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Gauss-Seidel Type Algorithms for a Class of Variational Inequalities

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Abstract. In this paper, we consider a new system of extended general quasi variational inequalities involving six nonlinear operators. Using projection operator technique, we show that system of extended general quasi variational inequalities is equivalent to a system of fixed point problems. Using this alternative equivalent formulation, we propose and analyze Gauss-Seidel type algorithms for solving a system of extended general quasi variational inequalities. Convergence of new method is discussed under some suitable conditions. Several special cases are discussed. Results obtained in this paper continue to hold for these problems.

1. Introduction

Variational inequalities introduced in the early sixties have played a critical and significant part in the study of several unrelated problems arising in finance, economics, network analysis, transportation, elasticity and optimization. Variational inequalities theory has witnessed an explosive growth in theoretical advances, algorithmic development and applications across all disciplines of pure and applied sciences, see [1-38]. It combines novel theoretical and algorithmic advances with new domain of applications. As a result of interaction between different branches of mathematical and engineering sciences, we now have a variety of techniques to suggest and analyze various iterative algorithms for solving variational inequalities and related optimization problems. Analysis of these problems requires a blend of techniques from convex analysis, functional analysis and numerical analysis.

In this paper, we introduce a new system of extended general quasi variational inequalities. Using projection technique, we prove that system of extended general quasi variational inequalities is equivalent to a system of nonlinear implicit projection equations. This alternative equivalent formulation help us to propose some new Gauss-Seidel type numerical schemes for solving a system of extended general quasi variational inequalities and its variant forms. We discuss the convergence of these methods under some mild conditions. Several special cases are also discussed. The comparison of these methods with other methods is a subject of future research.

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2. Preliminaries and Basic Results

Let H be a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let K_1, K_2 be two closed and convex sets in H.

For given nonlinear operators $T_1, T_2, g_1, g_2, h_1, h_2 : H \to H$ and two point-to-set mappings $K_1 : y \to K_1(y)$ and $K_2 : x \to K_2(x)$, which associate two closed and convex valued sets $K_1(y)$ and $K_2(x)$ with any elements $x, y \in H$, consider a problem of finding $x, y \in H : h_1(y) \in K_1(y), h_2(x) \in K_2(x)$ such that

$$\left.\begin{array}{l}\langle \rho_1 T_1 x + h_1(y) - g_1(x), g_1(v) - h_1(y)\rangle \geq 0, \quad \forall v \in H : g_1(v) \in K_1(y) \\ \langle \rho_2 T_2 y + h_2(x) - g_2(y), g_2(v) - h_2(x)\rangle \geq 0, \quad \forall v \in H : g_2(v) \in K_2(x)\end{array}\right\} \tag{1}$$

where $\rho_1 > 0$ and $\rho_2 > 0$ are constants. The system (1) is called a system of extended general quasi variational inequalities with six nonlinear operators.

We now list some special cases of the system of extended general quasi variational inequalities (1).

I. If $g_1 = g_2 = g$, $h_1 = h_2 = h$, $K_1(y) = K(y)$ and $K_2(x) = K(x)$, then problem (1) reduces to find $x, y \in H : h(y) \in K(y), h(x) \in K(x)$ such that

$$\left.\begin{array}{l}\langle \rho_1 T_1 x + h(y) - g(x), g(v) - h(y)\rangle \geq 0, \quad \forall v \in H : g(v) \in K(y) \\ \langle \rho_2 T_2 y + h(x) - g(y), g(v) - h(x)\rangle \geq 0, \quad \forall v \in H : g(v) \in K(x)\end{array}\right\}. \tag{2}$$

The problem of type (2) is called a system of extended general quasi variational inequalities with four nonlinear operators.

II. If $K_1(y) = K_1$ and $K_2(x) = K_2$, then problem (1) collapse to find $x, y \in H : h_1(y) \in K_1$, $h_2(x) \in K_2$ such that

$$\left.\begin{array}{l}\langle \rho_1 T_1 x + h_1(y) - g_1(x), g_1(v) - h_1(y)\rangle \geq 0, \quad \forall v \in H : g_1(v) \in K_1 \\ \langle \rho_2 T_2 y + h_2(x) - g_2(y), g_2(v) - h_2(x)\rangle \geq 0, \quad \forall v \in H : g_2(v) \in K_2\end{array}\right\}, \tag{3}$$

is a system of extended general variational inequalities involving six nonlinear operators. Recently, this problem was considered by Noor et al [35, 36].

III. If $h = g$, then problem (2) reduces to find $x, y \in H : g(y) \in K(y), g(x) \in K(x)$ such that

$$\left.\begin{array}{l}\langle \rho_1 T_1 x + g(y) - g(x), g(v) - g(y)\rangle \geq 0, \quad \forall v \in H : g(v) \in K(y) \\ \langle \rho_2 T_2 y + g(x) - g(y), g(v) - g(x)\rangle \geq 0, \quad \forall v \in H : g(v) \in K(x)\end{array}\right\}, \tag{4}$$

which is called a system of general quasi variational inequalities with three nonlinear operators.

IV. If $g_1 = g_2 = h_1 = h_2 = I$, identity operator, then system (1) reduces to find $y \in K_1(y)$ and $x \in K_2(x)$ such that

$$\left.\begin{array}{l}\langle \rho_1 T_1 x + y - x, v - y\rangle \geq 0, \quad \forall v \in K_1(y) \\ \langle \rho_2 T_2 y + x - y, v - x\rangle \geq 0, \quad \forall v \in K_2(x)\end{array}\right\}, \tag{5}$$

is called a system of quasi variational inequalities. This problem was introduced by Noor and Noor [33].

V. If $K_1(y) = K_2(x) = K$, then system (5) collapse to find $x, y \in K$ such that

$$\left.\begin{array}{l}\langle \rho_1 T_1 x + y - x, v - y\rangle \geq 0, \quad \forall v \in K \\ \langle \rho_2 T_2 y + y - x, v - x\rangle \geq 0, \quad \forall v \in K\end{array}\right\}, \tag{6}$$

which is a system of variational inequalities and has been studied extensively in recent years.

VI. If $T_1 = T_2 = T$ and $K(x) = K(y) = K$, then problem (2) reduces to find $u \in H : h(u) \in K$ such that

$$\langle \rho Tu + h(u) - g(u), g(v) - h(u) \rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{7}$$

which is called extended general variational inequality, introduced and studied by Noor [27].

VII. If $T_1 = T_2 = T$, then problem (4) reduces to find $u \in H : g(u) \in K(u)$ such that

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \quad \forall v \in H : g(v) \in K(u), \tag{8}$$

which is called general quasi variational inequality. This problem was introduced and studied by Noor [19].

VIII. If $K(u) = K$, then problem (8) reduces to find $u \in H : g(u) \in K$ such that

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \quad \forall v \in H : g(v) \in K. \tag{9}$$

This problem is called general variational inequality. This problem was introduced and studied by Noor [18], in 1988. It turned out that odd order and nonsymmetric obstacle, free, moving, unilateral and equilibrium problems arising in various branches of pure and applied sciences can be studied via general variational inequalities.

For suitable and appropriate choice of operators and spaces, one can obtain several new and known classes of variational inequalities.

We now summarize some basic properties and related definitions which are essential in the following discussions.

Definition 2.1. *A nonlinear operator $T : H \to H$ is said to be:*

(i) *strongly monotone, if there exists a constant $\alpha > 0$ such that*

$$\langle Tu - Tv, u - v \rangle \geq \alpha \|u - v\|^2, \quad \forall u, v \in H.$$

(ii) *Lipschitz continuous if there exists a constant $\beta > 0$ such that*

$$\|Tu - Tv\| \leq \beta \|u - v\|, \quad \forall u, v \in H.$$

Note that, if T satisfies (i) and (ii), then $\alpha \leq \beta$.

Lemma 2.2. *[30]Let $K(u)$ be a closed convex set in H. Then, for a given $z \in H, u \in K(u)$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0 \quad \forall v \in K(u),$$

if and only if

$$u = P_{K(u)}[z],$$

where $P_{K(u)}$ is the projection of H onto the closed convex valued set $K(u)$ in H.

We would like to point out that the implicit projection operator $P_{K(u)}$ is not nonexpansive. We shall assume that the implicit projection operator $P_{K(u)}$ satisfies the Lipschitz type continuity condition. This condition plays an important and fundamental role in the existence theory and in developing numerical methods for solving problem (1) and its variant forms.

Assumption 2.3. *[34] The implicit projection operator $P_{K(u)}$ satisfies the condition*

$$\| P_{K(u)}[w] - P_{K(v)}[w] \| \leq \nu \| u - v\|, \quad \forall u, v, w \in H, \tag{10}$$

where $\nu > 0$ is a constant.

In many important applications [16, 17] the convex-valued set $K(u)$ can be written as

$$K(u) = m(u) + K, \tag{11}$$

where $m(u)$ is a point-to-point mapping and K is a convex set. In this case, we have

$$P_{K(u)}[w] = P_{m(u)+K}[w] = m(u) + P_K[w - m(u)], \quad \forall u, v \in H. \tag{12}$$

We note that if $K(u)$ is as, defined by (11), and $m(u)$ is a Lipschitz continuous mapping with constant $\gamma > 0$, then using the relation (12), we have

$$\begin{aligned} \left\| P_{K(u)}[w] - P_{K(v)}[w] \right\| &= \| m(u) + P_K[w - m(u)] - m(v) - P_K[w - m(v)] \| \\ &\leq \| m(u) - m(v) \| + \| P_K[w - m(u)] - P_K[w - m(v)] \| \\ &\leq 2 \| m(u) - m(v) \| \\ &\leq 2\gamma \| u - v \|, \quad \forall u, v, w \in H, \end{aligned}$$

which shows that Assumption 2.3 holds with $\nu = 2\gamma > 0$.

Lemma 2.4. *[7] If $\{\delta_n\}_{n=0}^{\infty}$ is a nonnegative sequence satisfying the following inequality:*

$$\delta_{n+1} \leq (1 - \lambda_n)\delta_n + \sigma_n \quad \text{for all } n \geq 0,$$

with $0 \leq \lambda_n \leq 1$, $\sum_{n=0}^{\infty} \lambda_n = \infty$, and $\sigma_n = o(\lambda_n)$, then $\lim_{n\to\infty} \delta_n = 0$.

3. Main Results

In this section, we show that system of extended general quasi variational inequalities (1) is equivalent to a system of fixed point problems. This alternative equivalent formulation is used to suggest algorithms for solving problem (1), using the technique of Noor and Noor [33].

Lemma 3.1. *The system of extended general quasi variational inequalities* (1) *has a solution, $x, y \in H : h_1(y) \in K_1(y), h_2(x) \in K_2(x)$, if and only if, $x, y \in H : h_1(y) \in K_1(y), h_2(x) \in K_2(x)$ satisfies the relations*

$$\begin{aligned} h_1(y) &= P_{K_1(y)}[g_1(x) - \rho_1 T_1 x] \qquad (13) \\ h_2(x) &= P_{K_2(x)}[g_2(y) - \rho_2 T_2 y], \qquad (14) \end{aligned}$$

where $\rho_1 > 0$ and $\rho_2 > 0$ are constants.

Lemma 3.1 implies that the system (1) is equivalent to the fixed point problems (13) and (14). This alternative equivalent formulation is very useful from numerical and theoretical point of view. Using the fixed point formulations (13) and (14), we suggest and analyze some iterative algorithms.

We can rewrite (13) and (14) in the following equivalent forms:

$$\begin{aligned} y &= (1 - \beta_n) y + \beta_n \left\{ y - h_1(y) + P_{K_1(y)}[g_1(x) - \rho_1 T_1 x] \right\} \qquad (15) \\ x &= (1 - \alpha_n) x + \alpha_n \left\{ x - h_2(x) + P_{K_2(x)}[g_2(y) - \rho_2 T_2 y] \right\}, \qquad (16) \end{aligned}$$

where $0 \leq \alpha_n, \beta_n \leq 1$ for all $n \geq 0$.

This alternative equivalent formulation is used to suggest following algorithms for solving system of extended general quasi variational inequalities (1) and its variant forms.

Algorithm 3.2. *For given $x_0, y_0 \in H : h_1(y_0) \in K_1(y)$ and $h_2(x_0) \in K_2(x)$ find x_{n+1} and y_{n+1} by the iterative schemes*

$$y_{n+1} = (1-\beta_n) y_n + \beta_n \left\{ y_n - h_1(y_n) + P_{K_1(y_n)} \left[g_1(x_n) - \rho_1 T_1 x_n \right] \right\} \tag{17}$$

$$x_{n+1} = (1-\alpha_n) x_n + \alpha_n \left\{ x_n - h_2(x_n) + P_{K_2(x_n)} \left[g_2(y_{n+1}) - \rho_2 T_2 y_{n+1} \right] \right\}, \tag{18}$$

where $0 \le \alpha_n, \beta_n \le 1$ for all $n \ge 0$.

Algorithm 3.2 can be viewed as a Gauss-Seidel method for solving a system of extended general quasi variational inequalities. This is an implicit type method which is quite different from the Algorithm 3.2 of Noor et al [36].

We now discuss some special cases of Algorithm 3.2.

I. If $g_1 = g_2 = g$, $h_1 = h_2 = h$, $K_1(y) = K(y)$ and $K_2(x) = K(x)$, then Algorithm 3.2 reduces to following projection algorithm for solving the system (2).

Algorithm 3.3. *For given $x_0, y_0 \in H : h(x_0) \in K(x), h(y_0) \in K(y)$ find x_{n+1} and y_{n+1} by the iterative schemes*

$$y_{n+1} = (1-\beta_n) y_n + \beta_n \left\{ y_n - h(y_n) + P_{K(y_n)} \left[g(x_n) - \rho_1 T_1 x_n \right] \right\}$$

$$x_{n+1} = (1-\alpha_n) x_n + \alpha_n \left\{ x_n - h(x_n) + P_{K(x_n)} \left[g(y_{n+1}) - \rho_2 T_2 y_{n+1} \right] \right\},$$

where $0 \le \alpha_n, \beta_n \le 1$ for all $n \ge 0$.

II. If $h = g$, then Algorithm 3.3 reduces to the following algorithm for solving system (4).

Algorithm 3.4. *For given $x_0, y_0 \in H : g(x_0) \in K(x), g(y_0) \in K(y)$, compute sequences $\{x_n\}$ and $\{y_n\}$ by the iterative schemes*

$$y_{n+1} = (1-\beta_n) y_n + \beta_n \left\{ y_n - g(y_n) + P_{K(y_n)} \left[g(x_n) - \rho_1 T_1 x_n \right] \right\}$$

$$x_{n+1} = (1-\alpha_n) x_n + \alpha_n \left\{ x_n - g(x_n) + P_{K(x_n)} \left[g(y_{n+1}) - \rho_2 T_2 y_{n+1} \right] \right\},$$

where $0 \le \alpha_n, \beta_n \le 1$ for all $n \ge 0$.

III. If $K_1(y) = K_1$ and $K_2(x) = K_2$, then Algorithm 3.2 collapse to the following iterative method for solving system (3).

Algorithm 3.5. *[36]For arbitrary chosen initial points $x_0, y_0 \in H : h_1(y_0) \in K_1, h_2(x_0) \in K_2$, sequences $\{x_n\}$ and $\{y_n\}$ are computed by*

$$y_{n+1} = (1-\beta_n) y_n + \beta_n \{ y_n - h_1(y_n) + P_{K_1} [g_1(x_n) - \rho_1 T_1 x_n] \}$$

$$x_{n+1} = (1-\alpha_n) x_n + \alpha_n \{ x_n - h_2(x_n) + P_{K_2} [g_2(y_{n+1}) - \rho_2 T_2 y_{n+1}] \},$$

where $0 \le \alpha_n, \beta_n \le 1$ for all $n \ge 0$.

IV. If $T_1 = T_2 = T$, $g_1 = g_2 = h_1 = h_2 = I$, identity operator, then Algorithm 3.2 reduces to the following algorithm.

Algorithm 3.6. *For given $x_0 \in K_2(x), y_0 \in K_1(y)$ find the approximate solution x_n, y_n by the iterative schemes*

$$y_{n+1} = (1-\beta_n) y_n + \beta_n P_{K_1(y_n)} [x_n - \rho_1 T x_n]$$

$$x_{n+1} = (1-\alpha_n) x_n + \alpha_n P_{K_2(x_n)} [y_{n+1} - \rho_2 T y_{n+1}],$$

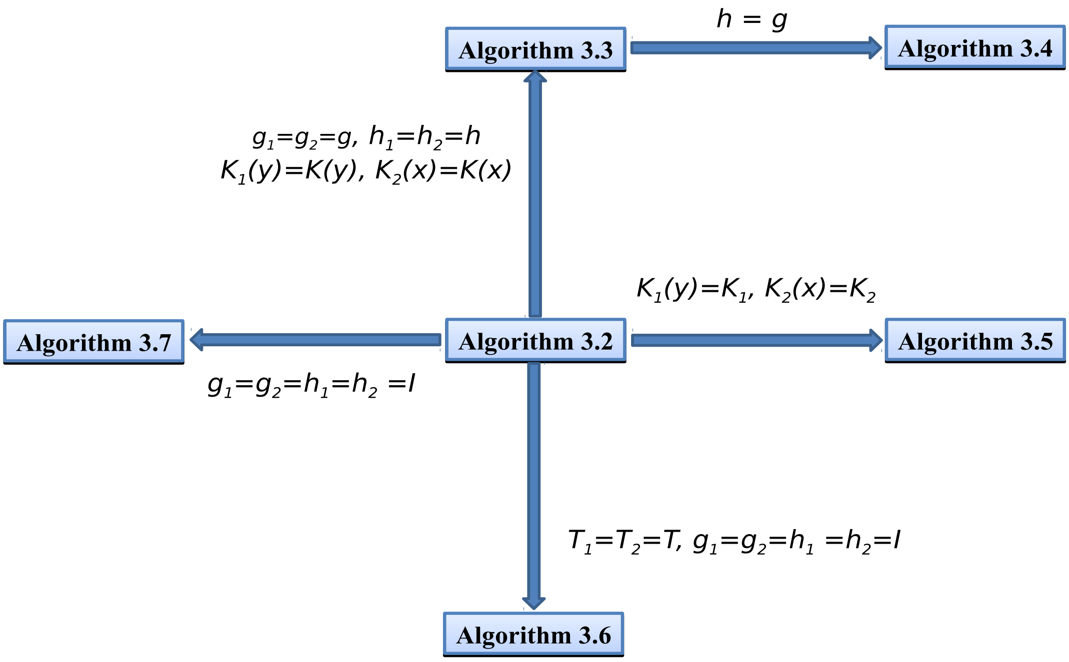
where $0 \le \alpha_n, \beta_n \le 1$ for all $n \ge 0$.

V. If $g_1 = g_2 = h_1 = h_2 = I$, identity operator, then Algorithm 3.2 reduces to:

Algorithm 3.7. *For given* $x_0 \in K_2(x)$, $y_0 \in K_1(y)$ *find the approximate solution* x_n, y_n *by the iterative schemes*

$$\begin{aligned} y_{n+1} &= (1-\beta_n)\, y_n + \beta_n P_{K_1(y_n)}\left[x_n - \rho_1 T_1 x_n\right] \\ x_{n+1} &= (1-\alpha_n)\, x_n + \alpha_n P_{K_2(x_n)}\left[y_{n+1} - \rho_2 T_2 y_{n+1}\right], \end{aligned}$$

where $0 \le \alpha_n, \beta_n \le 1$ *for all* $n \ge 0$.



For suitable and appropriate choice of operators and spaces, one can obtain several new and known iterative methods for solving system of extended general quasi variational inequalities and related problems.

We now investigate the convergence analysis of Algorithm 3.2. This is the main motivation of our next result.

Theorem 3.8. *Let operators* $T_1, T_2, g_1, g_2, h_1, h_2 : H \to H$ *be strongly monotone with constants* $\alpha_{T_1} > 0$, $\alpha_{T_2} > 0$, $\alpha_{g_1} > 0$, $\alpha_{g_2} > 0$, $\alpha_{h_1} > 0$, $\alpha_{h_2} > 0$ *and Lipschitz continuous with constants* $\beta_{T_1} > 0$, $\beta_{T_2} > 0$, $\beta_{g_1} > 0$, $\beta_{g_2} > 0$, $\beta_{h_1} > 0$, $\beta_{h_2} > 0$ *respectively. If Assumption 2.3 and following conditions hold:*

(i) $\theta_{T_1} = \sqrt{1 - 2\rho_1\alpha_{T_1} + \rho_1^2\beta_{T_1}^2} < 1.$

(ii) $\theta_{T_2} = \sqrt{1 - 2\rho_2\alpha_{T_2} + \rho_2^2\beta_{T_2}^2} < 1.$

(iii) $0 \le \alpha_n, \beta_n \le 1$ *for all* $n \ge 0$,

$$\begin{aligned} \alpha_n(1-\nu-\theta_{h_2}) - \beta_n\left(\theta_{g_1} + \theta_{T_1}\right) &\ge 0 \\ \beta_n(1-\nu-\theta_{h_1}) &\ge 0 \\ \alpha_n\left(\theta_{g_2} + \theta_{T_2}\right) &\ge 0 \quad , \end{aligned}$$

such that

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\alpha_n\left(1-\nu-\theta_{h_2}\right)-\beta_n\left(\theta_{g_1}+\theta_{T_1}\right)\right) &= \infty\\ \sum_{n=0}^{\infty}\beta_n\left(1-\nu-\theta_{h_1}\right) &= \infty\\ \sum_{n=0}^{\infty}\alpha_n\left(\theta_{g_2}+\theta_{T_2}\right) &= \infty,\end{aligned}$$

where

$$\theta_{g_1} = \sqrt{1-2\alpha_{g_1}+\beta_{g_1}^2},\quad \theta_{g_2} = \sqrt{1-2\alpha_{g_2}+\beta_{g_2}^2},$$

and

$$\theta_{h_1} = \sqrt{1-2\alpha_{h_1}+\beta_{h_1}^2},\quad \theta_{h_2} = \sqrt{1-2\alpha_{h_2}+\beta_{h_2}^2},$$

then sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 3.2 converge to x and y respectively.

Proof. Let $x, y \in H : h_1(y) \in K_1(y), h_2(x) \in K_2(x)$ be a solution of (1). Then from (16), (18) and using Assumption 2.3, we have

$$\begin{aligned}\|x_{n+1}-x\| &= \|(1-\alpha_n)x_n+\alpha_n\left\{x_n-h_2(x_n)+P_{K_2(x_n)}\left[g_2(y_{n+1})-\rho_2T_2y_{n+1}\right]\right\}\\ &\quad -(1-\alpha_n)x-\alpha_n\left\{x-h_2(x)+P_{K_2(x)}\left[g_2(y)-\rho_2T_2y\right]\right\}\|\\ &\le (1-\alpha_n)\|x_n-x\|+\alpha_n\|x_n-x-(h_2(x_n)-h_2(x))\|\\ &\quad +\alpha_n\left\|P_{K_2(x_n)}\left[g_2(y_{n+1})-\rho_2T_2y_{n+1}\right]-P_{K_2(x)}\left[g_2(y)-\rho_2T_2y\right]\right\|\\ &\le (1-\alpha_n)\|x_n-x\|+\alpha_n\|x_n-x-(h_2(x_n)-h_2(x))\|\\ &\quad +\alpha_n\left\|P_{K_2(x_n)}\left[g_2(y_{n+1})-\rho_2T_2y_{n+1}\right]-P_{K_2(x)}\left[g_2(y_{n+1})-\rho_2T_2y_{n+1}\right]\right\|\\ &\quad +\alpha_n\left\|P_{K_2(x)}\left[g_2(y_{n+1})-\rho_2T_2y_{n+1}\right]-P_{K_2(x)}\left[g_2(y)-\rho_2T_2y\right]\right\|\\ &\le (1-\alpha_n)\|x_n-x\|+\nu\alpha_n\|x_n-x\|+\alpha_n\|x_n-x-(h_2(x_n)-h_2(x))\|\\ &\quad +\alpha_n\left\|y_{n+1}-y-(g_2(y_{n+1})-g_2(y))\right\|\\ &\quad +\alpha_n\left\|y_{n+1}-y-\rho_2(T_2y_{n+1}-T_2y)\right\|. \end{aligned}\tag{19}$$

Since operator T_2 is strongly monotone and Lipschitz continuous with constants $\alpha_{T_2} > 0$ and $\beta_{T_2} > 0$, respectively. Then it follows that

$$\begin{aligned}\left\|y_{n+1}-y-\rho_2(T_2y_{n+1}-T_2y)\right\|^2 &= \left\|y_{n+1}-y\right\|^2-2\rho_2\langle T_2y_{n+1}-T_2y, y_{n+1}-y\rangle\\ &\quad +\left\|T_2y_{n+1}-T_2y\right\|^2\\ &\le \left(1-2\rho_2\alpha_{T_2}+\rho_2^2\beta_{T_2}^2\right)\left\|y_{n+1}-y\right\|^2. \end{aligned}\tag{20}$$

In a similar way, we have

$$\|x_n-x-(h_2(x_n)-h_2(x))\|^2 \le \left(1-2\alpha_{h_2}+\beta_{h_2}^2\right)\|x_n-x\|^2, \tag{21}$$

and

$$\left\|y_{n+1}-y-(g_2(y_{n+1})-g_2(y))\right\|^2 \le \left(1-2\alpha_{g_2}+\beta_{g_2}^2\right)\left\|y_{n+1}-y\right\|^2, \tag{22}$$

where we have used the strong monotonicity and Lipschitz continuity of operators g_2, h_2 with constants $\alpha_{g_2} > 0$, $\alpha_{h_2} > 0$ and $\beta_{g_2} > 0$, $\beta_{h_2} > 0$, respectively.

Combining (19) − (22), we obtain

$$\begin{aligned} \|x_{n+1} - x\| &\leq (1-\alpha_n)\|x_n - x\| + \nu\alpha_n\|x_n - x\| + \alpha_n\sqrt{1 - 2\alpha_{h_2} + \beta_{h_2}^2}\,\|x_n - x\| \\ &\quad + \alpha_n\sqrt{1 - 2\alpha_{g_2} + \beta_{g_2}^2}\,\|y_{n+1} - y\| + \alpha_n\sqrt{1 - 2\rho_2\alpha_{T_2} + \rho_2^2\beta_{T_2}^2}\,\|y_{n+1} - y\| \\ &= (1 - \alpha_n(1 - \nu - \theta_{h_2}))\|x_n - x\| + \alpha_n\left(\theta_{g_2} + \theta_{T_2}\right)\|y_{n+1} - y\|. \end{aligned} \tag{23}$$

Similarly, using strong monotonicity and Lipschitz continuity of operators T_1, g_1, h_1 with constants $\alpha_{T_1} > 0, \alpha_{g_1} > 0, \alpha_{h_1} > 0$ and $\beta_{T_1} > 0, \beta_{g_1} > 0, \beta_{h_1} > 0$, respectively. From (15), (17) and using Assumption 2.3, we have

$$\begin{aligned} \|y_{n+1} - y\| &= \|(1-\beta_n)y_n + \beta_n\left\{y_n - h_1(y_n) + P_{K_1(y_n)}[g_1(x_n) - \rho_1 T_1 x_n]\right\} \\ &\quad - (1-\beta_n)y - \beta_n\left\{y - h_1(y) + P_{K_1(y)}[g_1(x) - \rho_1 T_1 x]\right\}\| \\ &\leq (1-\beta_n)\|y_n - y\| + \beta_n\|y_n - y - (h_1(y_n) - h_1(y))\| \\ &\quad + \beta_n\left\|P_{K_1(y_n)}[g_1(x_n) - \rho_1 T_1 x_n] - P_{K_1(y)}[g_1(x) - \rho_1 T_1 x]\right\| \\ &\leq (1-\beta_n)\|y_n - y\| + \beta_n\|y_n - y - (h_1(y_n) - h_1(y))\| \\ &\quad + \beta_n\left\|P_{K_1(y_n)}[g_1(x_n) - \rho_1 T_1 x_n] - P_{K_1(y)}[g_1(x_n) - \rho_1 T_1 x_n]\right\| \\ &\quad + \beta_n\left\|P_{K_1(y)}[g_1(x_n) - \rho_1 T_1 x_n] - P_{K_1(y)}[g_1(x) - \rho_1 T_1 x]\right\| \\ &\leq (1-\beta_n)\|y_n - y\| + \beta_n\|y_n - y - (h_1(y_n) - h_1(y))\| \\ &\quad + \nu\beta_n\|y_n - y\| + \beta_n\|x_n - x - (g_1(x_n) - g_1(x))\| \\ &\quad + \beta_n\|x_n - x - \rho_1(T_1 x_n - T_1 x)\| \\ &= (1 - \beta_n(1 - \nu - \theta_{h_1}))\|y_n - y\| + \beta_n\left(\theta_{g_1} + \theta_{T_1}\right)\|x_n - x\|. \end{aligned} \tag{24}$$

Adding (23) and (24), we have

$$\begin{aligned} &\|x_{n+1} - x\| + \|y_{n+1} - y\| \\ &\leq (1 - \alpha_n(1 - \nu - \theta_{h_2}))\|x_n - x\| + \alpha_n\left(\theta_{g_2} + \theta_{T_2}\right)\|y_{n+1} - y\| \\ &\quad + (1 - \beta_n(1 - \nu - \theta_{h_1}))\|y_n - y\| + \beta_n\left(\theta_{g_1} + \theta_{T_1}\right)\|x_n - x\| \\ &= \left(1 - \alpha_n(1 - \nu - \theta_{h_2}) + \beta_n\left(\theta_{g_1} + \theta_{T_1}\right)\right)\|x_n - x\| \\ &\quad + \alpha_n\left(\theta_{g_2} + \theta_{T_2}\right)\|y_{n+1} - y\| + (1 - \beta_n(1 - \nu - \theta_{h_1}))\|y_n - y\|. \end{aligned}$$

From which, we have

$$\begin{aligned} &\|x_{n+1} - x\| + \left(1 - \alpha_n\left(\theta_{g_2} + \theta_{T_2}\right)\right)\|y_{n+1} - y\| \\ &\leq \left(1 - \alpha_n(1 - \nu - \theta_{h_2}) + \beta_n\left(\theta_{g_1} + \theta_{T_1}\right)\right)\|x_n - x\| + (1 - \beta_n(1 - \nu - \theta_{h_1}))\|y_n - y\|, \end{aligned}$$

which implies that

$$\begin{aligned} \|x_{n+1} - x\| + \nu_3\|y_{n+1} - y\| &\leq \max(\nu_1, \nu_2)\left(\|x_n - x\| + \|y_n - y\|\right) \\ &= \theta\left(\|x_n - x\| + \|y_n - y\|\right), \end{aligned} \tag{25}$$

where

$$\begin{aligned}\theta &= \max(\nu_1, \nu_2)\\ \nu_1 &= 1-\left(\alpha_n(1-\nu-\theta_{h_2})-\beta_n\left(\theta_{g_1}+\theta_{T_1}\right)\right)\\ \nu_2 &= 1-\beta_n(1-\nu-\theta_{h_1})\\ \nu_3 &= 1-\alpha_n\left(\theta_{g_2}+\theta_{T_2}\right).\end{aligned}$$

Using assumption (*iii*), we have $\theta < 1$. Thus from (25), it follows that

$$\lim_{n\to\infty}\left[\|x_{n+1}-x\|+\nu_3\left\|y_{n+1}-y\right\|\right]=0.$$

This implies that

$$\lim_{n\to\infty}\|x_{n+1}-x\|=0 \quad\text{and}\quad \lim_{n\to\infty}\left\|y_{n+1}-y\right\|=0.$$

This is the desired result. □

Using Lemma 3.1, one can easily show that $x, y \in H : h_1(y) \in K_1(y), h_2(x) \in K_2(x)$ is a solution of (1) if and only if, $x, y \in H : h_1(y) \in K_1(y), h_2(x) \in K_2(x)$ satisfies

$$h_1(y) = P_{K_1(y)}[z] \tag{26}$$
$$h_2(x) = P_{K_2(x)}[w] \tag{27}$$
$$z = g_1(x)-\rho_1 T_1 x \tag{28}$$
$$w = g_2(y)-\rho_2 T_2 y. \tag{29}$$

This alternative formulation can be used to suggest and analyze the following iterative methods for solving the system (1).

Algorithm 3.9. *For given $x_0, y_0 \in H : h_1(y_0) \in K_1(y), h_2(x_0) \in K_2(x)$ find x_{n+1} and y_{n+1} by the iterative schemes*

$$y_{n+1} = (1-\beta_n)y_n+\beta_n\left\{y_n-h_1(y_n)+P_{K_1(y_n)}[z_n]\right\} \tag{30}$$
$$x_{n+1} = (1-\alpha_n)x_n+\alpha_n\left\{x_n-h_2(x_n)+P_{K_2(x_n)}[w_n]\right\} \tag{31}$$
$$z_n = g_1(x_n)-\rho_1 T_1 x_n \tag{32}$$
$$w_n = g_2(y_{n+1})-\rho_2 T_2 y_{n+1}, \tag{33}$$

where $0 \le \alpha_n, \beta_n \le 1$ for all $n \ge 0$.

We now discuss some special cases of Algorithm 3.9.

I. If $g_1 = g_2 = g$, $h_1 = h_2 = h$, $K_1(y) = K(y)$ and $K_2(x) = K(x)$, then Algorithm 3.9 reduces to:

Algorithm 3.10. *For given $x_0, y_0 \in H : h(x_0), h(y_0) \in K$ find the approximate solutions x_{n+1} and y_{n+1} by the iterative schemes*

$$\begin{aligned}y_{n+1} &= (1-\beta_n)y_n+\beta_n\left\{y_n-h(y_n)+P_{K(y_n)}[z_n]\right\}\\ x_{n+1} &= (1-\alpha_n)x_n+\alpha_n\left\{x_n-h(x_n)+P_{K(x_n)}[w_n]\right\}\\ z_n &= g(x_n)-\rho_1 T_1 x_n\\ w_n &= g(y_{n+1})-\rho_2 T_2 y_{n+1},\end{aligned}$$

where $0 \le \alpha_n, \beta_n \le 1$ for all $n \ge 0$.

II. If $g_1 = g_2 = h_1 = h_2 = I$, identity operator, and $\alpha_n = \beta_n = 1$, then Algorithm 3.9 reduces to:

Algorithm 3.11. *For given $y_0 \in K_1(y), x_0 \in K_2(x)$, find the approximate solutions x_{n+1} and y_{n+1} by the iterative schemes*

$$\begin{aligned} y_{n+1} &= P_{K_1(y_n)}[z_n] \\ x_{n+1} &= P_{K_2(x_n)}[w_n] \\ z_n &= x_n - \rho_1 T_1 x_n \\ w_n &= y_{n+1} - \rho_2 T_2 y_{n+1}, \end{aligned}$$

for all $n \geq 0$.

For appropriate and suitable choice of operators and spaces, one can obtain several new and known iterative methods for solving system of extended general quasi variational inequalities and related optimization problems.

We now consider the convergence analysis of Algorithm 3.9, using the technique of Theorem 3.8. For the sake of completeness and to convey an idea, we include all the details.

Theorem 3.12. *Let operators $T_1, T_2, g_1, g_2, h_1, h_2 : H \to H$ be strongly monotone with constants $\alpha_{T_1} > 0$, $\alpha_{T_2} > 0$, $\alpha_{g_1} > 0$, $\alpha_{g_2} > 0$, $\alpha_{h_1} > 0$, $\alpha_{h_2} > 0$ and Lipschitz continuous with constants $\beta_{T_1} > 0$, $\beta_{T_2} > 0$, $\beta_{g_1} > 0$, $\beta_{g_2} > 0$, $\beta_{h_1} > 0$, $\beta_{h_2} > 0$ respectively. If Assumption 2.3 and following conditions hold:*

(i) $\theta_{T_1} = \sqrt{1 - 2\rho_1\alpha_{T_1} + \rho_1^2\beta_{T_1}^2} < 1.$

(ii) $\theta_{T_2} = \sqrt{1 - 2\rho_2\alpha_{T_2} + \rho_2^2\beta_{T_2}^2} < 1.$

(iii) $0 \leq \alpha_n, \beta_n \leq 1$ *for all $n \geq 0$,*

$$\begin{aligned} \alpha_n(1 - \nu - \theta_{h_2}) - \beta_n\left(\theta_{g_1} + \theta_{T_1}\right) &\geq 0 \\ \beta_n(1 - \nu - \theta_{h_1}) &\geq 0 \\ \alpha_n\left(\theta_{g_2} + \theta_{T_2}\right) &\geq 0 \quad , \end{aligned}$$

such that

$$\begin{aligned} \sum_{n=0}^{\infty}\left(\alpha_n(1 - \nu - \theta_{h_2}) - \beta_n\left(\theta_{g_1} + \theta_{T_1}\right)\right) &= \infty \\ \sum_{n=0}^{\infty}\beta_n(1 - \nu - \theta_{h_1}) &= \infty \\ \sum_{n=0}^{\infty}\alpha_n\left(\theta_{g_2} + \theta_{T_2}\right) &= \infty, \end{aligned}$$

where

$$\theta_{g_1} = \sqrt{1 - 2\alpha_{g_1} + \beta_{g_1}^2}, \quad \theta_{g_2} = \sqrt{1 - 2\alpha_{g_2} + \beta_{g_2}^2},$$

and

$$\theta_{h_1} = \sqrt{1 - 2\alpha_{h_1} + \beta_{h_1}^2}, \quad \theta_{h_2} = \sqrt{1 - 2\alpha_{h_2} + \beta_{h_2}^2},$$

then sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 3.9 converge to x and y respectively.

Proof. Let $x, y \in H : h_1(y) \in K_1(y), h_2(x) \in K_2(x)$ be a solution of (1). Then from (21), (27), (31) and using Assumption 2.3, we have

$$\begin{aligned} \|x_{n+1} - x\| &\leq (1-\alpha_n)\|x_n - x\| + \alpha_n \|x_n - x - (h_2(x_n) - h_2(x))\| \\ &\quad + \alpha_n \left\| P_{K_2(x_n)}[w_n] - P_{K_2(x)}[w] \right\| \\ &\leq (1-\alpha_n)\|x_n - x\| + \alpha_n \|x_n - x - (h_2(x_n) - h_2(x))\| \\ &\quad + \alpha_n \left\| P_{K_2(x_n)}[w_n] - P_{K_2(x)}[w_n] \right\| + \alpha_n \left\| P_{K_2(x)}[w_n] - P_{K_2(x)}[w] \right\| \\ &\leq (1-\alpha_n)\|x_n - x\| + \alpha_n\theta_{h_2}\|x_n - x\| + \alpha_n \nu \|x_n - x\| + \alpha_n \|w_n - w\| \\ &= (1 - \alpha_n(1 - \nu - \theta_{h_2}))\|x_n - x\| + \alpha_n \|w_n - w\|. \end{aligned} \tag{34}$$

Similarly, from (24), (26), (30) and using Assumption 2.3, we have

$$\begin{aligned} \|y_{n+1} - y\| &\leq (1-\beta_n)\|y_n - y\| + \beta_n \|y_n - y - (h_1(y_n) - h_1(y))\| \\ &\quad + \beta_n \left\| P_{K_1(y_n)}[z_n] - P_{K_1(y)}[z] \right\| \\ &\leq (1-\beta_n)\|y_n - y\| + \beta_n \|y_n - y - (h_1(y_n) - h_1(y))\| \\ &\quad + \beta_n \left\| P_{K_1(y_n)}[z_n] - P_{K_1(y)}[z_n] \right\| + \beta_n \left\| P_{K_1(y)}[z_n] - P_{K_1(y)}[z] \right\| \\ &\leq (1-\beta_n)\|y_n - y\| + \beta_n\theta_{h_1}\|y_n - y\| + \beta_n \nu \|y_n - y\| + \beta_n \|z_n - z\| \\ &= (1 - \beta_n(1 - \nu - \theta_{h_1}))\|y_n - y\| + +\beta_n \|z_n - z\|. \end{aligned} \tag{35}$$

From (20), (22), (29) and (33), we have

$$\begin{aligned} \|w_n - w\| &= \|g_2(y_{n+1}) - \rho_2 T_2 y_{n+1} - g_2(y) + \rho_2 T_2 y\| \\ &\leq \|y_{n+1} - y - (g_2(y_{n+1}) - g_2(y))\| + \|y_{n+1} - y - \rho_2(T_2 y_{n+1} - T_2 y)\| \\ &\leq \left(\theta_{g_2} + \theta_{T_2}\right)\|y_{n+1} - y\|. \end{aligned} \tag{36}$$

Similarly, from (24), (28) and (32), we have

$$\begin{aligned} \|z_n - z\| &= \|g_1(x_n) - \rho_1 T_1 x_n - g_1(x) + \rho_1 T_1 x\| \\ &\leq \|x_n - x - (g_1(x_n) - g_1(x))\| + \|x_n - x - \rho_1(T_1 x_n - T_1 x)\| \\ &\leq \left(\theta_{g_1} + \theta_{T_1}\right)\|x_n - x\|. \end{aligned} \tag{37}$$

Combining (34), (36) and (35), (37), we have

$$\|x_{n+1} - x\| \leq (1 - \alpha_n(1 - \nu - \theta_{h_2}))\|x_n - x\| + \alpha_n\left(\theta_{g_2} + \theta_{T_2}\right)\|y_{n+1} - y\|, \tag{38}$$

and

$$\|y_{n+1} - y\| \leq (1 - \beta_n(1 - \nu - \theta_{h_1}))\|y_n - y\| + \beta_n\left(\theta_{g_1} + \theta_{T_1}\right)\|x_n - x\|. \tag{39}$$

Adding (38) and (39), we have

$$\begin{aligned} &\|x_{n+1} - x\| + \|y_{n+1} - y\| \\ &\leq (1 - \alpha_n(1 - \nu - \theta_{h_2}))\|x_n - x\| + \alpha_n\left(\theta_{g_2} + \theta_{T_2}\right)\|y_{n+1} - y\| \\ &\quad + (1 - \beta_n(1 - \nu - \theta_{h_1}))\|y_n - y\| + \beta_n\left(\theta_{g_1} + \theta_{T_1}\right)\|x_n - x\| \\ &= \left(1 - \alpha_n(1 - \nu - \theta_{h_2}) + \beta_n\left(\theta_{g_1} + \theta_{T_1}\right)\right)\|x_n - x\| \\ &\quad + \alpha_n\left(\theta_{g_2} + \theta_{T_2}\right)\|y_{n+1} - y\| + (1 - \beta_n(1 - \nu - \theta_{h_1}))\|y_n - y\|. \end{aligned}$$

From which, we have

$$\|x_{n+1} - x\| + \left(1 - \alpha_n \left(\theta_{g_2} + \theta_{T_2}\right)\right) \left\|y_{n+1} - y\right\|$$

$$\leq \left(1 - \alpha_n (1 - \nu - \theta_{h_2}) + \beta_n \left(\theta_{g_1} + \theta_{T_1}\right)\right) \|x_n - x\| + (1 - \beta_n (1 - \nu - \theta_{h_1})) \left\|y_n - y\right\|,$$

which implies that

$$\begin{aligned} \|x_{n+1} - x\| + \nu_3 \left\|y_{n+1} - y\right\| &\leq \max(\nu_1, \nu_2) \left(\|x_n - x\| + \left\|y_n - y\right\|\right) \\ &= \theta \left(\|x_n - x\| + \left\|y_n - y\right\|\right), \end{aligned} \quad (40)$$

where

$$\begin{aligned} \theta &= \max(\nu_1, \nu_2) \\ \nu_1 &= 1 - \left(\alpha_n (1 - \nu - \theta_{h_2}) - \beta_n \left(\theta_{g_1} + \theta_{T_1}\right)\right) \\ \nu_2 &= 1 - \beta_n (1 - \nu - \theta_{h_1}) \\ \nu_3 &= 1 - \alpha_n \left(\theta_{g_2} + \theta_{T_2}\right). \end{aligned}$$

Using assumption (*iii*), we have $\theta < 1$. Thus from (40), it follows that

$$\lim_{n\to\infty} \left[\|x_{n+1} - x\| + \nu_3 \left\|y_{n+1} - y\right\|\right] = 0.$$

This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x\| = 0,$$

and

$$\lim_{n\to\infty} \left\|y_{n+1} - y\right\| = 0.$$

This is the required result. □

4. Conclusion

In this paper, we have considered a new system of extended general quasi variational inequalities. It has been shown that system of extended general quasi variational inequalities is equivalent to a system of fixed point problems. These equivalent formulations have been used to propose and analyze several Gauss-Seidel type algorithms for solving system of extended general quasi variational inequalities and their variant forms. Several special cases are also discussed. The idea and technique of this paper may motivate for further research in this area. The researchers are encouraged to explore the novel and innovative applications of the system of extended general quasi variational inequalities and their variant forms in pure and applied sciences.

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