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Unified (p, q)-analog of Apostol Type Polynomials of Order α

Ugur Duran[a], **Mehmet Acikgoz**[b], **Serkan Araci**[c]

[a]*Department of the Basic Concepts of Engineering, Faculty of Engineering and Natural Sciences, İskenderun Technical University, TR-31200 Hatay, Turkey*
[b]*Department of Mathematics, Faculty of Arts and Science, University of Gaziantep, TR-27310 Gaziantep, Turkey*
[c]*Department of Economics, Faculty of Economics, Administrative and Social Sciences, Hasan Kalyoncu University, TR-27410 Gaziantep, Turkey*

Abstract. In this work, we introduce a class of a new generating function for (p, q)-analog of Apostol type polynomials of order α including Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials of order α. By making use of their generating function, we derive some useful identities. We also introduce (p, q)-analog of Stirling numbers of second kind of order v by which we construct a relation including aforementioned polynomials.

1. Introduction

Throughout of the paper we make use of the following notations: $\mathbb{N} := \{1, 2, 3, \cdots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. The (p, q)-number is defined by $[n]_{p,q} = \frac{p^n - q^n}{p-q} \quad (p \neq q)$. Obviously that when $p = 1$, we have $[n]_q = \frac{1-q^n}{1-q}$ that stands for q-number. One can see that (p, q)-number is closely related to q-number with this relation $[n]_{p,q} = p^{n-1} [n]_{\frac{q}{p}}$. By appropriately using this obvious relation between the q-notation and its variant, the (p, q)-notation, most (if not all) of the (p, q)-results can be derived from the corresponding known q-results by merely changing the parameters and variables involved.

Let us now brief some tools in (p, q)-calculus which will be useful in deriving the results of the paper. The (p, q)-derivative operator given by

$$D_{p,q;x} f(x) := D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p-q)x} \quad (x \neq 0) \text{ with } \left(D_{p,q} f\right)(0) = f'(0). \tag{1}$$

The (p, q)-power basis is also defined by

$$(x + a)^n_{p,q} = \sum_{k=0}^{n} \binom{n}{k}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} x^k a^{n-k}.$$

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Email addresses: mtdrnugur@gmail.com (Ugur Duran), acikgoz@gantep.edu.tr (Mehmet Acikgoz), mtsrkn@hotmail.com (Serkan Araci)

Here the notations $\binom{n}{k}_{p,q}$ and $[n]_{p,q}!$ are $\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}$ $(n \geq k)$ and $[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q}\cdots[2]_{p,q}[1]_{p,q}$ $(n \in \mathbb{N})$ with the initial condition $[0]_{p,q}! = 1$.

Let

$$e_{p,q}(x) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!} \text{ and } E_{p,q}(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!}$$

denote two types of exponential functions satisfying relations $e_{p,q}(x)E_{p,q}(-x) = 1$ and $e_{p^{-1},q^{-1}}(x) = E_{p,q}(x)$ which also have the following (p,q)-derivative representations

$$D_{p,q}e_{p,q}(x) = e_{p,q}(px) \text{ and } D_{p,q}E_{p,q}(x) = E_{p,q}(qx). \tag{2}$$

The definite (p,q)-integral for a function f is defined by

$$\int_0^a f(x)\, d_{p,q}x = (p-q)\, a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}}a\right). \tag{3}$$

For further information (p,q)-calculus used in this paper, one can look at [2, 4, 5] and cited references therein.

Apostol type polynomials and numbers firstly introduced by Apostol [1] and also Srivastava [21]. Some relationships between Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials were introduced and studied extensively by Luo and Srivastava [10-13], Lu and Srivastava [4] and Srivastava [19-25]. Motivated by their works, many mathematicians have studied and investigated Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials and numbers, *cf.* [1, 6, 7, 8, 11-14, 17]. Also q-analogs of Apostol type polynomials and numbers were introduced and discussed by several authors, see [3, 5, 9, 15, 16]. Moreover, (p,q)-Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x,y;\lambda : p,q)$, (p,q)-Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x,y;\lambda : p,q)$ and (p,q)-Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x,y;\lambda : p,q)$ were defined by Duran and Acikgoz in [5], as follows:

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x,y;\lambda : p,q) \frac{z^n}{[n]_{p,q}!} = \left(\frac{z}{\lambda e_{p,q}(z) - 1}\right)^{\alpha} e_{p,q}(xz) E_{p,q}(yz),$$
$$\left(|z| < 2\pi \text{ when } \lambda = 1; |z| < \left|\log \lambda\right| \text{ when } \lambda \neq 1\right)$$
$$\sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x,y;\lambda : p,q) \frac{z^n}{[n]_{p,q}!} = \left(\frac{2}{\lambda e_{p,q}(z) + 1}\right)^{\alpha} e_{p,q}(xz) E_{p,q}(yz)$$
$$\left(|z| < \pi \text{ when } \lambda = 1; |z| < \left|\log(-\lambda)\right| \text{ when } \lambda \neq 1\right)$$

and

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x,y;\lambda : p,q) \frac{z^n}{[n]_{p,q}!} = \left(\frac{2z}{\lambda e_{p,q}(z) + 1}\right)^{\alpha} e_{p,q}(xz) E_{p,q}(yz)$$
$$\left(|z| < \pi \text{ when } \lambda = 1; |z| < \left|\log(-\lambda)\right| \text{ when } \lambda \neq 1\right)$$

where $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), $\alpha \in \mathbb{N}_0$ a nonnegative integer, and $p,q \in \mathbb{C}$ with the condition $0 < |q| < |p| \leq 1$.

In the next section, we perform to define the family of unified (p,q)-analog of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials of order α and to investigate some properties of them. Moreover, we consider (p,q) analog of a new generalization of Stirling numbers of the second kind of order v by which we derive a relation including unified (p,q)-analog of Apostol type polynomials of order α.

2. Unified (p,q)-Analog of Apostol Type Polynomials of Order α

Inspired by the generating function [25]

$$f_{a,b}(x;t;k,\beta) := \frac{2^{1-k}t^k e^{xt}}{\beta^b e^t - a^b} = \sum_{n=0}^{\infty} P_{n,\beta}(x;k,a,b)\frac{t^n}{n!}$$
$$\left(|t| < 2\pi \text{ when } \beta = a; |t| < \left|\beta \log\left(\frac{b}{a}\right)\right| \text{ when } \beta \neq a; \alpha, k \in \mathbb{N}_0; a, b \in \mathbb{R}\backslash\{0\}; \beta \in \mathbb{C}\right)$$

in this paper, we consider the following Definition 2.1 based on (p,q)-numbers.

Definition 2.1. *Unified (p,q)-analog of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials of order α is defined as follows:*

$$\Upsilon_{a,b}^{(\alpha)}(x,y;z;k,\beta:p,q) = \sum_{n=0}^{\infty} \mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)\frac{z^n}{[n]_{p,q}!} = \left(\frac{2^{1-k}z^k}{\beta^b e_{p,q}(z) - a^b}\right)^{\alpha} e_{p,q}(xz)E_{p,q}(yz)$$
$$\left(|z| < 2\pi \text{ when } \beta = a; |z| < \left|\beta \log\left(\frac{b}{a}\right)\right| \text{ when } \beta \neq a; \alpha, k \in \mathbb{N}_0; a, b \in \mathbb{R}\backslash\{0\}; \beta \in \mathbb{C}\right).$$

We note that $\mathcal{P}_{n,\beta}^{(1)}(x,y,k,a,b:p,q) := \mathcal{P}_{n,\beta}(x,y,k,a,b:p,q)$ which are called unified (p,q)-analog of Apostol type polynomials.

Remark 2.2. *When $p = \alpha = 1$, as $q \to 1$, in Definition 2.1, it was studied systematically by Ozden et al. [18].*

We now give here some basic properties for $\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)$ by the following four Lemmas 2.3-2.6 without proofs, since they can be proved by using Definition 2.1.

Lemma 2.3. *We have*

$$\begin{aligned}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q) &= \sum_{j=0}^{n}\binom{n}{j}_{p,q}\mathcal{P}_{j,\beta}^{(\alpha)}(0,y,k,a,b:p,q)\,x^{n-j}p^{\binom{n-j}{2}},\\ &= \sum_{j=0}^{n}\binom{n}{j}_{p,q}\mathcal{P}_{j,\beta}^{(\alpha)}(x,0,k,a,b:p,q)\,y^{n-j}q^{\binom{n-j}{2}},\\ &= \sum_{j=0}^{n}\binom{n}{j}_{p,q}\mathcal{P}_{j,\beta}^{(\alpha)}(0,0,k,a,b:p,q)\,(x+y)_{p,q}^{n-j}. \qquad (4)\end{aligned}$$

Lemma 2.4. *(Addition property) For $\alpha, \mu \in \mathbb{N}_0$, $\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)$ satisfies the following relation:*

$$\mathcal{P}_{n,\beta}^{(\alpha+\mu)}(x,y,k,a,b:p,q) = \sum_{j=0}^{n}\binom{n}{j}_{p,q}\mathcal{P}_{j,\beta}^{(\alpha)}(x,0,k,a,b:p,q)\mathcal{P}_{n-j,\beta}^{(\mu)}(0,y,k,a,b:p,q).$$

It immediately follows from Lemma 2.4 that $\mathcal{P}_{n,\beta}^{(0)}(x,y,k,a,b:p,q) = (x+y)_{p,q}^{n}$.

Lemma 2.5. *(Derivative properties) We have*

$$\begin{aligned}D_{p,q;x}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q) &= [n]_{p,q}\mathcal{P}_{n-1,\beta}^{(\alpha)}(px,y,k,a,b:p,q),\\ D_{p,q;y}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q) &= [n]_{p,q}\mathcal{P}_{n-1,\beta}^{(\alpha)}(x,qy,k,a,b:p,q).\end{aligned}$$

Lemma 2.6. *(Difference equation) We have*

$$\mathcal{P}_{n,\beta}^{(\alpha-1)}(0,y,k,a,b:p,q)=\frac{2^{k-1}[n]_{p,q}!}{[n+k]_{p,q}!}\Big(\beta^b\mathcal{P}_{n+k,\beta}^{(\alpha)}(1,y,k,a,b:p,q)-a^b\mathcal{P}_{n+k,\beta}^{(\alpha)}(0,y,k,a,b:p,q)\Big), \tag{5}$$

$$\mathcal{P}_{n,\beta}^{(\alpha-1)}(x,-1,k,a,b:p,q)=\frac{2^{k-1}[n]_{p,q}!}{[n+k]_{p,q}!}\Big(\beta^b\mathcal{P}_{n+k,\beta}^{(\alpha)}(x,0,k,a,b:p,q)-a^b\mathcal{P}_{n+k,\beta}^{(\alpha)}(x,-1,k,a,b:p,q)\Big).$$

From Lemma 2.3 and Lemma 2.5, we obtain the following Theorem 2.7.

Theorem 2.7. *We have*

$$\frac{[n+k]_{p,q}!}{2^{k-1}[n]_{p,q}!}\mathcal{P}_{n,\beta}^{(\alpha-1)}(0,y,k,a,b:p,q)=\beta^b\sum_{j=0}^{n+k}\binom{n+k}{j}_{p,q}p^{\binom{n+k-j}{2}}\mathcal{P}_{j,\beta}^{(\alpha)}(0,y,k,a,b:p,q)-a^b\mathcal{P}_{n+k,\beta}^{(\alpha)}(0,y,k,a,b:p,q). \tag{6}$$

Corollary 2.8. *Upon setting $\alpha=1$ in Eq. (6) gives the following relation*

$$y^n=\frac{2^{k-1}[n]_{p,q}!}{q^{\binom{n}{2}}[n+k]_{p,q}!}\left(\beta^b\sum_{j=0}^{n+k}\binom{n+k}{j}_{p,q}p^{\binom{n+k-j}{2}}\mathcal{P}_{j,\beta}(0,y,k,a,b:p,q)-a^b\mathcal{P}_{n+k,\beta}(0,y,k,a,b:p,q)\right).$$

Here is a recurrence relation of unified (p,q)-analog of Apostol type polynomials by the following theorem.

Theorem 2.9. *The following relationship holds true for $\mathcal{P}_{n,\beta}(x,y,k,a,b:p,q)$:*

$$a^b\mathcal{P}_{n,\beta}(x,y,k,a,b:p,q)=\beta^b\sum_{j=0}^{n}\binom{n}{j}_{p,q}q^{\binom{n-j}{2}}\mathcal{P}_{j,\beta}(x,y,k,a,b:p,q)-\frac{[n]_{p,q}!}{[n-k]_{p,q}!}2^{1-k}(x+y)_{p,q}^{n-k}.$$

Proof. Since

$$\frac{a^b}{\left(\beta^b e_{p,q}(z)-a^b\right)e_{p,q}(z)}=\frac{\beta^b}{\beta^b e_{p,q}(z)-a^b}-\frac{1}{e_{p,q}(z)},$$

we have

$$\begin{aligned}\frac{2^{1-k}z^k a^b e_{p,q}(xz)E_{p,q}(yz)}{\left(\beta^b e_{p,q}(z)-a^b\right)e_{p,q}(z)} &= \frac{2^{1-k}z^k\beta^b e_{p,q}(xz)E_{p,q}(yz)}{\beta^b e_{p,q}(z)-a^b}-\frac{2^{1-k}z^k e_{p,q}(xz)E_{p,q}(yz)}{e_{p,q}(z)},\\ a^b\frac{2^{1-k}z^k}{\beta^b e_{p,q}(z)-a^b}e_{p,q}(xz)E_{p,q}(yz) &= \beta^b\frac{2^{1-k}z^k e_{p,q}(xz)E_{p,q}(yz)}{\beta^b e_{p,q}(z)-a^b}e_{p,q}(z)-2^{1-k}z^k e_{p,q}(xz)E_{p,q}(yz).\end{aligned}$$

From here we derive that

$$\begin{aligned}& a^b\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta}(x,y,k,a,b:p,q)\frac{z^n}{[n]_{p,q}!}\\ =\ & \beta^b\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta}(x,y,k,a,b:p,q)\frac{z^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{z^n}{[n]_{p,q}!}-2^{1-k}\sum_{n=0}^{\infty}(x+y)_{p,q}^n\frac{z^{n+k}}{[n]_{p,q}!}.\end{aligned}$$

Using Cauchy product and then equating the coefficients of $\frac{z^n}{[n]_{p,q}}$ completes the proof. □

We provide now the following explicit formula for unified (p,q)-analog of Apostol type polynomials of order α.

Theorem 2.10. *The unified polynomial* $\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)$ *holds the following relation:*

$$\begin{aligned}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q) &= \sum_{j=0}^{n}\binom{n}{j}_{p,q}\frac{2^{k-1}[n]_{p,q}!}{[n+k]_{p,q}!}\mathcal{P}_{n-j,\beta}^{(\alpha)}(0,0,k,a,b:p,q)\\ &\cdot\left(\beta^{b}\sum_{s=0}^{j+k}\binom{j+k}{s}_{p,q}p^{\binom{j+k-s}{2}}\mathcal{P}_{s,\beta}(x,y,k,a,b:p,q)-a^{b}\mathcal{P}_{j+k,\beta}(x,y,k,a,b:p,q)\right).\end{aligned}$$

Proof. The proof of this theorem is derived from the Eq. (4) and Theorem 2.9. So we omit the proof. □

The (p,q)-integral representations of $\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)$ are given in the following theorem.

Theorem 2.11. *(Integral representations)We have*

$$\begin{aligned}\int_{u}^{v}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)\,d_{p,q}x &= p\frac{\mathcal{P}_{n+1,\beta}^{(\alpha)}\left(\frac{v}{p},y,k,a,b:p,q\right)-\mathcal{P}_{n+1,\beta}^{(\alpha)}\left(\frac{u}{p},y,k,a,b:p,q\right)}{[n+1]_{p,q}},\\ \int_{u}^{v}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)\,d_{p,q}y &= p\frac{\mathcal{P}_{n+1,\beta}^{(\alpha)}\left(x,\frac{v}{q},k,a,b:p,q\right)-\mathcal{P}_{n+1,\beta}^{(\alpha)}\left(x,\frac{u}{q},k,a,b:p,q\right)}{[n+1]_{p,q}}.\end{aligned}$$

Proof. By using Lemma 2.5 and Eq. (3), the proof can be easily proved. So we omit it. □

The following theorem involves in the recurrence relationship for unified (p,q)-analog of Apostol type polynomials of order α.

Theorem 2.12. *(Recurrence relationship) The following equality is true for* $n,k\in\mathbb{N}_0$*:*

$$\begin{aligned}&\beta^{b}\sum_{j=0}^{n}\binom{n}{j}_{p,q}p^{\binom{n-j}{2}}m^{j}\mathcal{P}_{j,\beta}^{(\alpha)}(x,0,k,a,b:p,q)-a^{b}\sum_{j=0}^{n}\binom{n}{j}_{p,q}p^{\binom{n-j}{2}}m^{j}\mathcal{P}_{j,\beta}^{(\alpha)}(x,-1,k,a,b:p,q) \quad (7)\\ &=\frac{2^{1-k}[n]_{p,q}!}{[n-k]_{p,q}!}\sum_{j=0}^{n-k}\binom{n-k}{j}_{p,q}p^{\binom{n-k-j}{2}}m^{j+k}\mathcal{P}_{j,\beta}^{(\alpha-1)}(x,-1,k,a,b:p,q).\end{aligned}$$

Proof. Based on the proof technique of Mahmudov in [16], the proof can be made. □

Now we are in a position to state some recurrence relationships for the unified (p,q)-analog of Apostol type polynomials as follows.

Theorem 2.13. *The following recurrence relation holds true for* $n,k\in\mathbb{N}_0$ *and* $x,y\in\mathbb{R}$*:*

$$\begin{aligned}&\mathcal{P}_{n+1,\beta}(x,y,k,a,b:p,q)=yq^{k}p^{n-k}\mathcal{P}_{n,\beta}\left(\frac{q}{p}x,\frac{q}{p}y,k,a,b:p,q\right) \quad (8)\\ &+p^{n+1-k}\frac{[k]_{p,q}}{[n+1]_{p,q}}\mathcal{P}_{n+1,\beta}(x,y,k,a,b:p,q)+xq^{k}p^{n-k}\mathcal{P}_{n,\beta}(x,y,k,a,b:p,q)\\ &-2^{k-1}\beta^{b}\frac{[n]_{p,q}!}{[n+k]_{p,q}!}\sum_{j=0}^{n+k}\binom{n+k}{j}_{p,q}\mathcal{P}_{j,\beta}(x,y,k,a,b:p,q)\,q^{j}p^{n-j}\mathcal{P}_{n+k-j,\beta}(1,0,k,a,b:p,q),\end{aligned}$$

Proof. By using the same method of Kurt's work [9], for $\alpha=1$ *in Definition 2.1, applying* (p,q)*-derivative operator to* $\mathcal{P}_{n,\beta}(x,y,k,a,b:p,q)$*, with respect to z, yields to desired result.* □

We now give the following Theorem 2.14.

Theorem 2.14. *For $n \in \mathbb{N}_0$ and $x, y \in \mathbb{R}$, the following formulas are valid:*

$$\begin{aligned}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q) &= \frac{2^{k-1}[n]_{p,q}!}{[n+k]_{p,q}!}\sum_{s=0}^{n+k}\binom{n+k}{s}_{p,q}\mathcal{P}_{n+k-s,\beta}(0,my,k,a,b:p,q)\,m^{s-n}\\ &\cdot\left\{\beta^b\sum_{j=0}^{s}\binom{s}{j}_{p,q}p^{\binom{j}{2}}m^{-j}\mathcal{P}_{s-j,\beta}^{(\alpha)}(x,0,k,a,b:p,q)-a^b\mathcal{P}_{s,\beta}^{(\alpha)}(x,0,k,a,b:p,q)\right\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q) &= \frac{2^{k-1}[n]_{p,q}!}{[n+k]_{p,q}!}\sum_{s=0}^{n+k}\binom{n+k}{s}_{p,q}\mathcal{P}_{n+k-s,\beta}(mx,0,k,a,b:p,q)\,m^{s-n}\\ &\cdot\left\{\beta^b\sum_{j=0}^{s}\binom{s}{j}_{p,q}\mathcal{P}_{s-j,\beta}^{(\alpha)}(0,y,k,a,b:p,q)\,p^{\binom{j}{2}}m^{-j}-a^b\mathcal{P}_{s,\beta}^{(\alpha)}(0,y,k,a,b:p,q)\right\}.\end{aligned}$$

Proof. *This proof can be made by using the same method of Mahmudov [16]. So we omit it.* □

Combining Theorem 2.12 with Theorem 2.14 gives the following theorem.

Theorem 2.15. *We have*

$$\begin{aligned}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q) &= \frac{2^{k-1}[n]_{p,q}!}{[n+k]_{p,q}!}\sum_{s=0}^{n+k}\binom{n+k}{s}_{p,q}\mathcal{P}_{n+k-s,\beta}(0,my,k,a,b:p,q)\,m^{s-n}\\ &\cdot\left\{\frac{2^{1-k}[s]_{p,q}!}{m^s[s-k]_{p,q}!}\sum_{j=0}^{s-k}\binom{s-k}{j}_{p,q}p^{\binom{s-k-j}{2}}m^{j+k}\mathcal{P}_{j,\beta}^{(\alpha-1)}(x,-1,k,a,b:p,q)\right.\\ &\left.+a^b\sum_{j=0}^{s}\binom{s}{j}_{p,q}p^{\binom{s-j}{2}}m^j\mathcal{P}_{j,\beta}^{(\alpha)}(x,-1,k,a,b:p,q)-a^b\mathcal{P}_{s,\beta}^{(\alpha)}(x,0,k,a,b:p,q)\right\}.\end{aligned}$$

In the case when $\alpha = 1$ in Theorem 2.15, we have the following corollary.

Corollary 2.16. *We have*

$$\begin{aligned}\mathcal{P}_{n,\beta}(x,y,k,a,b:p,q) &= \frac{2^{k-1}[n]_{p,q}!}{[n+k]_{p,q}!}\sum_{s=0}^{n+k}\binom{n+k}{s}_{p,q}\mathcal{P}_{n+k-s,\beta}(0,my,k,a,b:p,q)\,m^{s-n}\\ &\cdot\left\{\frac{2^{1-k}[s]_{p,q}!}{m^s[s-k]_{p,q}!}\sum_{j=0}^{s-k}\binom{s-k}{j}_{p,q}p^{\binom{s-k-j}{2}}m^{j+k}(x-1)_{p,q}^j\right.\\ &\left.+a^b\sum_{j=0}^{s}\binom{s}{j}_{p,q}p^{\binom{s-j}{2}}m^j\mathcal{P}_{j,\beta}(x,-1,k,a,b:p,q)-a^b\mathcal{P}_{s,\beta}(x,0,k,a,b:p,q)\right\}.\end{aligned}$$

Let us define (p,q)-analog of Stirling numbers of the second kind of order v as follows.

Definition 2.17. *(p,q)-analog of Stirling numbers $S_{p,q}(n,v;a,b,\beta)$ of the second kind of order v is defined by means of the following generating function:*

$$\sum_{n=0}^{\infty} S_{p,q}(n,v;a,b,\beta)\frac{z^n}{[n]_{p,q}!} = \frac{\left(\beta^b e_{p,q}(z) - a^b\right)^v}{[v]_{p,q}!}.$$

A correlation between the family of unified polynomials $\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)$ and the generalized (p,q)-Stirling numbers $S_{p,q}(n,v;a,b,\beta)$ of the second kind of order v is presented in following Theorem 2.18.

Theorem 2.18. *The following relationship*

$$\mathcal{P}_{n-vk,\beta}^{(\alpha)}(x,y,k,a,b:p,q) = 2^{(k-1)v}\frac{[v]_{p,q}!}{[vk]_{p,q}!}\sum_{j=0}^{n}\frac{\binom{n}{j}_{p,q}}{\binom{n}{kv}_{p,q}}\mathcal{P}_{j,\beta}^{(\alpha+v)}(x,y,k,a,b:p,q)\,S_{p,q}(n-j,v;a,b,\beta)$$

is true.

Proof. It follows from Definition 2.17. □

In the case when $\alpha = 0$ in Theorem 2.18, we have the following corollary.

Corollary 2.19. *The following correlation holds true:*

$$2^{(1-k)v}\frac{[vk]_{p,q}!}{[v]_{p,q}!}(x+y)_{p,q}^{n-vk} = \sum_{j=0}^{n}\frac{\binom{n}{j}_{p,q}}{\binom{n}{vk}_{p,q}}\mathcal{P}_{j,\beta}^{(v)}(x,y,k,a,b:p,q)\,S_{p,q}(n-j,v;a,b,\beta).$$

3. Conclusion

In this paper, we have introduced unified (p,q)-analog of Apostol type polynomials of order α. We have also analyzed some properties of them including addition property, derivative properties, recurrence relationships, integral representations and so on. By defining the generalized (p,q)-Stirling numbers of the second kind of order v, a correlation between these numbers and unified (p,q)-analog of Apostol type polynomials of order α is obtained. We note that the results obtained here reduce to known results of unified q-polynomials when $p = 1$. Also, when $q \to p = 1$, our results in this paper turn into the unified Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials.

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