies $\mathcal{F}(\mu) = \bot$(cf. [21]); $\mathcal{F}_A$ is called the *trace of* $\mathcal{F}$ *on* A, in which case $\mathcal{F}_A$ exists. If $\mathcal{G} \in \mathcal{F}^s_{\mathbb{H}}(A)$, then $[\mathcal{G}] = \imath_A^{\Rightarrow}(\mathcal{G}) \in \mathcal{F}^s_{\mathbb{H}}(X)$, where $[\mathcal{G}](\nu) = \mathcal{G}(\nu_{|_A})$. Note that $\mathcal{F} \le [\mathcal{F}_A]$, and if $\mathcal{F}(\top_A) = \top$, then $\mathcal{F} = [\mathcal{F}_A]$. Also, note that if $\mathcal{U} \in \mathcal{F}^{su}_{\mathbb{H}}(X)$, then $\mathcal{U}_A \in \mathcal{F}^s_{\mathbb{H}}(A)$ if and only if $\mathcal{U}(\top_A) = \top$.

Lemma 2.3. ([22]) *Let $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$ and let $f : X \longrightarrow Y$. If $\mathcal{U} \ge f^{\Rightarrow}(\mathcal{F})$ is a stratified $\mathbb{H}$-ultrafilter on Y, then there exists a $\mathcal{G} \in \mathcal{F}^{su}_{\mathcal{H}}(X)$ such that $\mathcal{G} \ge \mathcal{F}$ and $f^{\Rightarrow}(\mathcal{G}) = \mathcal{U}$.*

If $(X, \cdot)$ is a semigroup, and $A, B \subseteq X$, then one defines $A \cdot B$(or simply AB) by $A \cdot B = \{ab\colon a \in A, b \in B\}$. In particular, $\{a\}B$ is denoted just by aB and similarly, $A\{b\}$ by Ab (cf. [9, 12, 44]).

If $(X, \cdot)$ is a group and $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$, then $\mathcal{F}^{-1}$ is defined by $\mathcal{F}^{-1}(\nu) = \mathcal{F}(\nu^{-1})$, where $\nu^{-1}\colon X \to \mathbb{H}, x \mapsto \nu(x^{-1})$. Since for any $\nu \in \mathbb{H}^X$, $\jmath^{\Rightarrow}(\mathcal{F})(\nu) = \mathcal{F}(\jmath^{\leftarrow}(\nu)) = \mathcal{F}(\nu^{-1}) = \mathcal{F}^{-1}(\nu)$, where $\jmath\colon X \longrightarrow X, x \mapsto x^{-1}$, $\jmath$ is known as *inversion* mapping, we have $\mathcal{F}^{-1} \in \mathcal{F}^s_{\mathbb{H}}(X)$. Also, if $m\colon X \times X \longrightarrow X, (x, y) \mapsto xy$, the semigroup or group operation on X, then for any $\nu_1, \nu_2 \in \mathbb{H}^X$ and $z \in X$, $m^{\rightarrow}(\nu_1 \times \nu_2)(z) = \bigvee_{m(x,y)=z}(\nu_1 \times \nu_2)(x, y)$ $= \bigvee_{xy=z}(\nu_1 \circ pr_X \wedge \nu_2 \circ pr_X)(x, y) = \bigvee_{xy=z} \nu_1 \circ pr_X(x, y) \wedge \nu_2 \circ pr_X(x, y) = \bigvee_{xy=z} \nu_1(x) \wedge \nu_2(y) = \nu_1 \odot \nu_2(z)$.

If $(X, \cdot)$ is a semigroup, and $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_{\mathbb{H}}(X)$, then the map $\mathcal{F} \odot \mathcal{G}\colon \mathbb{H}^X \longrightarrow \mathbb{H}$ is defined for any $\nu \in \mathbb{H}^X$ by: $\mathcal{F} \odot \mathcal{G}(\nu) = \bigvee\{\mathcal{F}(\nu_1) \wedge \mathcal{G}(\nu_2)|\nu_1, \nu_2 \in \mathbb{H}^X, \nu_1 \odot \nu_2 \le \nu\}$. It is shown in Proposition 3.3[2](see also, Lemma 2.9 and Proposition 2.10 [3]) that $\mathcal{F} \odot \mathcal{G} = m^{\Rightarrow}(\mathcal{F} \times \mathcal{G})$ is a stratified $\mathbb{H}$-filter on X.

We just recall below the notion of stratified $\mathbb{H}$-neighborhood system, for further details including the notion of stratified $\mathbb{H}$-topology, we refer to [19] (see also [20]).

Definition 2.4. ([19]) A pair $(X, \mathfrak{N} = (\mathfrak{N}^x)_{x\in X})$ is called a *stratified* $\mathbb{H}$*-neighborhood topological space*, where $\mathfrak{N}$ is a family of stratified $\mathbb{H}$-filters on a nonempty set X satisfying the following:

(HN1) $\forall x \in X, \mathfrak{N}^x \le [x]$;

(HN2) $\forall x \in X$, and $\forall \nu \in \mathbb{H}^X$, $\mathfrak{N}^x(\nu) = \bigvee\{\mathfrak{N}^x(\xi)|\xi \in \mathbb{H}^X, \xi(y) \le \mathfrak{N}^y(\nu), \forall y \in X\}$.

A map $f\colon (X, \mathfrak{N}) \longrightarrow (X', \mathfrak{N}')$ between stratified $\mathbb{H}$-neighborhood topological spaces is said to be *continuous* at $x \in X$ if $\mathfrak{N}'^{f(x)} \le f^{\Rightarrow}(\mathfrak{N}^x)$. It is continuous if it is continuous at each point of X.

Definition 2.5. ([21, 25]) Let X be a nonempty set and $\lim : \mathcal{F}^s_{\mathbb{H}}(X) \longrightarrow \mathbb{H}^X$ a map satisfying the following:

(LGC1) $\forall x \in X$: $\lim[x](x) = \top$;

(LGC2) $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}^s_{\mathbb{H}}(X)$ with $\mathcal{F} \le \mathcal{G}$ implies $\lim \mathcal{F}(x) \le \lim \mathcal{G}(x)$, $\forall x \in X$,

then the pair $(X, \lim)$ is called a *stratified* $\mathbb{H}$*-generalized convergence space*.

• A stratified $\mathbb{H}$-generalized convergence space is called *stratified* $\mathbb{H}$*-convergence space* if it satisfies

(LCS) $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}^s_{\mathbb{H}}(X)$, $\forall x \in X$; $\lim \mathcal{F}(x) \wedge \lim \mathcal{G}(x) \le \lim(\mathcal{F} \wedge \mathcal{G})(x)$.

• A stratified $\mathbb{H}$-generalized convergence space $(X, \lim)$ is called *stratified* $\mathbb{H}$*-Choquet convergence space* if it satisfies

(LCC) $\forall \mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$, $\forall x \in X$, $\lim \mathcal{F}(x) = \bigwedge\{\lim \mathcal{U}(x) : \mathcal{U} \in \mathcal{F}^{su}_{\mathbb{H}}(X), \mathcal{U} \ge \mathcal{F}\}$.

• A stratified $\mathbb{H}$-generalized convergence space $(X, \lim)$ is called *stratified* $\mathbb{H}$*-pretopological convergence space* if it satisfies

(Lp) $\forall \mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$, $\forall x \in X : \lim \mathcal{F}(x) = \bigwedge_{\nu\in\mathbb{H}^X}(\mathfrak{N}^x(\nu) \to \mathcal{F}(\nu))$,

where $\forall \nu \in \mathbb{H}^X$: $\mathfrak{N}^x(\nu) = \bigwedge_{\mathcal{G}\in\mathcal{F}^s_{\mathbb{H}}(X)}(\lim \mathcal{G}(x) \to \mathcal{G}(\nu))$.

A map $f : (X, \lim) \longrightarrow (X', \lim')$ between stratified $\mathbb{H}$-generalized convergence spaces (resp. stratified $\mathbb{H}$-convergence spaces, stratified $\mathbb{H}$-pretopological convergence spaces, stratified $\mathbb{H}$-Choquet convergence spaces), is said to be *continuous* if and only if $\forall \mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$, $\forall x \in X$: $\lim \mathcal{F}(x) \leq \lim' f^{\Rightarrow}(\mathcal{F})(f(x))$. Note that every stratified $\mathbb{H}$-pretopological convergence space is a stratified $\mathbb{H}$-convergence space (cf.[23]).

For a given source $\left(f_j : X \longrightarrow \left(X, \lim_j\right)\right)_{j \in J}$, the *initial structure* on X is defined in [21] for any $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$ and $x \in X$ by: $\lim \mathcal{F}(x) = \bigwedge_{j \in J} \lim_j f_j(\mathcal{F})(f_j(x))$.

Special examples of such a structure are *subspaces* and *product spaces*. As for subspace, consider $(X, \lim)$, a stratified $\mathbb{H}$-generalized convergence space, $A \subseteq X$ and $\iota_A : A \hookrightarrow X$, $x \mapsto x$, an inclusion mapping, then the *initial structure* on A written as $\lim|_A$ is given for any $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(A)$ and $x \in X$ by: $\lim|_A \mathcal{F}(x) = \lim \iota_A(\mathcal{F})(\iota_A(x)) = \lim[\mathcal{F}](x)$.

For product space, we just consider the projection mappings $pr_i : \prod_{j \in J} X_j \longrightarrow X_i$. In particular, if $(X, \lim)$, $(Y, \lim)$ are stratified $\mathbb{H}$-generalized convergence spaces, then *their product* $(X \times Y, \lim_X \times \lim_Y)$ is a stratified $\mathbb{H}$-generalized convergence space [21], where $\lim_X \times \lim_Y : \mathcal{F}^s_{\mathbb{H}}(X \times Y) \longrightarrow \mathbb{H}^{X \times Y}$ is defined for any $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X \times Y)$ by:

$$(\lim_X \times \lim_Y)\mathcal{F} = pr_X^{\leftarrow}(\lim_X pr^{\Rightarrow}(\mathcal{F})) \wedge pr_Y^{\leftarrow}\left(\lim_Y pr_Y^{\Rightarrow}(\mathcal{F})\right).$$

Definition 2.6. ([25]) Let $(X, \lim)$ be a stratified $\mathbb{H}$-generalized convergence space. Then $(X, \lim)$ is called *Hausdorff-separated* or *T_2-space* if and only if for all $x, y \in X$, $\forall \mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$, $\lim \mathcal{F}(x) = \lim \mathcal{F}(y) = \top$ implies $x = y$.

We recall a characterization of Hausdorff-separated space from [Lemma 4.4[25]]: A stratified $\mathbb{H}$-pretopological convergence space $(X, \lim)$ is Hausdorff-separated if and only if $\forall x, y \in X$: $\mathfrak{N}^x \vee \mathfrak{N}^y \in \mathcal{F}^s_{\mathbb{H}}(X)$ implies $x = y$.

Definition 2.7. ([27, 28, 30]) (1) If $(X, \lim)$ is a stratified $\mathbb{H}$-generalized convergence space and $A \subseteq X$, and $x \in X$, then the lim-*closure* of A, denoted by $\overline{A}^{\lim}$, is defined as follows:

$x \in \overline{A}^{\lim}$ if there exists a $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$ such that $\lim \mathcal{F}(x) = \top$ and $\mathcal{F}(\top_A) = \top$.

(2) $A \subseteq X$ is called $\top$-*closed* in $(X, \lim)$ if for $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$, $\lim \mathcal{F}(x) = \top$ and $\mathcal{F}(\top_A) = \top$ implies $x \in A$.

Remark that A is $\top$-closed if and only if $\overline{A}^{\lim} \subseteq A$. Furthermore, note that as it is pointed out in [28], $\top$-closedness of A can be characterized by stratified $\mathbb{H}$-ultrafilters, in which case A is $\top$-closed if and only if for all $\mathcal{U} \in \mathcal{F}^{su}_{\mathbb{H}}(X)$, $\lim \mathcal{U}(x) = \top$ and $\mathcal{U}(\top_A) = \top$ implies $x \in A$.

Definition 2.8. ([27]) A stratified $\mathbb{H}$-generalized convergence space $(X, \lim)$ is called *compact* if and only if for all $\mathcal{U} \in \mathcal{F}^{su}_{\mathbb{H}}(X)$ there exists $x \in X$ such that $\lim \mathcal{U}(x) = \top$. If $A \subseteq X$, then A is called *compact* if the subspace $(A, \lim_{|_A})$ of $(X, \lim)$ is compact (see cf. [27]).

Lemma 2.9. ([27]) *Let $(X, \lim)$ be a stratified $\mathbb{H}$-generalized convergence space and $A \subseteq X$. Then A is compact if and only if for every stratified $\mathbb{H}$-ultrafilter $\mathcal{U} \in \mathcal{F}^{su}_{\mathbb{H}}(X)$ with $\mathcal{U}(\top_A) = \top$ there is an $x \in A$ such that $\lim \mathcal{U}(x) = \top$*

Lemma 2.10. ([26, 27]) *Let $(X, \lim)$ and $(Y, \lim')$ be stratified $\mathbb{H}$-generalized convergence spaces, and $f : (X, \lim) \longrightarrow (Y, \lim')$ be continuous. If $A \subseteq X$ is compact, then $f(A)$ is a compact subset of $(Y, \lim')$.*

Definition 2.11. ([9, 12]) Let $(X, \cdot)$ be a semigroup. A *subsemigroup* T of a semigroup X is a non-empty subset T of X such that $TT \subseteq T$. A *subgroup* of a semigroup X is a nonempty subset A of X such that $xA = Ax = A$ for each $x \in A$. An element e in X is called an *idempotent* if and only if $e^2 = e$. The set of all idempotents is denoted by $\mathbf{E}(X)$. If X contains an idempotent e, then $\{e\}$ is a subgroup of X, and is contained in a maximal subgroup. We denote $\mathbf{H}(e)$ the maximal subgroup of X containing the idempotent e.

Definition 2.12. ([9, 12]) A semigroup $(X, \cdot)$ is called *right simple* if $aX = X$ for all $a \in X$; equivalently, for all $a, b \in X$ there exists $x \in X$ such that $ax = b$; it is called *left simple* if $Xa = X$ for all $a \in X$; equivalently, for all $a, b \in X$ there exists $x \in X$ such that $xa = b$. It is called *simple* if it is both left simple and right simple.

A semigroup $(X, \cdot)$ is called *right group* if and only if X is both left cancellative, i.e., $zx = zy$ implies that $x = y$, and right simple.

3. Complete Heyting Algebra-Valued Convergence Semigroups

Definition 3.1. A triple $(X, \cdot, \lim)$ is called a *stratified* $\mathbb{H}$*-generalized convergence semigroup*(resp. *stratified* $\mathbb{H}$*-convergence semigroup*, *stratified* $\mathbb{H}$*-pretopological semigroup*, *stratified* $\mathbb{H}$*-Choquet convergence semigroup*) if the following are fulfilled:

(LCSG1) $(X, \cdot)$ is a semigroup;

(LCSG2) $(X, \lim)$ is a stratified $\mathbb{H}$-generalized convergence space (resp. stratified $\mathbb{H}$-convergence space, stratified $\mathbb{H}$-pretopological convergence space, stratified $\mathbb{H}$-Choquet convergence space);

(LCSG3) the multiplication $m : X \times X \longrightarrow X, (x, y) \mapsto xy$ is continuous; equivalently, for all $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_L(X)$, for all $x, y \in X$: $\lim \mathcal{F}(x) \wedge \mathcal{G}(y) \leq \lim \mathcal{F} \odot \mathcal{G}(xy)$.

Lemma 3.2. *Let* $(X, \cdot, \lim)$ *be a stratified* $\mathbb{H}$*-generalized convergence semigroup, and* $a \in X$*. Then both the left and the right translations by* a*,* ${}^a\varrho \colon X \longrightarrow X, x \mapsto ax$*, and* $\varrho_a \colon X \longrightarrow X, x \mapsto xa$ *are continuous.*

Proof. Fix $a \in X$. Then for any $x \in X$ and $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$, $\lim \mathcal{F}(x) = \lim[a](a) \wedge \lim \mathcal{F}(x) \leq \lim\left([a] \odot \mathcal{F}\right)(ax) \leq \lim\ {}^a\varrho^{\Rightarrow}(\mathcal{F})({}^a\varrho(x))$; in fact, upon using stratification (S$\mathbb{H}$), for any $\nu \in \mathbb{H}^X$, $\left([a] \odot \mathcal{F}\right)(\nu) = \bigvee_{\nu_1 \odot \nu_2 \leq \nu} [a](\nu_1) \wedge \mathcal{F}(\nu_2) = \bigvee_{\nu_1 \odot \nu_2 \leq \nu} \nu_1(a) \wedge \mathcal{F}(\nu_2) \leq \bigvee_{\nu_1 \odot \nu_2 \leq \nu} \mathcal{F}\left(\nu_1(a) \wedge \nu_2\right) \leq \bigvee_{\nu_1(a) \wedge \nu_2 \leq \nu \circ {}^a\varrho} \mathcal{F}\left(\nu_1(a) \wedge \nu_2\right) \leq \mathcal{F}(\nu \circ {}^a\varrho) = {}^a\varrho^{\Rightarrow}(\mathcal{F})(\nu)$. Similarly, the continuity of ϱ_a follows from the observation: $\lim \mathcal{F}(x) = \lim \mathcal{F}(x) \wedge \lim[a](a) \leq \lim\left(\mathcal{F} \odot [a]\right)(xa) \leq \lim \varrho_a^{\Rightarrow}(\mathcal{F})(\varrho_a(x))$. □

Lemma 3.3. *Let* $(X, \cdot, \lim)$ *be a Hausdorff-separated stratified* $\mathbb{H}$*-generalized convergence semigroup and* A, B *are subsets of* X*. Then the following holds:*

(a) If B *is* $\top$*-closed in* $(X, \lim)$*, then* $\{x \in X : xA \subseteq B\}$ *is* $\top$*-closed.*

(b) If B *is compact, then* $\{x \in X : A \subseteq xB\}$ *is* $\top$*-closed.*

(c) If B *is compact, then* $\{x \in X : xA \subseteq Bx\}$ *is* $\top$*-closed.*

Proof. (a) Denote $C := \{x \in X : xA \subseteq B\}$ with B is $\top$-closed in $(X, \lim)$, and let $x \in \overline{C}^{\lim}$. Then there exists $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$ such that $\lim \mathcal{F}(x) = \top$ with $\mathcal{F}(\top_C) = \top$. Let $a \in A$. Then it follows from the continuity of $x \mapsto xa$ that $\lim(\mathcal{F} \odot [a])(xa) = \top$. Now note that due to Lemma 2.6[28], as $\{x\}$ is $\top$-closed, one can deduce $\mathcal{F}(\top_{\{x\}}) = \top$, and since $\top_{\{x\}} \cdot \top_A = \top_{xA} \leq \top_B$, we obtain: $\left(\mathcal{F} \odot [a]\right)(\top_B) = \bigvee\{\mathcal{F}(\nu_1) \wedge [a](\nu_2) \colon \nu_1 \odot \nu_2 \leq \top_B\} \geq \mathcal{F}(\top_{\{x\}}) \wedge \top_A(a) = \top$, i.e., $\left(\mathcal{F} \odot [a]\right)(\top_B) = \top$. As B is $\top$-closed, we have $xa \in B$. This means that $x \in C$.

(b) Let $K := \{x \in X : A \subseteq xB\}$, and B be a compact subset of X. We show K is $\top$-closed. For, let $x \in \overline{K}^{\lim}$. Then there exists a $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$ such that $\lim \mathcal{F}(x) = \top$ and $\mathcal{F}(\top_K) = \top$. Let $\mathcal{U} \in \mathcal{F}^{su}_{\mathbb{H}}(X)$ such that $\mathcal{U} \geq \mathcal{F}$. Take $a \in A$, then $\{a\}$ is $\top$-closed, and so, $\lim \mathcal{F}(a) = \top$ but then $\lim \mathcal{U}(a) = \top$. Since B is compact, xB compact subset of X, and hence there exists $z \in xB$ such that $\lim \mathcal{U}(z) = \top$. As $z \in xB$, there exits $b \in B$ such that $z = xb$. Then $\lim \mathcal{U}(xb) = \top$. But X is Hausdorff-separated so, $a = xb$. This means $A \subseteq xB$ and hence $x \in K$.

(c) Set $F := \{x \in X : xA \subseteq Bx\}$, and let $x \in \overline{F}^{\lim}$. Then there exists $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$ such that $\lim \mathcal{F}(x) = \top$ and $\mathcal{F}(\top_F) = \top$. If $y \in xA$, then there is $a \in A$ such that $y = xa$. Choose $\mathcal{U} \in \mathcal{F}^{su}_{\mathbb{H}}(X)$ such that $\mathcal{U} \geq \mathcal{F}$. Then since B is compact, there exists $z \in B$ such that $\lim \mathcal{U}(z) = \top$. Then due to continuity of $x \mapsto zx$, $\lim \mathcal{U} \odot \mathcal{F}(zx) = \top$. Also, we have $\lim \mathcal{U} \odot \mathcal{F}(xa) = \top$. Since X is Hausdorff-separated, it follows from $\lim \mathcal{U} \odot \mathcal{F}(zx) = \lim \mathcal{U} \odot \mathcal{F}(xa) = \top$ implies $zx = xa$, but then $y = zx$, showing $xA \subseteq Bx$, hence $x \in F$. □

In classical theory of topological semigroups, the following result is known as Ellis's Lemma[14](see also [41]). This result has been incorporated for classical convergence semigroup in [17].

Theorem 3.4. *Let* $(X, \cdot, \lim)$ *be a compact Hausdorff-separated stratified* $\mathbb{H}$*-generalized convergence semigroup. Then it contains an idempotent element.*

Proof. Let $\mathcal{S}$ denote the set of $\top$-closed subsemigroups of X, i.e., $\mathcal{S} = \{A \subseteq X \colon AA \subseteq A, \overline{A}^{\lim} = A\}$. Since it follows from Lemma 3.5[27] that X itself is a $\top$-closed subsemigroup, we have $X \in \mathcal{S}$; so $\mathcal{S} \neq \emptyset$. If $\mathcal{S}$ is partially ordered by reverse inclusion, then by Zorn's lemma, there exists a minimal element K in $\mathcal{S}$. Let

$y \in K$. Then since K is $\top$-closed subsemigroup of X, by Corollary 3.4[27], K is compact, and by continuity of the mapping $f\colon X \longrightarrow X, x \mapsto yx$, yK is compact subset of X, and therefore, again by Lemma 3.5[27], yK is a $\top$-closed subsemigroup of X such that $yK \subseteq K$. That yK is a subsemigroup follows from $(yK)(yK) \subseteq yK$, which is true because of the fact that $(yx_1)(yx_2) = y(x_1yx_2) = yx_3 \in yK$, $x_1, x_2, x_3 \in K$. Hence by minimality of K, we get $yK = K$. Similarly, $Ky = K$. Thus, K is a subgroup of X. If e is the identity of K, then again by minimality, $K = \{e\}$. $\square$

The following lemma in classical topological semigroups is known as *Swelling Lemma* (see Lemma 1.9[9]). We generalize it in the context of compact Hausdorff-separated stratified $\mathbb{H}$-generalized convergence semigroup, in which case, a part of the classical proof will remain as it is.

Lemma 3.5. *Let $(X, \cdot, \lim)$ be a compact Hausdorff-separated stratified $\mathbb{H}$-generalized convergence semigroup. If A is a $\top$-closed subset of X, $t \in X$ and $A \subseteq tA$, then $A = tA$.*

Proof. Let $K := \{x \in X \colon tA \subseteq xA\}$. Then clearly T is a subsemigroup of X and in view of Preceding Lemma 3.3(c), we have K is $\top$-closed, and hence by Theorem 3.4, it contains an idempotent e. Hence the rest of the proof, mainly, algebraic part follows from the proof of the classical Swelling Lemma 1.9[9]. $\square$

Theorem 3.6. *Let $(X, \cdot, \lim)$ be a compact Hausdorff-separated stratified $\mathbb{H}$-pretopological convergence semigroup. Then*

(a) The set of all idempotents $\mathbf{E}(X)$ of X is $\top$-closed, and hence compact.

(b) Every maximal subgroup $\mathbf{H}(e)$ of X is $\top$-closed and hence compact.

Proof. (a) Let $x \in \overline{\mathbf{E}(X)}^{\lim}$ and $x^2 \neq x$. Then by Lemma 4.4[25], $\mathfrak{N}^{x^2} \vee \mathfrak{N}^x \notin \mathcal{F}^s_{\mathbb{H}}(X)$. This implies that there are $\nu_1, \nu_2 \in \mathbb{H}^X$ such that $\nu_1 \wedge \nu_2 = \perp_X$ and $\mathfrak{N}^{x^2}(\nu_1) \wedge \mathfrak{N}^x(\nu_2) \neq \perp$... ... ...(I).

Since $x \in \overline{\mathbf{E}(X)}^{\lim}$, there exists $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$ such that $\lim \mathcal{F}(x) = \top$ and $\mathcal{F}(\top_{\mathbf{E}(X)}) = \top$. Then $\mathfrak{N}^x \leq \mathcal{F}$; and since the map $(x, x) \mapsto x^2$ is continuous, we have $\lim \mathcal{F}(x) \wedge \lim \mathcal{F}(x) \leq \lim (\mathcal{F} \odot \mathcal{F})(x^2)$, which implies that $\lim (\mathcal{F} \odot \mathcal{F})(x^2) = \top$. Consequently, $\mathfrak{N}^{x^2} \leq \mathcal{F} \odot \mathcal{F}$. But it follows that $\mathcal{F} \odot \mathcal{F} \leq \mathcal{F}$, which by (LGC2), $\lim \mathcal{F}(x^2) = \top$, and so, $\mathfrak{N}^{x^2} \leq \mathcal{F}$. In fact, for any $\xi \in \mathbb{H}^X$, $\mathcal{F} \odot \mathcal{F}(\xi) = \bigvee\{\mathcal{F}(\xi_1) \wedge \mathcal{F}(\xi_2) \colon \xi_1 \odot \xi_2 \leq \xi\} \leq \bigvee\{\mathcal{F}(\xi_1 \wedge \xi_2) \colon \xi_1 \wedge \xi_2 \leq \xi_1 \odot \xi_2 \leq \xi\} \leq \bigvee\{\mathcal{F}(\xi_1 \wedge \xi_2) \colon \xi_1 \wedge \xi_2 \leq \xi\} \leq \mathcal{F}(\xi)$. This is so, because of idempotency, and the fact that for any $z \in X$, $\xi_1 \odot \xi_2(z) = \bigvee_{st=z} \xi_1(s) \wedge \xi_2(t) \geq \bigvee_{s=s^2=z} \xi_1(s) \wedge \xi_2(s) = \xi_1 \wedge \xi_2(z)$; i.e., $\xi_1 \wedge \xi_2 \leq \xi_1 \odot \xi_2$. Thus, we have $\mathfrak{N}^{x^2}(\nu_1) \wedge \mathfrak{N}^x(\nu_2) \leq \mathcal{F}(\nu_1) \wedge \mathcal{F}(\nu_2) \leq \mathcal{F}(\nu_1 \wedge \nu_2) = \mathcal{F}(\perp_X) = \perp$ implying that $\mathfrak{N}^{x^2}(\nu_1) \wedge \mathfrak{N}^x(\nu_2) = \perp$ ... ...(II), a contradiction with (I), and hence by Lemma 4.4[25], we have $x^2 = x$, i.e., $x \in \mathbf{E}(X)$. The compactness of $\mathbf{E}(X)$ follows from Corollary 3.4[27].

(b) For the fact that $\mathbf{H}(e) = \{x \in eXe : (\exists y \in eXe)\, xy = yx = e\}$ is the largest subgroup of X having e as its identity follows from [40]. We only show that $\mathbf{H}(e)$ is $\top$-closed. To this end, let $x \in \overline{\mathbf{H}(e)}^{\lim}$. Then there exists $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$ such that $\lim \mathcal{F}(x) = \top$ and $\mathcal{F}\left(\top_{\mathbf{H}(e)}\right) = \top$. Choose $\mathcal{U} \in \mathcal{F}^{su}_{\mathbb{H}}(X)$ such that $\mathcal{U} \geq \mathcal{F}$. As X is compact, there exists $y \in X$ such that $\lim \mathcal{U}(y) = \top$. Because of continuity of the mapping $m : X \times X \longrightarrow X, (x, y) \mapsto xy$, $\lim \mathcal{F}(x) \wedge \lim \mathcal{U}(y) \leq \lim(\mathcal{F} \odot \mathcal{U})(xy) \Rightarrow \lim \mathcal{F} \odot \mathcal{U}(xy) = \top$, $x, y \in X$. Since $\varphi : X \longrightarrow X, x \mapsto exe$ is continuous, it follows from [26] that eXe is compact subset of X, and by Lemma 3.5[27], eXe is $\top$-closed; as $\mathbf{H}(e) \subseteq eXe$, we have $\mathcal{U}(\top_{eXe}) = \top$, therefore, $y \in eXe$. Also, note that $\lim \mathcal{F} \odot \mathcal{U}(e) = \top$. Thus $\lim \mathcal{F} \odot \mathcal{U}(xy) = \lim \mathcal{F} \odot \mathcal{U}(e) = \top$, But as X is Hausdorff-separated, $xy = e$; similarly, $yx = e$ implying that $x \in \mathbf{H}(e)$, showing that $\overline{\mathbf{H}(e)}^{\lim} \subseteq \mathbf{H}(e)$. So, $\mathbf{H}(e)$ is $\top$-closed, and hence by Corollary 3.4[27], it is is compact. $\square$

Theorem 3.7. *Let $(X, \cdot, \lim)$ and $(Y, \cdot, \lim')$ be compact Hausdorff-separated stratified $\mathbb{H}$-convergence semigroups and $f\colon X \longrightarrow Y$ be a surjective continuous semigroup-homomorphism. Then $f(\mathbf{E}(X)) = \mathbf{E}(Y)$.*

Proof. We only check $\mathbf{E}(Y) \subseteq f(\mathbf{E}(X))$. For, let $e \in \mathbf{E}(Y)$. Then since $\{e\}$ is $\top$-closed by Lemma 2.6[28]; so, by Corollary 2.13[30], $f^{-1}(\{e\})$ is $\top$-closed which in view of Corollary 3.4[27] is compact, and then a subsemigroup of X. Consequently, by Theorem 3.4, it contains an idempotent element, say e'. So, $f(e') = e$. Therefore, $\mathbf{E}(Y) \subseteq f(\mathbf{E}(X))$. $\square$

Proposition 3.8. *Let $(X, \cdot, \lim)$ be a stratified $\mathbb{H}$-convergence semigroup, and A be a subsemigroup of X. Then $\overline{A}^{\lim}$ is a subsemigroup of X.*

Proof. Let $x, y \in \overline{A}^{\lim}$. Then there are $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_{\mathbb{H}}(X)$ such that $\lim \mathcal{F}(x) = \top$, $\lim \mathcal{G}(y) = \top$ and $\mathcal{F}(\top_A) = \top$ and $\mathcal{G}(\top_A) = \top$. Since $m : X \times X \longrightarrow X$, $(x, y) \mapsto xy$ is continuous, $\lim \mathcal{F}(x) \wedge \lim \mathcal{G}(y) \leq \lim \mathcal{F} \odot \mathcal{G}(xy)$ implying that $\lim \mathcal{F} \odot \mathcal{G}(xy) = \top$. Since A is a subsemigroup, we have $AA \subseteq A$ and hence $\top_A \odot \top_A = \top_{AA} \leq \top_A$ which implies that $\mathcal{F} \odot \mathcal{G}(\top_A) = \bigvee\{\mathcal{F}(\nu_1) \wedge \mathcal{G}(\nu_2) : \nu_1, \nu_2 \in \mathbf{H}^X, \nu_1 \odot \nu_2 \leq \top_A\} \geq \mathcal{F}(\top_A) \wedge \mathcal{G}(\top_A) = \top$ yields that $\mathcal{F} \odot \mathcal{G}(\top_A) = \top$. But in view of Proposition 3.3[2], we know $\mathcal{F} \odot \mathcal{G} \in \mathcal{F}^s_{\mathbf{H}}(X)$, by setting $\mathcal{H} := \mathcal{F} \odot \mathcal{G} \in \mathcal{F}^s_{\mathbf{H}}(X)$, we have $\lim \mathcal{H}(xy) = \top$ and $\mathcal{H}(\top_A) = \top$ which implies that $xy \in \overline{A}^{\lim}$. $\square$

Corollary 3.9. *Let $(X, \cdot, \lim)$ be a stratified $\mathbb{H}$-convergence semigroup and $A \subset X$ be a maximal subsemigroup of X. Then A is $\top$-closed.*

Proof. By Proposition 3.8, $\overline{A}^{\lim}$ is a subsemigroup of X. Due to Lemma 2.7(2)[30], $A \subseteq \overline{A}^{\lim}$, which by maximality of A coincides with $\overline{A}^{\lim}$. Hence A is $\top$-closed. $\square$

Theorem 3.10. *Let $(X, \cdot, \lim)$ be a Hausdorff-separated stratified $\mathbb{H}$-convergence semigroup and G a subgroup of X. Then $\overline{G}^{\lim}$ is a subsemigroup of X with identity.*

Proof. Due to Proposition 3.8, it follows immediately that $\overline{G}^{\lim}$ is a subsemigroup of X. To show the remaining part, let $e \in G$ be the identity element of G, and assume $x \in \overline{G}^{\lim}$. Then there is $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$ such that $\lim \mathcal{F}(x) = \top$. and $\mathcal{F}(\top_G) = \top$. By continuity of the map $(e, x) \mapsto ex$, we have $\top = \lim[e](e) \wedge \lim \mathcal{F}(x) \leq \lim([e] \odot \mathcal{F})(ex)$ implies $\lim([e] \odot \mathcal{F})(ex) = \top$. Now as $\top_e \odot \top_G = \top_{eG} = \top_G$, we have $([e] \odot \mathcal{F})(\top_G) = \bigvee\{[e](\nu_1) \wedge \mathcal{F}(\nu_2) : \nu_1, \nu_2 \in \mathbb{H}^X, \nu_1 \odot \nu_2 \leq \top_G\} \geq [e](\top_e) \wedge \mathcal{F}(\top_G) = \top$, implying $([e] \odot \mathcal{F})(\top_G) = \top$. Also, for any $\nu \in \mathbb{H}^X$, we have $([e] \odot \mathcal{F})(\nu) = \bigvee\{[e](\nu_1) \wedge \mathcal{F}(\nu_2) : \nu_1, \nu_2 \in \mathbb{H}^X, \nu_1 \odot \nu_2 \leq \nu\} \geq [e](\top_e) \wedge \mathcal{F}(\nu) = \top_e(e) \wedge \mathcal{F}(\nu) = \top \wedge \mathcal{F}(\nu) = \mathcal{F}(\nu)$ implies that $\mathcal{F} \leq ([e] \odot \mathcal{F})$. Then by using (LGC2), $\lim \mathcal{F}(x) \leq \lim([e] \odot \mathcal{F})(x)$ for any $x \in X$. This yields that $\lim([e] \odot \mathcal{F})(x) = \top$. Thus we arrive at $\lim([e] \odot \mathcal{F})(ex) = \top$ and also, $\lim([e] \odot \mathcal{F})(x) = \top$. Since $[e] \odot \mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$, and X is Hausdorff, we have $ex = x$. Similarly, $xe = x$, which in conjunction with Lemma 2.7(2)[30] ensures that $e \in \overline{G}^{\lim}$. $\square$

Theorem 3.11. *Let $(X, \cdot, \lim)$ be a compact Hausdorff-separated stratified $\mathbb{H}$-convergence semigroup and Y a subgroup of X. Then $\overline{Y}^{\lim}$ is also a subgroup of X.*

Proof. In view of Proposition 3.8, it suffices to show that, if $x \in \overline{Y}^{\lim}$, then there exists $x^{-1} \in \overline{Y}^{\lim}$ such that $x^{-1}x = x^{-1}x = e$. Let $y \in \overline{Y}^{\lim}$. Then there exists $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$ such that $\lim \mathcal{F}(y) = \top$ and $\mathcal{F}(\top_Y) = \top$. Clearly, $\jmath^{\Rightarrow}(\mathcal{F}) = \mathcal{F}^{-1} \in \mathcal{F}^s_{\mathbb{H}}(X)$. Choose $\mathcal{U} \geq \mathcal{F}^{-1}$. Since $(X, \lim)$ is compact, there exists an $h \in X$ such that $\lim \mathcal{U}(h) = \top$. As because Y is a subgroup of X, we have $\mathcal{U}(\top_Y) \geq \mathcal{F}^{-1}(\top_Y) = \mathcal{F}(\top_{Y^{-1}}) \geq \mathcal{F}(\top_Y) = \top$, i.e., $\mathcal{U}(\top_Y) = \top$. This together with $\lim \mathcal{U}(h) = \top$ imply that $h \in \overline{Y}^{\lim}$. Now we have $\lim \mathcal{U} \odot \mathcal{F}(hy) \geq \lim \mathcal{U}(h) \wedge \lim \mathcal{F}(y) = \top$, implying $\lim \mathcal{U} \odot \mathcal{F}(hy) = \top$. Since $\jmath^{\Rightarrow}(\mathcal{F}) \leq \mathcal{U}$, then in view of Lemma 2.3, there exists $\mathcal{G} \in \mathcal{F}^{su}_{\mathbb{H}}(Y)$ such that $\mathcal{G} \geq \mathcal{F}$ and $\jmath^{\Rightarrow}(\mathcal{G}) = \mathcal{U}$. Then it follows that $\mathcal{U} \odot \mathcal{F} = \jmath^{\Rightarrow}(\mathcal{G}) \odot \mathcal{F} \leq \mathcal{G}^{-1} \odot \mathcal{G} \leq [e]$ which yields that $\lim[e](hy) \geq \lim \mathcal{U} \odot \mathcal{F}(hy) = \top$, i.e., $\lim[e](hy) = \top$ but as we know $\lim[e](e) = \top$ implies $hy = e$ because of Hausdorffness. Similarly, $yh = e$, and hence $h = y^{-1} \in Y$. This proves $\overline{Y}^{\lim}$ is a subgroup of X. $\square$

Corollary 3.12. *Let $(X, \cdot, \lim)$ be a compact Hausdorff-separated stratified $\mathbb{H}$-convergence semigroup and A be a $\top$-closed subsemigroup of X, then A is a subgroup of X*

Theorem 3.13. *Let $(X, \cdot, \lim)$ be a compact Hausdorff-separated stratified $\mathbb{H}$-Choquet convergence semigroup such that $(X, \cdot)$ is algebraically a group. Then $(X, \cdot, \lim)$ is a stratified $\mathbb{H}$-convergence group.*

Proof. Let $\jmath : X \longrightarrow X, x \mapsto x^{-1}$, $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$ and $x \in X$. Then $\jmath^{\Rightarrow}(\mathcal{F}) \in \mathcal{F}^s_{\mathcal{H}}(X)$. Let $\mathcal{U} \in \mathcal{F}^{su}_{\mathbb{H}}(X)$ such that $\mathcal{U} \geq \jmath^{\Rightarrow}(\mathcal{F})$. Since $(X, \lim)$ is compact, there exists $y \in X$ such that $\lim \mathcal{U}(y) = \top$, and due to continuity of semigroup operation, we have $\top = \lim[x](x) \wedge \lim \mathcal{U}(y) \leq \lim \mathcal{F}(x) \wedge \lim \mathcal{U}(y) \leq \lim \mathcal{F} \odot \mathcal{U}(xy)$ implying that $\lim \mathcal{F} \odot \mathcal{U}(xy) = \top$. Now it ensures from $\mathcal{U} \geq \jmath^{\Rightarrow}(\mathcal{F})$ in conjunction with Lemma 2.3 (see Lemma 3.7[22]) that there exists $\mathcal{G} \in \mathcal{F}^{su}_{\mathbb{H}}(X)$ such that $\mathcal{G} \geq \mathcal{F}$ implying $\jmath^{\Rightarrow}(\mathcal{G}) = \mathcal{U}$. This yields that $\mathcal{F} \odot \mathcal{G}^{-1} \leq \mathcal{G} \odot \mathcal{G}^{-1} \leq [e]$. But then $\mathcal{F} \odot \mathcal{U} = \mathcal{F} \odot \mathcal{G}^{-1} \leq [e]$. So, $\lim[e](xy) \geq \lim \mathcal{F} \odot \mathcal{U}(xy) = \top$ implying $\lim[e](xy) = \top$. But we know that $\lim[e](e) = \top$, hence $xy = e$, since $(X, \lim)$ is Hausdorff-separated. Similarly, one obtains $yx = e$ which implies $y = x^{-1}$. Now as $(X, \lim)$ is a stratified $\mathbb{H}$-Choquet convergence space, we have $\lim \mathcal{F}(x) \leq \top = \bigwedge_{\mathcal{U} \in \mathcal{F}^{su}_{\mathbb{H}}, \mathcal{U} \geq \jmath^{\Rightarrow}(\mathcal{F})} \lim \mathcal{U}(y) = \lim \jmath^{\Rightarrow}(\mathcal{F})(y)$, i.e., $\lim \mathcal{F}(x) \leq \lim \jmath^{\Rightarrow}(\mathcal{F})(x^{-1})$, proving that $\jmath$ is continuous. Hence $(X, \cdot, \lim)$ is a stratified $\mathbb{H}$-convergence group. $\square$

Proposition 3.14. *If $(X, \cdot, \lim)$ is a compact Hausdorff-separated stratified $\mathbb{H}$-convergence semigroup, then each maximal subgroup is $\top$-closed.*

Proof. Let $(X, \cdot, \lim)$ be a compact stratified $\mathbb{H}$-convergence semigroup and Y a maximal subgroup of X. Then in view of Theorem 3.11, $\overline{Y}^{\lim}$ is a subgroup of X and $Y \subseteq \overline{Y}^{\lim} \subseteq X$. Since $\overline{Y}^{\lim}$ is a subgroup of X and Y is maximal, we have $\overline{Y}^{\lim} = Y$. This shows that Y is $\top$-closed. $\square$

Definition 3.15. Let X be a right group. For each idempotent e of X, let $\varphi_e : X \longrightarrow X$ be defined by $\varphi_e(x) = (ex)^{-1}$. Then $(X, \cdot, \lim)$ is called a *stratified $\mathbb{H}$-convergence right group* if and only if φ_e is continuous for every idempotent e in X.

Theorem 3.16. *Let $(X, \cdot, \lim)$ be a compact Hausdorff-separated stratified $\mathbb{H}$-Choquet convergence semigroup. If X is right simple, then $(X, \cdot, \lim)$ is a stratified $\mathbb{H}$-convergence right group.*

Proof. By a result in Clifford-Preston [12] in conjunction with Theorem 3.4, X is a right group since it is right simple and contains an idempotent. For each idempotent e of X, let $\varphi_e : X \longrightarrow X$ be defined by $\varphi_e(x) = (ex)^{-1}$. Let $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$ and $x \in X$. Then $\varphi_e^{\Rightarrow}(\mathcal{F}) \in \mathcal{F}^s_{\mathbb{H}}(Xe)$. Let $\mathcal{U}$ be an ultrafilter on Xe such that $\mathcal{U} \geq \varphi_e^{\Rightarrow}(\mathcal{F})$. Since Xe is compact, because of X is compact, and $X = Xe$, as X is a right group. Then there exists $y \in Xe$ such that $\lim \mathcal{U}(y) = \top$. Now $\lim(\mathcal{U} \odot \mathcal{F})(yx) = \top$ and also, $\lim(\mathcal{U} \odot \mathcal{F})(e) = \top$ implying $yx = e$ since X is Hausdorff-separated. Thus $y = (ex)^{-1}$. Now as $(X, \lim)$ is stratified $\mathbb{H}$-Choquet convergence space, we have $\lim \mathcal{F}(x) \leq \top = \bigwedge_{\mathcal{U} \geq \varphi_e^{\Rightarrow}(\mathcal{F})} \lim \mathcal{U}(y) = \lim \varphi_e^{\Rightarrow}(\mathcal{F})(y)$. Hence $\lim \mathcal{F}(x) \leq \lim \varphi_e^{\Rightarrow}(\mathcal{F})((ex)^{-1})$. $\square$

Definition 3.17. Let $(X, \cdot)$ be a semigroup and for any $x \in X$, let $O(x) = \{x, x^2, x^3, ...\}$. Define $\Gamma(x) = \overline{O(x)}^{\lim}$.

Remark 3.18. Note that $O(x)$ is a subsemigroup of X and $\Gamma(x)$ is also a subsemigroup of X by Proposition 3.11.

Proposition 3.19. *Let $(X, \cdot, \lim)$ be a Hausdorff-separated stratified $\mathbb{H}$-generalized convergence semigroup. Then $(X, \cdot)$ has an idempotent if and only if there exists $x \in X$ such that $\Gamma(x)$ is compact.*

Proof. Let $(X, \cdot, \lim)$ have an idempotent e. Then $O(e) = \{e\}$, so $\Gamma(e) = \{e\}$ which is compact. Next, let there be an $x \in X$ such that $\Gamma(x)$ is compact. Then by a classical result on semigroups $\Gamma(x)$ has a kernel $\mathcal{K}$, and $\Gamma(x)$ is commutative. Consequently, it is a group and the identity of the kernel $\mathcal{K}$ is the desired idempotent. $\square$

4. Ideals in Complete Heyting Algebra-Valued Convergence Semigroups

Definition 4.1. ([9, 12, 44]) A nonempty subset A of a semigroup $(X, \cdot)$ is called *left ideal* if $XA \subseteq A$; it is called *right ideal* if $AX \subseteq A$. It is called *ideal* if it is both left and right ideal.

Definition 4.2. ([9, 12, 44]) Let $(X, \cdot)$ be a semigroup and $a \in X$. Let $J(a) = \{a\} \cup Xa \cup aX \cup SaS$, $L(a) = \{a\} \cup Xa$ and $R(a) = \{a\} \cup aX$, whence $J(a)$ is the smallest ideal of X containing a; and $L(a)$ and $R(a)$ are the smallest left and right ideals of X containing a. Furthermore, if A is a subset of a semigroup X, then $L(A) = A \cup XA$, $R(A) = A \cup AX$ and $J(A) = A \cup XA \cup AX \cup XAX$.

If $A \subseteq X$, then $J_0(A)$ is defined to be the empty set if A contains no ideal of X and $J_0(A)$ is the union of all ideals contained in A in the contrary case. Similarly, $L_0(A)$ and $R_0(A)$ are defined.

Proposition 4.3. *Let $(X, \cdot, \lim)$ be a Hausdorff-separated stratified $\mathbb{H}$-convergence semigroup and A be an ideal of X. Then $\overline{A}^{\lim}$ is an ideal of X.*

Proof. Let A be an ideal, $x \in X$ and $z \in \overline{A}^{\lim}$. We show that $xz \in \overline{A}^{\lim}$. From $z \in \overline{A}^{\lim}$ it follows that there exists $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$ such that $\lim \mathcal{F}(z) = \top$ and $\mathcal{F}(\top_A) = \top$. Now since $\psi_x : X \longrightarrow X, z \mapsto xz$ is continuous, we have $\lim \mathcal{F}(z) \leq \lim \psi_x^{\Rightarrow}(\mathcal{F})(\psi_x(z))$ which implies that $\lim \psi_x^{\Rightarrow}(\mathcal{F})(xz) = \lim \psi_x^{\Rightarrow}(\mathcal{F})(\psi_x(z)) = \top$, and $\psi_x^{\Rightarrow}(\mathcal{F}) \in \mathcal{F}^s_{\mathbb{H}}(X)$. Now $\psi_x^{\Rightarrow}(\mathcal{F})(\top_A) = \mathcal{F}(\psi_x^{\Leftarrow}(\top_A)) = \mathcal{F}(\top_A \circ \psi_x) = \top$, since $\top_A \circ \psi_x(z) = \top_A(xz) = \top$ and $xz \in A$ being A a left ideal of X. This proves that $xz \in \overline{A}^{\lim}$, meaning $\overline{A}^{\lim}$ is a left ideal of X. Similarly, we can prove that it is right ideal. Hence it is an ideal of X. □

Lemma 4.4. *If $(X, \cdot, \lim)$ is a Hausdorff-separated stratified $\mathbb{H}$-convergence semigroup and $A \subseteq X$ is $\top$-closed, then $J_0(A)$, $L_0(A)$ and R_0 are $\top$-closed.*

Proof. We only prove for the case $J_0(A)$. Let $J_0(A) \neq \emptyset$. Since $J_0(A)$ is the largest ideal of X contained in A, i.e., $J_0(A) \subseteq A$, we have by Lemma 2.7(3)[30], $\overline{J_0(A)}^{\lim} \subseteq \overline{A}^{\lim}$. By Lemma 4.3, $\overline{J_0(A)}^{\lim}$ is an ideal of X. Now since A is $\top$-closed, we have $\overline{A}^{\lim} = A$, and so, $\overline{J_0(A)}^{\lim} \subseteq J_0(A)$. This proves that $\overline{J_0(A)}^{\lim}$ is $\top$-closed. □

Proposition 4.5. *Let $(X, \cdot, \lim)$ be a compact Hausdorff-separated stratified $\mathbb{H}$-convergence semigroup and let A be a compact subset of X. Then $L(A)$, $R(A)$ and $J(A)$ are all compact.*

Proof. Since X and A are compact, $L(A) = A \cup XA$ is compact. Similarly, other parts follow. □

Proposition 4.6. *Let $(X, \cdot, \lim)$ be a compact Hausdorff-separated stratified $\mathbb{H}$-convergence semigroup with identity. Then $J(a)$ is compact for each $a \in X$. The same holds for $L(a)$ and $R(a)$.*

Proof. Since X is compact, $\{a\}$ is compact being $\top$-closed subset of Hausdorff-separated space X; aX, Xa and XaX are compact under continuous mappings. □

Definition 4.7. ([30]) Let $\mathbb{E}$ be a class of stratified $\mathbb{H}$-generalized convergence spaces $(E, \lim_E)$ which contains a space with at least two points. A stratified $\mathbb{H}$-generalized convergence space $(X, \lim)$ is called *$\mathbb{E}$-connected* if for any $(E, \lim_E) \in \mathbb{E}$, every continuous mapping $f : (X, \lim) \longrightarrow (E, \lim_E)$ is constant.

Proposition 4.8. *Let $(X, \cdot, \lim)$ be an $\mathbb{E}$-connected stratified $\mathbb{H}$-convergence semigroup with identity and let A be an $\mathbb{E}$-connected subset of X. Then $L(A)$, $R(A)$ and $J(A)$ are all $\mathbb{E}$-connected.*

Proof. In view of Lemma 5.1[30], XA being the continuous image of $\mathbb{E}$-connected sets X and A under $m : X \times X \longrightarrow X, (x, y) \mapsto xy$ is $\mathbb{E}$-connected. Hence exploiting Lemma 5.5[30], we deduce that $L(A) = A \cup XA$ is $\mathbb{E}$-connected. Similar arguments show that $R(A)$ and $J(A)$ are $\mathbb{E}$-connected. □

5. Examples: Stratified $\mathbb{H}$-Convergence Semigroups, Approach Convergence Semigroups and Probabilistic Convergence Semigroups

Example 5.1. Let $(X, \cdot)$ be a semigroup and $(X, \lim_i)$ be an indiscrete stratified $\mathbb{H}$-convergence space, where $\lim_i \mathcal{F}(x) = \top \ \forall \mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$ and $x \in X$ [24]. Then $(X, \cdot, \lim_i)$ is an indiscrete stratified $\mathbb{H}$-convergence semigroup.

Example 5.2. let $(X, \cdot)$ be a semigroup and $(X, \lim_d)$ be a discrete stratified $\mathbb{H}$-convergence space, where

$$\lim_d \mathcal{F}(x) = \begin{cases} \top, & \text{if } \mathcal{F} \geq [x]; \\ \bot, & \text{if } \mathcal{F} \not\geq [x]. \end{cases}$$

Then $(X, \cdot, \lim_d)$ is a discrete stratified $\mathbb{H}$-convergence semigroup.

Definition 5.3. A triple $(X, \cdot, \mathfrak{N} = (\mathfrak{N}^x)_{x\in X})$ is called a *stratified* $\mathbb{H}$*-neighborhood topological semigroup* if the following conditions are satisfied:

(HTSG1) $(X, \cdot)$ is a semigroup;

(HTSG2) $(X, \mathfrak{N})$ is a stratified $\mathbb{H}$-neighborhood topological space;

(HTSGM) the mapping $m : (X \times X, \mathfrak{N} \times \mathfrak{N}) \longrightarrow (X, \mathfrak{N}), (x, y) \mapsto xy$ is continuous, where the product stratified $\mathbb{H}$-neighborhood system $\mathfrak{N} \times \mathfrak{N}$ on $X \times X$, is given for any $\nu \in \mathbb{H}^{X\times X}$ by: $(\mathfrak{N} \times \mathfrak{N})^{(x,y)}(\nu) = \bigvee\{\mathfrak{N}^x(\nu_1) \wedge \mathfrak{N}^y(\nu_2) : \nu_1, \nu_2 \in \mathbb{H}^X, \nu_1 \circ pr_1 \wedge \nu_2 \circ pr_2 \leq \nu\}$.

Proposition 5.4. *Every stratified* $\mathbb{H}$*-neighborhood topological semigroup is a stratified* $\mathbb{H}$*-convergence semigroup.*

Proof. Let $(X, \cdot, \mathfrak{N})$ be a stratified $\mathbb{H}$-neighborhood topological semigroup, we show that $(X, \cdot, \lim_{\mathfrak{N}})$ is a stratified $\mathbb{H}$-convergence semigroup. Let $m : (X \times X, \mathfrak{N} \times \mathfrak{N}) \longrightarrow (X, \mathfrak{N}), (x, y) \mapsto xy$ be continuous, and $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_{\mathbb{H}}(X)$. Then for any $x, y \in X$, we have

$\lim_{\mathfrak{N}} \mathcal{F} \odot \mathcal{G}(xy) = \bigwedge_{\nu\in\mathbb{H}^X} (\mathfrak{N}^{xy}(\nu) \to (\mathcal{F} \odot \mathcal{G})(\nu))$

$\geq \bigwedge_{\nu\in\mathbb{H}^X} \left(\bigvee\{\mathfrak{N}^x(\nu_1) \wedge \mathfrak{N}^y(\nu_2) : \nu_1, \nu_2 \in \mathbb{H}^X, \nu_1 \odot \nu_2 \leq \nu\} \to \mathcal{F} \odot \mathcal{G})(\nu)\right)$

$= \bigwedge_{\nu\in\mathbb{H}^X} \bigwedge_{\nu_1\odot\nu_2\leq\nu} ((\mathfrak{N}^x(\nu_1) \wedge \mathfrak{N}^y(\nu_2)) \to (\mathcal{F} \odot \mathcal{G})(\nu))$

$\geq \bigwedge_{\nu_1,\nu_2\in\mathbb{H}^X} ((\mathfrak{N}^x(\nu_1) \wedge \mathfrak{N}^y(\nu_2)) \to (\mathcal{F}(\nu_1) \wedge \mathcal{G}(\nu_2)))$

$\geq \bigwedge_{\nu_1,\nu_2\in\mathbb{H}^X} (\mathfrak{N}^x(\nu_1) \to \mathcal{F}(\nu_1)) \wedge \bigwedge_{\nu_2\in\mathbb{H}^X} (\mathfrak{N}^y(\nu_2) \to \mathcal{G}(\nu_2))$

$= \lim_{\mathfrak{N}} \mathcal{F}(x) \wedge \lim_{\mathfrak{N}} \mathcal{G}(y)$,

that is, $\lim_{\mathfrak{N}} \mathcal{F}(x) \wedge \lim \mathcal{G}_{\mathfrak{N}}(y) \leq \lim(\mathcal{F} \odot \mathcal{G})(xy)$. □

Definition 5.5. ([15]) Let X be a nonempty set and $\mathcal{U}$ a stratified $\mathbb{H}$-filter on $X \times X$. If $\mathcal{U}$ satisfies the properties below, then it is called *stratified* $\mathbb{H}$*-quasi-uniformity* on X.

(QUS1) $\mathcal{U}(d) \leq \bigwedge_{x\in X} d(x, x)$, $\forall d \in \mathbb{H}^{X\times X}$;

(QUS2) $\mathcal{U}(d) \leq \bigvee\{\mathcal{U}(d_1) \wedge \mathcal{U}(d_2) : d_1 \circ d_2 \leq d\}$, $\forall d \in \mathbb{H}^{X\times X}$,

where $d_1 \circ d_2(x, y) = \bigvee_{z\in X} d_1(x, z) \wedge d_2(z, y)$,$\forall (x, y) \in X \times X$.

The pair $(X, \mathcal{U})$ is called *stratified* $\mathbb{H}$*-quasi-uniform space*.

It follows from [15] that given a stratified $\mathbb{H}$-quasi-uniformity on X, one can obtain a stratified $\mathbb{H}$-neighborhood system for each $x \in X$ and $\nu \in \mathbb{H}^X$: $\mathfrak{N}^x_{\mathcal{U}}(\nu) = \bigvee\{\mathcal{U}(d) | d \in \mathbb{H}^{X\times X}, d(x, -) \leq \nu\}$, where $d(x, -) : X \longrightarrow \mathbb{H}, y \mapsto d(x, -)(y) = d(x, y)$. This stratified $\mathbb{H}$-neighborhood system then yields a stratified $\mathbb{H}$-convergence structure on X: $\lim_{\mathcal{U}} \mathcal{F}(x) = \bigwedge_{\nu\in\mathbb{H}^X} \left(\mathfrak{N}^x_{\mathcal{U}}(\nu) \to \mathcal{F}(\nu)\right)$(cf. [24]).

Proposition 5.6. *Let* $(X, \cdot)$ *be a commutative and cancelation semigroup and let* $\mathcal{U}$ *be a stratified* $\mathbb{H}$*-quasi-uniformity on* X *such that for all* $d \in \mathbb{H}^{X\times X}$ *and for all* $x, y, z \in X$*, the property that* $d(x, y) \leq d(xz, yz)$ *holds. Then* $(X, \cdot, \lim_{\mathfrak{N}^{\mathcal{U}}})$ *is a stratified* $\mathbb{H}$*-convergence semigroup.*

Proof. Let $x, y \in X$ and $\nu \in \mathbb{H}^X$. Then we have

$\mathfrak{N}^{m(x,y)}_{\mathcal{U}}(\nu) = \mathfrak{N}^{xy}_{\mathcal{U}}(\nu)$

$= \bigvee\{\mathcal{U}(d) | d \in \mathbb{H}^{X\times X}, d(xy, -) \leq \nu\}$

$\leq \bigvee\{\bigvee\{\mathcal{U}(d_1) \wedge \mathcal{U}(d_2) | d_1, d_2 \in \mathbb{H}^{X\times X}, d_1 \circ d_2 \leq d\} | d \in \mathbb{H}^{X\times X}, d(xy, -) \leq \nu\}$

$\leq \bigvee\{\bigvee\{\mathcal{U}(d_1) \wedge \mathcal{U}(d_2) | d_1, d_2 \in \mathbb{H}^{X\times X}, d_1(x, -) \times d_2(y, -)(x', y') \leq d(xy, x'y') \leq \nu(x'y'), \forall x', y' \in X\}$

$\leq \bigvee\{\bigvee\{\mathcal{U}(d_1) \wedge \mathcal{U}(d_2) | d_1, d_2 \in \mathbb{H}^{X\times X}, d_1(x, -) \times d_2(y, -)(x', y') \leq \nu(x'y'), \forall x', y' \in X\}$

$\leq \bigvee\{\bigvee\{\mathcal{U}(d'_1) | d'_1(x, -) \leq d_1(x, -)\} \wedge \bigvee\{\mathcal{U}(d'_2) | d'_2(y, -) \leq d_2(y, -)\} | d_1(x, -) \times d_2(y, -)(x', y') \leq \nu(x'y'), \forall x', y' \in X\}$

$\leq \bigvee\{\mathfrak{N}^x_{\mathcal{U}}(d_1(x,-)) \wedge \mathfrak{N}^y_{\mathcal{U}}(d_2(y,-)) | d_1(x,-), d_2(y,-) \in \mathbb{H}^X, d_1(x,-) \times d_2(y,-)(x',y')$
$\leq m^{\leftarrow}(\nu)(x',y'), \forall x',y' \in X\}$
$= \left(\mathfrak{N}^x_{\mathcal{U}} \times \mathfrak{N}^y_{\mathcal{U}}\right)(m^{\leftarrow}(\nu))$.

In fact, upon using the given property, one obtains: for any $x',y' \in X$, $(d_1(x,-) \times d_2(y,-))(x',y') = d_1(x,x') \wedge d_2(y,y') \leq d_1(xy,x'y) \wedge d_2(x'y,x'y') \leq d_1 \circ d_2(xy,x'y') \leq d(xy,x'y') \leq \nu(x'y')$; note that X is a semigroup, and so, $x'y \in X$.

Thus, we have $\mathfrak{N}^{m(x,y)}_{\mathcal{U}}(\nu) \leq \left(\mathfrak{N}^x_{\mathcal{U}} \times \mathfrak{N}^y_{\mathcal{U}}\right)(m^{\leftarrow}(\nu))$, showing that $(x,y) \mapsto xy$ is continuous, and hence $\left(X,\cdot,\mathfrak{N} = \left(\mathfrak{N}^x_{\mathcal{U}}\right)_{x\in X}\right)$ is a stratified $\mathbb{H}$-neighborhood topological semigroup, which in conjunction with Proposition 5.4 proves that $(X,\cdot,\lim_{\mathfrak{N}^{\mathcal{U}}})$ is a stratified $\mathbb{H}$-convergence semigroup. □

Definition 5.7. ([5,37,38]) An *ultra approach convergence structure* λ on a set X is a function $\lambda : \mathbb{F}(X) \to [0,\infty]^X$ which satisfies the following conditions:

(uALS1) $\lambda(\dot{x})(x) = 0, \forall x \in X$, where $\dot{x} = \{A \subseteq X | x \in A\} \in \mathbb{F}(X)$.

(uACS2) If $\mathbb{F},\mathbb{G} \in \mathbb{F}(X)$ with $\mathbb{F} \leq \mathbb{G}$, then $\lambda(\mathbb{G}) \leq \lambda(\mathbb{F})$.

(uACS3) $\forall \mathbb{F},\mathbb{G} \in \mathbb{F}(X)$, $\lambda(\mathbb{F} \wedge \mathbb{G}) = \lambda(\mathbb{F}) \vee \lambda(\mathbb{G})$.

The pair (X,λ) is called an *ultra approach convergence space*.

A mapping $f : (X,\lambda) \to (X',\lambda')$ between ultra approach convergence spaces is called a *contraction* if and only if $\forall \mathbb{F} \in \mathbb{F}(X)$ and $x \in X$, $\lambda'(f(\mathbb{F}))(f(x)) \leq \lambda(\mathbb{F})(x)$.

Definition 5.8. (see also, [38]) Let $(X,\cdot)$ be a semigroup and (X,λ) be an ultra approach convergence space. Then the triple $(X,\cdot,\lambda)$ is called an *ultra approach convergence semigroup* if and only if the following conditions are fulfilled:

(uACGM) $\forall \mathbb{F},\mathbb{G} \in \mathbb{F}(X), x,y \in X$: $\lambda(\mathbb{F} \odot \mathbb{G})(xy) \leq \lambda(\mathbb{F})(x) \vee \lambda(\mathbb{G})(y)$

If $\mathcal{F} \in \mathcal{F}^s_{[0,1]}(X)$, then $\Phi_{\mathcal{F}}$ is a filter defined by: $\Phi_{\mathcal{F}} = \{A \subseteq X | \mathcal{F}(1_A) = 1\}$, (cf. [29]).

Now let $S : [0,1] \to [0,\infty]$ be a strictly decreasing surjective mapping such that $S(1) = 0$, which is also order reversing and satisfies that $S\left(\bigwedge_{j\in J} \alpha_j\right) = \bigvee_{j\in J} S(\alpha_j)$ and $S\left(\bigvee_{j\in J} \alpha_j\right) = \bigwedge_{j\in J} S(\alpha_j)$. For this map S, there exists inverse $S^{-1} : [0,\infty] \to [0,1]$ which is strictly decreasing and surjective.

Proposition 5.9. ([29]) *If (X,λ) is a convergence approach space, then $(X,\lim_\lambda)$ is a stratified $[0,1]$-convergence space, where $\lim_\lambda \mathcal{F}(x) = S^{-1}(\lambda(\Phi_{\mathcal{F}})(x))$, $\forall \mathcal{F} \in \mathcal{F}^s_{[0,1]}(X)$.*

Proposition 5.10. ([29]) *If $f : (X,\lambda) \to (X',\lambda')$ is a contraction, then $f : (X,\lim_\lambda) \to (X',\lim_{\lambda'})$ is continuous.*

Proposition 5.11. *If $(X,\cdot,\lambda)$ is an ultra approach convergence semigroup, then $(X,\cdot,\lim_\lambda)$ is a $[0,1]$-convergence semigroup, where for any $\mathcal{F} \in \mathcal{F}^s_{[0,1]}(X)$, $\lim_\lambda(\mathcal{F})(x) = S^{-1}(\lambda(\mathbb{F}_{\mathcal{F}})(x))$.*

Proof. Only we need to prove that if $\mathcal{F},\mathcal{G} \in \mathcal{F}^s_{[0,1]}(X)$, and $x,y \in X$, then $\lim_\lambda \mathcal{F}(x) \wedge \lim_\lambda \mathcal{G}(y) \leq \lim_\lambda(\mathcal{F} \odot \mathcal{G})(xy)$, i.e., the multiplication, $m : X \times X \to X, (x,y) \mapsto xy$ is continuous. We have

$\lim_\lambda(\mathcal{F} \odot \mathcal{G})(xy) = S^{-1}\left(\lambda\left(\Phi_{\mathcal{F}\odot\mathcal{G}}\right)(xy)\right)$.
$\geq S^{-1}(\lambda(\Phi_{\mathcal{F}} \odot \Phi_{\mathcal{G}})(xy))$ (as both S^{-1} and λ are order reversing)
$\geq S^{-1}(\lambda(\Phi_{\mathcal{F}})(x) \vee \lambda(\Phi_{\mathcal{G}})(y))$ (since S^{-1} is order reversing, applying (uACGM))
$= S^{-1}(\lambda(\Phi_{\mathcal{F}})(x)) \wedge S^{-1}(\lambda(\Phi_{\mathcal{G}})(y))$
$= \lim_\lambda \mathcal{F}(x) \wedge \lim_\lambda \mathcal{G}(y)$.

In fact, if $\mathcal{F},\mathcal{G} \in \mathcal{F}^s_{[0,1]}(X)$, then we have: $\Phi_{\mathcal{F}} \odot \Phi_{\mathcal{G}} \leq \Phi_{\mathcal{F}\odot\mathcal{G}}$.

Indeed, if $A \in \Phi_{\mathcal{F}} \odot \Phi_{\mathcal{G}}$, then there are $F \in \Phi_{\mathcal{F}}$ and $G \in \Phi_{\mathcal{G}}$ such that $F \cdot G \subseteq A$. These mean that there are F,G with $\mathcal{F}(1_F) = 1$ and $\mathcal{G}(1_G) = 1$ such that $F \cdot G \subseteq A$. Since $1_F \cdot 1_G = 1_{F\cdot G} \subseteq 1_A$, we have $1 = \mathcal{F}(1_F) \wedge \mathcal{G}(1_G) \leq \bigvee\{\mathcal{F}(\nu_1) \wedge \mathcal{G}(\nu_2) : \nu_1 \odot \nu_2 \leq 1_A\} = \mathcal{F} \odot \mathcal{G}(1_A)$, i.e., $\mathcal{F} \odot \mathcal{G}(1_A) = 1$, which yields that $A \in \Phi_{\mathcal{F}\odot\mathcal{G}}$. □

Remark 5.12. Lowen-Windels approach semigroup (see [38, Proposition 5.1]] is a convergence approach semigroup (see Definition 6.4, and [38, Proposition 6.5]] according to their notions of approach space and approach convergence space [39]. However, if we consider $(X, +, \lambda)$ (cf. [38, Definition 6.4]), as convergence approach semigroup, then replacing $\odot$ by $\oplus$, and making some notational readjustments, one can deduce that the triple $(X, +, \lim_\lambda)$ is an example of stratified $[0,1]$-convergence semigroup.

Definition 5.13. ([21]) Let $(X, \lim_X)$ and $(Y, \lim_Y)$ be stratified $\mathbb{H}$-convergence spaces. If $C(X,Y) = \{f | f : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ is continuous$\}$, then the *convergence structure of continuous convergence* is defined for any $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(C(X,Y))$ and $f \in C(X,Y)$ by

$$c - \lim \mathcal{F}(f) = \bigwedge_{\mathcal{G}\in\mathcal{F}^s_{\mathbb{H}}(X)} \bigwedge_{x\in X} (\lim_X \mathcal{G}(x) \to \lim_Y ev(\mathcal{F} \times \mathcal{G})(f(x)))$$

Proposition 5.14. *Let $(X, \lim)$ be a stratified $\mathbb{H}$-convergence space and $(Y, \cdot, \lim')$ be a Hausdorff-separated stratified $\mathbb{H}$-convergence semigroup. Then $(C(X,Y), \cdot, c - \lim)$ is a Hausdorff-separated stratified $\mathbb{H}$-convergence semigroup.*

Proof. This follows from Proposition 4.11[3] in conjunction with Corollary 5.7[27]. □

Definition 5.15. ([16]) A pair $(X, \mathbb{C})$ with $\mathbb{C} = (\mathbf{c}_x)_{x\in X}$, where $\mathbf{c}_x : \mathbb{F}(X) \longrightarrow [0,1]$ is called a *probabilistic convergence space under* $\wedge$ if and only if the following conditions are fulfilled.

(PC1) $\forall x \in X$: $\mathbf{c}_x(\dot{x}) = 1$.

(PC2) $\forall x \in X$, $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$ with $\mathbb{F} \leq \mathbb{G}$ implies $\mathbf{c}_x(\mathbb{F}) \leq \mathbf{c}_x(\mathbb{G})$.

(PC3) $\forall x \in X$, $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(X)$, $\mathbf{c}_x(\mathbb{F}) \wedge \mathbf{c}_x(\mathbb{G}) \leq \mathbf{c}_x(\mathbb{F} \wedge \mathbb{G})$.

A mapping $f : \left(X, \mathbb{C}^X\right) \longrightarrow \left(Y, \mathbb{C}^Y\right)$ is called continuous between probabilistic limit spaces $\left(X, \mathbb{C}^X\right)$ and $\left(Y, \mathbb{C}^Y\right)$ if and only if for all $x \in X$ and for all $\mathbb{F} \in \mathbb{F}(X)$, $\mathbf{c}^X_x(\mathbb{F}) \leq \mathbf{c}^Y_{f(x)}(f(\mathbb{F}))$.

Definition 5.16. A triple $(X, \cdot, \mathbb{C} = (\mathbf{c}_x)_{x\in X})$ is called a *probabilistic convergence semigroup under* $\wedge$ if and only if the following conditions are satisfied:

(PCG1) $(X, \cdot)$ is a semigroup.

(PCG2) $(X, \mathbb{C} = (\mathbf{c}_x)_{x\in X})$ is a probabilistic convergence space under $\wedge$.

(PCGM) $\forall\, \mathbb{F}, \mathbb{G} \in \mathbb{F}(X), \forall x, y \in X$: $\mathbf{c}_x(\mathbb{F}) \wedge \mathbf{c}_y(\mathbb{G}) \leq \mathbf{c}_{xy}(\mathbb{F} \odot \mathbb{G})$.

Proposition 5.17. *If $(X, \cdot, \mathbb{C} = (\mathbf{c}_x)_{x\in X})$ is a probabilistic convergence semigroup under $\wedge$, then $(X, \cdot, \lim_{\mathbb{C}})$ is a stratified $[0,1]$-convergence semigroup, where $\lim_{\mathbb{C}} \mathcal{F}(x) = \mathbf{c}_x(\Phi_{\mathcal{F}})$, for any $\mathcal{F} \in \mathcal{F}^s_{[0,1]}(X)$ and $x \in X$.*

Proof. The pair $(X, \lim_{\mathbb{C}})$ is a stratified $[0,1]$-convergence space. Let $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_{[0,1]}(X)$ and $x, y \in X$, $\lim_{\mathbb{C}} \mathcal{F}(x) \wedge \lim_{\mathcal{G}}(y) \leq \lim_{\mathbb{C}}(\mathcal{F} \odot \mathcal{G})(xy)$. Let $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_{[0,1]}(X)$ and $x, y \in X$. Then we have

$$\lim_{\mathbb{C}}(\mathcal{F} \odot \mathcal{G})(xy) = \mathbf{c}_{xy}\left(\Phi_{\mathcal{F}\odot\mathcal{G}}\right) \geq \mathbf{c}_{xy}(\Phi_{\mathcal{F}} \odot \Phi_{\mathcal{G}}) \geq \mathbf{c}_x(\Phi_{\mathcal{F}}) \wedge \mathbf{c}_y(\Phi_{\mathcal{G}})\,(by(PCGM)) = \lim_{\mathbb{C}} \mathcal{F}(x) \wedge \lim_{\mathbb{C}} \mathcal{G}(y).$$ □

6. Stratified $\mathbb{H}$-QUCS-Uniformization of Stratified $\mathbb{H}$-Convergence Semigroups

Definition 6.1. ([24]) Let X be a non-empty set. A map $\mathfrak{U} : \mathcal{F}^s_{\mathbb{H}}(X \times X) \longrightarrow \mathbb{H}$ is called a *stratified* $\mathbb{H}$*-quasi-uniform convergence structure* if and only if $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}^s_{\mathbb{H}}(X \times X)$

(HQUS1) $\forall x \in X$, $\mathfrak{U}([(x,x)]) = \top$.

(HQUS2) $\mathcal{F} \leq \mathcal{G}$ implies $\mathfrak{U}(\mathcal{F}) \leq \mathfrak{U}(\mathcal{G})$.

(HQUS3) $\mathfrak{U}(\mathcal{F}) \wedge \mathfrak{U}(\mathcal{G}) \leq \mathfrak{U}(\mathcal{F} \wedge \mathcal{G})$.

(HQUS4) $\mathfrak{U}(\mathcal{F}) \wedge \mathfrak{U}(\mathcal{G}) \leq \mathfrak{U}(\mathcal{F} \circ \mathcal{G})$, whenever $\mathcal{F} \circ \mathcal{G}$ exists,

are fulfilled. Then the pair $(X, \mathfrak{U})$ is called a *stratified* $\mathbb{H}$*-quasi-uniform convergence space*.

If, moreover, there is a stratified $\mathbb{H}$-filter $\mathcal{V} \in \mathcal{F}^s_{\mathbb{H}}(X \times X)$ such that

(PHQUS) $\forall \mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X \times X)$, $\mathfrak{U}(\mathcal{F}) = \bigwedge_{\eta\in\mathbb{H}^{X\times X}} (\mathcal{V}(\eta) \to \mathcal{F}(\eta))$,

then the pair $(X, \mathfrak{U})$ is called a *principal stratified* $\mathbb{H}$*-quasi-uniform convergence space*.

A map $f : (X, \mathfrak{U}) \longrightarrow (Y, \mathfrak{U}')$ is called *quasi-uniformly continuous* if $\forall \mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X \times X)$, $\mathfrak{U}(\mathcal{F}) \leq \mathfrak{U}'((f \times f)^{\Rightarrow}(\mathcal{F}))$.

Let $\mathbb{H} = (\mathbb{H}, \leq, \wedge)$ be a complete Heyting algebra and $(X, \cdot)$ be a semigroup with identity element e. If $\nu \in \mathbb{H}^X$, define a map $\Upsilon^l_\nu \colon X \times X \longrightarrow \mathbb{H}$ for any $x, y \in X$ by $\Upsilon^l_\nu(x, y) = x \odot \nu(y)$, where $x \odot \nu(y) = \bigvee_{xz=y, z\in X} \nu(z)$(analogously, define $\Upsilon^r_\nu \colon X \times X \longrightarrow \mathbb{H}$ for any $x, y \in X$ by $\Upsilon^r_\nu(x, y) = \nu \odot x(y)$, where $\nu \odot x(y) = \bigvee_{zx=y, z\in X} \nu(z)$.

Also, for any $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$, we define a map $\Upsilon^l_{\mathcal{F}} \colon \mathbb{H}^{X\times X} \longrightarrow \mathbb{H}$ defined by

$$\Upsilon^l_{\mathcal{F}}(d) = \bigvee\{\mathcal{F}(\nu) \colon \nu \in \mathbb{H}^X, \Upsilon^l_\nu \leq d\}$$

(analogously, for any $\mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X)$, we define a map $\Upsilon^r_{\mathcal{F}} \colon \mathbb{H}^{X\times X} \longrightarrow \mathbb{H}$ defined by

$$\Upsilon^r_{\mathcal{F}}(d) = \bigvee\{\mathcal{F}(\nu) \colon \nu \in \mathbb{H}^X, \Upsilon^r_\nu \leq d\}.)$$

Theorem 6.2. *Every stratified $\mathbb{H}$-convergence semigroup with identity element e gives rise to a stratified $\mathbb{H}$-quasi-uniform convergence space. That is, every stratified $\mathbb{H}$-convergence semigroup with identity element e is* ***S***$\mathbb{H}$-***QUCS****-uniformizable.*

The proof of this theorem follows from the following construction, for details see Theorem 5.6[3].

If we define the map $\mathfrak{U}^l \colon \mathcal{F}^s_{\mathbb{H}}(X \times X) \longrightarrow \mathbb{H}$ for any $\mathcal{G} \in \mathcal{F}^s_{\mathbb{H}}(X \times X)$ by

$$\mathfrak{U}^l(\mathcal{G}) = \bigvee\{\lim \mathcal{F}(e) \colon \mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X), \Upsilon^l_{\mathcal{F}} \leq \mathcal{G}\},$$

then the pair $\left(X, \mathfrak{U}^l\right)$ is a stratified left $\mathbb{H}$-quasi-uniform convergence space (analogously, $(X, \mathfrak{U}^r)$ is also a stratified right $\mathbb{H}$-quasi-uniform convergence space, where

$$\mathfrak{U}^r(\mathcal{G}) = \bigvee\{\lim \mathcal{F}(e) \colon \mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}(X), \Upsilon^r_{\mathcal{F}} \leq \mathcal{G}\}.)$$

Definition 6.3. A triple $(X, \cdot, \mathfrak{U})$ is called a *stratified $\mathbb{H}$-quasi-uniform convergence semigroup* if the following conditions are fulfilled:

(QUCG1) $(X, \cdot)$ is a semigroup;

(QUCS2) $(X, \mathfrak{U})$ is a stratified $\mathbb{H}$-quasi-uniform convergence space;

(QUCS3) The mapping $m \colon X \times X \longrightarrow X, (x, y) \mapsto xy$ is quasi-uniformly continuous,

where the product stratified $\mathbb{H}$-quasi-uniform convergence structure $\mathfrak{U} \times \mathfrak{U}$ on $X \times X$ is given by: $\mathfrak{U} \times \mathfrak{U}(\mathcal{F}) = \mathfrak{U}\left((pr_1 \times pr_1)^{\Rightarrow}(\mathcal{F})\right) \wedge \mathfrak{U}\left((pr_2 \times pr_2)^{\Rightarrow}(\mathcal{F})\right), \forall \mathcal{F} \in \mathcal{F}^s_{\mathbb{H}}\left((X \times X) \times (X \times X)\right)$.

Proposition 6.4. *Every principal stratified $\mathbb{H}$-quasi-uniform convergence semigroup is a stratified $\mathbb{H}$-convergence semigroup.*

Proof. This follows from Proposition 7.6[1]. □

Proposition 6.5. *Let $(X, \cdot)$ be a commutative and cancelation semigroup, and let $\mathcal{U}$ be a stratified $\mathbb{H}$-quasi-uniformity on X such that for all $d \in \mathbb{H}^{X\times X}$ and for all $x, y, z \in X$, the property that $d(x, y) \leq d(xz, yz)$ holds. Then $(X, \cdot, \lim_{\mathfrak{U}^{\mathcal{U}}})$ is a stratified $\mathbb{H}$-convergence semigroup.*

Proof. It follows from Lemma 5.5[24] that $(X, \mathfrak{U}_{\mathcal{U}})$ is a principal stratified $\mathbb{H}$-quasi-uniform convergence space. Now it remains to be shown that $(X, \cdot, \mathfrak{U}_{\mathcal{U}})$ is a principal stratified $\mathbb{H}$-quasi-uniform convergence semigroup, i.e., we show that for any $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_{\mathbb{H}}(X \times X)$: $\mathfrak{U}_{\mathcal{U}}(\mathcal{F}) \wedge \mathfrak{U}_{\mathcal{U}}(\mathcal{G}) \leq \mathfrak{U}_{\mathcal{U}}\left((m \times m)^{\Rightarrow}(\mathcal{F} \times \mathcal{G})\right)$.

To prove this, let $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_{\mathbb{H}}(X \times X)$. Then

$$\mathfrak{U}_{\mathcal{U}}\left((m \times m)^{\Rightarrow}(\mathcal{F} \times \mathcal{G})\right) = \bigwedge_{d\in\mathbb{H}^{X\times X}} \left(\mathcal{U}(d) \to (m \times m)^{\Rightarrow}(\mathcal{F} \times \mathcal{G})(d)\right)$$

$$\geq \bigwedge_{d\in\mathbb{H}^{X\times X}} \left(\underbrace{\left(\bigvee\{\mathcal{U}(d_1) \wedge \mathcal{U}(d_2) | d_1, d_2 \in \mathbb{H}^{X\times X}, d_1 \circ d_2 \leq d\}\right)} \to (m \times m)^{\Rightarrow}(\mathcal{F} \times \mathcal{G})(d) \right) \quad (\sharp)$$

Upon using the stated property, for any $(x, y), (x', y') \in X \times X$, $(d_1 \otimes d_2)\left((x, y), (x', y')\right) = d_1(x, x') \wedge d_2(y, y') \leq d_1(xy, x'y) \wedge d_2(x'y, x'y') \leq d_1 \circ d_2(xy, x'y') \leq d(xy, x'y') = d(m(x, y), m(x', y')) = (m \times m)^{\leftarrow}(d)\left((x, y), (x', y')\right)$; note that $x'y \in X$ as $(X, \cdot)$ is a semigroup. We have

$\underbrace{\left(\bigvee\{\mathcal{U}(d_1)\wedge\mathcal{U}(d_2)|d_1,d_2\in\mathbb{H}^{X\times X}, d_1\circ d_2\leq d\}\right)}$

$\leq \left(\bigvee\{\mathcal{U}(d_1)\wedge\mathcal{U}(d_2)|d_1,d_2\in\mathbb{H}^{X\times X}, d_1\otimes d_2\leq (m\times m)^{\leftarrow}(d)\}\right)$ $(\natural)$

Now using $(\natural)$ in $(\sharp)$, one obtains that

$(\sharp)\ \geq \bigwedge_{d\in\mathbb{H}^{X\times X}}\left(\left(\bigvee\{\mathcal{U}(d_1)\wedge\mathcal{U}(d_2)|d_1,d_2\in\mathbb{H}^{X\times X}, d_1\otimes d_2\leq (m\times m)^{\leftarrow}(d)\}\right)\to (m\times m)^{\Rightarrow}(\mathcal{F}\times\mathcal{G})(d)\right)$,

that is,

$\geq \bigwedge_{d\in\mathbb{H}^{X\times X}}\left(\underbrace{\left(\bigvee\{\mathcal{U}(d_1)\wedge\mathcal{U}(d_2)|d_1,d_2\in\mathbb{H}^{X\times X}, d_1\circ d_2\leq d\}\right)}\to (m\times m)^{\Rightarrow}(\mathcal{F}\times\mathcal{G})(d)\right)$

$\geq \bigwedge_{d\in\mathbb{H}^{X\times X}}\left(\left(\bigvee\{\mathcal{U}(d_1)\wedge\mathcal{U}(d_2)|d_1,d_2\in\mathbb{H}^{X\times X}, d_1\otimes d_2\leq (m\times m)^{\leftarrow}(d)\}\right)\to (m\times m)^{\Rightarrow}(\mathcal{F}\times\mathcal{G})(d)\right)$

$= \bigwedge_{d\in\mathbb{H}^{X\times X}}\left(\left(\bigvee\{\mathcal{U}(d_1)\wedge\mathcal{U}(d_2)|d_1,d_2\in\mathbb{H}^{X\times X}, d_1\otimes d_2\leq (m\times m)^{\leftarrow}(d)\}\right)\to (\mathcal{F}\times\mathcal{G})\left((m\times m)^{\leftarrow}(d)\right)\right)$

$\geq \bigwedge_{d\in\mathbb{H}^{X\times X}}\bigwedge_{d_1,d_2\in\mathbb{H}^{X\times X}, d_1\otimes d_2\leq (m\times m)^{\leftarrow}(d)}((\mathcal{U}(d_1)\wedge\mathcal{U}(d_2))\to(\mathcal{F}(d_1)\wedge\mathcal{G}(d_2))$

$\geq \bigwedge_{d_1,d_2\in\mathbb{H}^{X\times X}}((\mathcal{U}(d_1)\wedge\mathcal{U}(d_2)\to\mathcal{F}(d_1))\wedge(\mathcal{U}(d_1)\wedge\mathcal{U}(d_2))\to\mathcal{G}(d_2))$ (applying Lemma 2.1(ii) and (iv))

$\geq \bigwedge_{d_1\in\mathbb{H}^{X\times X}}(\mathcal{U}(d_1)\to\mathcal{F}(d_1))\wedge\bigwedge_{d_2\in\mathbb{H}^{X\times X}}(\mathcal{U}(d_2)\to\mathcal{G}(d_2))$

$= \mathfrak{U}_{\mathcal{U}}(\mathcal{F})\wedge\mathfrak{U}_{\mathcal{U}}(\mathcal{G})$,

showing that $\mathfrak{U}_{\mathcal{U}}(\mathcal{F})\wedge\mathfrak{U}_{\mathcal{U}}(\mathcal{G})\leq\mathfrak{U}_{\mathcal{U}}\left((m\times m)^{\Rightarrow}(\mathcal{F}\times\mathcal{G})\right)$, i.e., $m\colon (x,y)\mapsto xy$ is quasi-uniformly continuous. Hence the result follows from the Proposition 6.4 in conjunction with the Lemma 6.2[24]. □

7. Conclusion

In this article we have introduced a notion of complete Heyting algebra-valued convergence semigroup, and presented various basic facts including a wide variety of natural examples in an attempt to develop a theory on lattice-valued convergence semigroups. One of the interesting problems that received much attention over the years in the area of classical topological semigroups is the embedding of a topological semigroup into a topological group(cf. [11, 14, 36, 43]) besides developing the theory itself which have many interesting and useful applications (cf.[40]). It is an interesting topic to discuss the embedding problem in our case. Another point we would like to mention here that in Proposition 5.6, we used a property $d(x,y)\leq d(xz,yz)$ which leads to a stratified $\mathbb{H}$-quasi-uniformity on a commutative and cancelation semigroup to be a stratified $\mathbb{H}$-convergence semigroup. As pointed out by the referee and we also believe that it is a nice question whether one can provide some examples of spaces which satisfy this property, and some spaces which do not. Unfortunately, at this stage we could not worked out such examples, and leave it as an open question. We intend to look into this question in one of our future papers along with the embedding problem that we have cited above.

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