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Finite Difference Method for Bitsadze-Samarskii Type Overdetermined Elliptic Problem with Dirichlet Conditions

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Abstract. In this paper, we apply finite difference method to Bitsadze-Samarskii type overdetermined elliptic problem with Dirichlet conditions. Stability, coercive stability inequalities for solution of the first and second order of accuracy difference schemes (ADSs) are proved. Then, established abstract results are applied to get stable difference schemes for Bitsadze-Samarskii type overdetermined elliptic multidimensional differential problems with multipoint nonlocal boundary conditions. Finally, numerical results with explanation on the realization in two dimensional and three dimensional cases are presented.

1. Introduction

Recent years, theory and methods of solving inverse problems of determining unknown parameter of differential equations have been comprehensively studied by several researchers (see [2–5, 11–21, 24–26, 28–30, 32] and the bibliography therein). The papers [3, 11–15, 17–19] are devoted to study of well-posedness of various overdetermined problems for elliptic differential and difference equations. In [3, 11, 18, 19], overdetermined problems with Dirichlet type overdetermination were investigated. Inverse problems with Neumann type overdetermination were studied in papers [12, 14, 15]. In the present paper, we discuss Bitsadze-Samarskii type overdetermined elliptic problem with Dirichlet conditions.

In papers [1, 6–9, 13, 16, 17, 22, 29, 30], the Bitsadze-Samarskii type nonlocal boundary value problems and generalizations such type problems to various differential and difference elliptic equations have been investigated.

Assume that $k_1, \ldots, k_q, \lambda_0, \lambda_1, \ldots, \lambda_q$ are known nonnegative real numbers satisfying the conditions

$$\sum_{i=1}^{q} k_i = 1, k_i \geq 0, i = 1, \ldots, q, 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_q < T, \lambda_0 \in (0, T). \tag{1}$$

Let A be a selfadjoint and positive definite operator in an arbitrary Hilbert space H, and let smooth function $g(t)$, the elements $\phi, \zeta, \eta \in D(A)$, and the numbers $k_1, \ldots, k_q, \lambda_0, \lambda_1, \ldots, \lambda_q$ be given.

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In this work, we apply finite difference method to following Bitsadze-Samarskii type inverse elliptic multipoint problem of finding an element $p \in H$ and a function $v \in C^2([0,T], H) \cap C([0,T], D(A))$:

$$\begin{cases} -v_{tt}(t) + Av(t) = g(t) + p, \ t \in (0,T), \\ v(0) = \phi, \ v(T) = \sum\limits_{i=1}^{q} \alpha_i v(\lambda_i) + \eta, v(\lambda_0) = \zeta. \end{cases} \tag{2}$$

In papers [29, 30], theorem on solvability and uniqueness of solution for problem (2) has been proved. Well-posedness of problem (2) was studied in [16]. In [16], the stability inequalities for solution of problem (2) was aplied to study the following multidimensional elliptic problem with overdetermination

$$\begin{cases} -v_{tt}(t,x) - \sum\limits_{r=1}^{n} (a_r(x) v_{x_r})_{x_r} + \sigma v(x) = g(t,x) + p(x), x = (x_1, ...x_n) \in \Omega, 0 < t < T, \\ v(0,x) = \phi(x), v(T,x) - \sum\limits_{i=1}^{q} k_i v(\lambda_i, x) = \eta(x), \ v(\lambda_0, x) = \zeta(x), x \in \overline{\Omega}, \\ v(t,x) = 0, x \in S, \end{cases} \tag{3}$$

where $\Omega = (0,\ell)^n$ is the open cube in R_n with boundary S, $\overline{\Omega} = \Omega \cup S$ and nonnegative real numbers σ, λ_0, λ_i, $i = 1, \dots, q$, and coefficients k_i, $i = 1, \dots, q$ under condition (1) are known, smooth functions a_r, ϕ, η, ζ, and f are given , $a_r(x) > 0$, $\forall x \in \Omega$.

Stability, coercive stability inequalities for the solution of problem (3) and three overdetermined problems for the multidimensional elliptic equation with different boundary conditions were established in [16].

In this paper, we apply finite difference method to problem (2). A first and a second order of ADSs are constructed. Stability, coercive stability estimates for their solutions are established. Then, we study the first and second order ADSs for overdetermined problem (3) and obtain the stability estimates for its solution.

Let $\left\{t_k = k\tau, \ k = \overline{0,N}, \ N\tau = T\right\}$ be the set of grid points, $u_k = u(t_k)$, $g_k = g(t_k)$, $k = \overline{0,N}$. Consider difference scheme

$$\begin{cases} -\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k = g_k, \quad k = \overline{1, N-1}, \\ u_0 \text{ and } u_N \text{ are given .} \end{cases} \tag{4}$$

Let I be identity operator. So, $A > \delta I$ for some positive number δ. Due to the fact that A is a selfadjoint positive definite operator the operator $C = \frac{1}{2}(\tau A + \sqrt{4A + \tau^2 A^2})$ has the same property ([10, 27]). Introduce next notations

$$R = (I + \tau C)^{-1}, P = (I - R^{2N})^{-1}, \ D = (I + \tau C)(2I + \tau C)^{-1} C^{-1}.$$

It is known that the bounded operator R is defined on the whole H.

Solution of difference problem (4) is defined by formula ([10])

$$\begin{aligned} u_k &= P\left[\left(R^k - R^{2N-k}\right) u_0 + \left(R^{N-k} - R^{N+k}\right) u_N\right] - P\left(R^{N-k} - R^{N+k}\right) D \sum_{j=1}^{N-1} \left(R^{N-j} - R^{N+j}\right) g_j \tau \\ &+ D \sum_{j=1}^{N-1} \left(R^{|k-j|} - R^{k+j}\right) g_j \tau \ \ (k = \overline{1, N-1}). \end{aligned} \tag{5}$$

The rest of present paper is planned as follows: In Section 2, we construct a first and a second order of ADSs for problem (1.1) and establish stability, coercive stability inequalities for their solutions. In Section 3, we give stability inequalities for corresponding solutions of the first and second order of ADSs to Bitsadze-Samarskii type overdetermined elliptic multidimensional differential problem with multipoint nonlocal boundary conditions (NLBCs). In Section 4, numerical results with explanation on the realization in two dimensional and three dimensional cases are presented. Finally, the conclusion is presented in Section 5.

2. A First and a Second Order of ADSs for Problem (2)

Denote

$$l_i = [\frac{\lambda_i}{\tau}],\ \mu_i = \frac{\lambda_i}{\tau} - l_i, i = 0, 1, \dots, q,$$

where [·] is notation of greatest integer function. Now, let us to give some lemmas that will be used in further.

Lemma 2.1. ([10]) *The following estimates hold:*

$$\| R^k \|_{H\to H} \le M(\delta)(1 + \delta^{\frac{1}{2}}\tau)^{-k},\ \ \| CR^k \|_{H\to H} \le \frac{M(\delta)}{k\tau}, k \ge 1, \| P \|_{H\to H} \le M(\delta),\ \delta > 0. \quad (6)$$

Lemma 2.2. *Suppose that assumptions (1) are satisfied, then the operator*

$$\Delta_1 = \left[I - R^{2N} - R^{l_0} + R^{2N-l_0}\right]\left[I - R^{2N} - \sum_{i=1}^{q} k_i\left(R^{N-l_i} - R^{N+l_i}\right)\right] - \left[R^{N-l_0} - R^{N+l_0}\right]\left[\sum_{i=1}^{q} k_i\left(R^{l_i} - R^{2N-l_i}\right)\right] \quad (7)$$

has an inverse S_1 and its norm is bounded, i.e.

$$\| S_1 \|_{H\to H} \le M\left(\delta, \lambda_1, \dots, \lambda_q\right). \quad (8)$$

Proof. By using operator calculus it can be shown that

$$\Delta_1 = \left(I - R^{2N}\right)(I - R^{l_0})\left(I - \sum_{i=1}^{q} k_i R^{N-l_i}\right)\left(I - \sum_{i=1}^{q} k_i R^{N-(l_0-l_i)}\right). \quad (9)$$

Thus, Δ_1 has an inverse and estimate (8) follows from (1), (9), and estimates (6). □

Lemma 2.3. *Suppose that assumptions (1) are satisfied, then the operator*

$$\begin{aligned}\Delta_2 &= \left[I - R^{2N} + (\mu_0 - 1)\left(R^{l_0} - R^{2N-l_0}\right) - \mu_0\left(R^{l_0+1} - R^{2N-l_0-1}\right)\right] \\ &\times\left[I - R^{2N} + \sum_{i=1}^{q} k_i(\mu_i - 1)\left(R^{N-l_i} - R_i^{N+l}\right) - \sum_{i=1}^{q} k_i\mu_i\left(R^{N-l_i-1} - R^{N+l_i+1}\right)\right] \\ &- \left[(\mu_0 - 1)\left(R^{N-l_0} - R^{N+l_0}\right) - \mu_0\left(R^{N-l_0-1} - R^{N+l_0+1}\right)\right] \\ &\times\left[\sum_{i=1}^{q} k_i(\mu_i - 1)\left(R^{l_i} - R^{2N-l_i}\right) - \sum_{i=1}^{q} k_i\mu_i\left(R^{l_i+1} - R^{2N-l_i-1}\right)\right]\end{aligned} \quad (10)$$

has an inverse S_2 and its norm is bounded, i.e.

$$\| S_2 \|_{H\to H} \le M\left(\delta, \lambda_1, \dots, \lambda_q\right). \quad (11)$$

Proof. It is easy to see that

$$S_2 - S_1 = S_2 S_1 K, \quad (12)$$

where

$$\begin{aligned}K &= \mu_0\left(R^{l_0} - R^{2N-l_0} - R^{l_0+1} + R^{2N-l_0-1}\right) \\ &\times\left[I - R^{2N} - \sum_{i=1}^{q} k_i\left(R^{N-l_i} - R^{N+l_i}\right) + \sum_{i=1}^{q} k_i\mu_i\left(R^{N-l_i} - R_i^{N+l} - R^{N-l_i-1} + R^{N+l_i+1}\right)\right] \\ &+ \left(I - R^{2N} - R^{l_0} + R^{2N-l_0}\right)\left[\sum_{i=1}^{q} k_i\mu_i\left(R^{N-l_i} - R_i^{N+l} - R^{N-l_i-1} + R^{N+l_i+1}\right)\right] \\ &- \mu_0\left(R^{N-l_0} - R^{N+l_0} - R^{N-l_0-1} + R^{N+l_0+1}\right)\left[\sum_{i=1}^{q} k_i\left(R^{l_i} - R^{2N-l_i}\right) + \sum_{i=1}^{q} k_i\mu_i\left(R^{l_i} - R^{2N-l_i} - R^{l_i+1} + R^{2N-l_i-1}\right)\right] \\ &- \left[R^{N-l_0} - R^{N+l_0}\right]\sum_{i=1}^{q} k_i\mu_i\left(R^{l_i} - R^{2N-l_i} - R^{l_i+1} + R^{2N-l_i-1}\right).\end{aligned} \quad (13)$$

Applying Cauchy-Schwarz, triangle inequalities, and estimates (6), we obtain

$$\begin{aligned}
&\| K \|_{H\to H} \leq \left|\mu_0\right| \ \| R^{l_0} - R^{2N-l_0} - R^{l_0+1} + R^{2N-l_0-1} \|_{H\to H} \\
&\times\left[\| I - R^{2N} - \sum_{i=1}^{q} k_i\left(R^{N-l_i} - R^{N+l_i}\right) \|_{H\to H} + \| \sum_{i=1}^{q} k_i\mu_i\left(R^{N-l_i} - R_i^{N+l} - R^{N-l_i-1} + R^{N+l_i+1}\right) \|_{H\to H}\right] \\
&+ \| I - R^{2N} - R^{l_0} + R^{2N-l_0} \|_{H\to H} \max_{1\leq i\leq q}\left|\mu_i\right| \ \| \sum_{i=1}^{q} k_i\left(R^{N-l_i} - R_i^{N+l} - R^{N-l_i-1} + R^{N+l_i+1}\right) \|_{H\to H} \\
&+ \| R^{N-l_0} - R^{N+l_0} \|_{H\to H} \max_{1\leq i\leq q}\left|\mu_i\right| \ \| \sum_{i=1}^{q} k_{ii}\left(R^{l_i} - R^{2N-l_i} - R^{l_i+1} + R^{2N-l_i-1}\right) \|_{H\to H} \\
&\leq M_1\left(\delta, \lambda_1, \ldots, \lambda_q\right)\tau.
\end{aligned} \tag{14}$$

By using the triangle inequality, formula (12), estimates (8), (14), we get

$$\| S_2 \|_{H\to H} \leq \| S_1 \|_{H\to H} + \| S_2 \|_{H\to H} \| S_1 \|_{H\to H} \leq M\left(\delta, \lambda_1, \ldots, \lambda_q\right) + \| S_2 \|_{H\to H} M_1\left(\delta, \lambda_1, \ldots, \lambda_q\right)\tau$$

for any parameter $\tau > 0$. From that estimate (11) follows. □

By using the approximation formulas $v(\lambda_i) = v_{l_i} + o(\tau)$ and $v(\lambda_i) = v_{l_i} + \mu_i(v_{l_i+1} - v_{l_i}) + o(\tau^2)$ for each $i = 0, 1, \ldots, q$ to $v(\lambda_i)$, inverse problem (2) can be corresponded to the first order of ADS

$$\begin{cases} -\tau^{-2}(v_{k+1} - 2v_k + v_{k-1}) + Av_k = g(t_k) + p, \quad 1 \leq k \leq N-1, \\ v_0 = \phi, v_N = \sum_{i=1}^{q} k_i v_{l_i} + \eta, v_{l_0} = \zeta, \end{cases} \tag{15}$$

and the second order of ADS

$$\begin{cases} -\tau^{-2}(v_{k+1} - 2v_k + v_{k-1}) + Av_k = g(t_k) + p, \ 1 \leq k \leq N-1, \\ v_0 = \varphi, \ v_N = \sum_{i=1}^{q} k_i\left(v_{l_i} + \mu_i\left(v_{l_i+1} - v_{l_i}\right)\right) + \eta, \ v_{l_0} + \mu_0\left(v_{l_0+1} - v_{l_0}\right) = \zeta. \end{cases} \tag{16}$$

Applying the substitution

$$v_k = u_k + A^{-1}(p), \tag{17}$$

we get the following auxiliary difference schemes

$$\begin{cases} -\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k = g_k, \quad 1 \leq k \leq N-1, \\ u_0 - u_{l_0} = \phi - \zeta, u_N = \sum_{i=1}^{q} k_i u_{l_i} + \eta, \end{cases} \tag{18}$$

and

$$\begin{cases} -\tau^{2}\left(u_{k+1} - 2u_k + u_{k-1}\right) + Au_k = g_k, 1 \leq k \leq N-1, \\ u_0 + (\mu_0 - 1)\,u_{l_0} - \mu_0 u_{l_0+1} = \phi - \zeta, u_N + \sum_{i=1}^{q} k_i\left[(\mu_i - 1)\,u_{li} - \mu_i u_{l_i+1}\right] = \eta \end{cases} \tag{19}$$

to find $\{u_k\}_0^N$, correspondingly.

To find solution of problem (2), we consider the algorithm which includes three stages. In the first stage, we find $\{u_k\}_0^N$ as solution of (18) or (19). Putting $k = l_0$ and $k = l_0 + 1$, we get u_{l_0} and u_{l_0}, respectively. In the second stage, we obtain p by

$$p = A\zeta - Au_{l_0} \text{ or } p = A\zeta - A\left[(1 - \mu_0)\,u_{l_0} + \mu_0 u_{l_0+1}\right]. \tag{20}$$

In the third stage, we use formula (17) to obtain the solution $\{u_k\}_0^N$ of problems (15) and (16).

Let $\alpha \in (0,1)$ be a given number. Denote by $C_\tau(H), C_\tau^\alpha(H)$, and $C_\tau^{\alpha,\alpha}(H)$ the Banach spaces of H-valued grid functions $g_\tau = \{g_k\}_{k=1}^{N-1}$ with the corresponding norms,

$$\left\|g_\tau\right\|_{C_\tau(H)} = \max_{1\le k\le N-1}\left\|g_k\right\|_H, \ \left\|g_\tau\right\|_{C_\tau^\alpha(H)} = \left\|g_\tau\right\|_{C_\tau(H)} + \sup_{1\le k<k+n\le N-1} \frac{\|g_{k+n}-g_k\|_H}{(n\tau)^\alpha},$$
$$\left\|g_\tau\right\|_{C_\tau^{\alpha,\alpha}(H)} = \left\|g_\tau\right\|_{C_\tau(H)} + \sup_{1\le k<k+n\le N-1} \frac{(k\tau+n\tau)^\alpha(1-k\tau)^\alpha\|g_{k+n}-g_k\|_H}{(n\tau)^\alpha}.$$

Theorem 2.4. *Suppose that assumptions (1) are satisfied, $\phi,\eta,\zeta \in D(A)$ and $g_\tau \in C_\tau^{\alpha,\alpha}(H)$, then the solution $\left(\{v_k\}_{k=1}^{N-1}, p\right)$ of difference problem (15) obeys the following stability estimates*

$$\left\|\{v\}_{k=1}^{N-1}\right\|_{C_\tau(H)} \le M\left(\delta,\lambda_1,\ldots,\lambda_q\right)\left[\left\|\phi\right\|_H + \left\|\eta\right\|_H + \|\zeta\|_H + \left\|g_\tau\right\|_{C_\tau(H)}\right], \tag{21}$$

$$\left\|A^{-1}p\right\|_H \le M\left(\delta,\lambda_1,\ldots,\lambda_q\right)\left[\left\|\phi\right\|_H + \left\|\eta\right\|_H + \|\zeta\|_H + \left\|g_\tau\right\|_{C_\tau(H)}\right], \tag{22}$$

$$\left\|p\right\|_H \le M\left(\delta,\lambda_1,\ldots,\lambda_q\right)\left[\left\|A\phi\right\|_H + \left\|A\eta\right\|_H + \|A\zeta\|_H + \frac{1}{\alpha(1-\alpha)}\left\|g_\tau\right\|_{C_\tau^{\alpha,\alpha}(H)}\right], \tag{23}$$

where $M\left(\delta,\lambda_1,\ldots,\lambda_q\right)$ does not depend on $\phi,\zeta,\eta,\tau,\alpha$, and g_τ.

Proof. By using (5), NLBCs of problem (18), one can get sistem of equations

$$\begin{cases} \left(I - R^{2N} - R^{l_0} + R^{2N-l_0}\right)u_0 - \left(R^{N-l_0} - R^{N+l_0}\right)u_N = F_1, \\ \sum\limits_{i=1}^{q} k_i\left(R^{l_i} - R^{2N-l_i}\right)u_0 + \left[I - R^{2N} + \sum\limits_{i=1}^{q} k_i(R^{N-l_i} - R^{N+l_i})\right]u_N = F_2 \end{cases} \tag{24}$$

to find u_0 and u_N, where

$$F_1 = P^{-1}\left(\phi-\zeta\right) + \left(R^{N-l_0} - R^{N+l_0}\right)D\sum_{j=1}^{N-1}\left(R^{N-j-1} - R^{N+j-1}\right)g_j\tau - P^{-1}D\sum_{j=1}^{N-1}\left(R^{|l_0-j|-1} - R^{l_0+j-1}\right)g_j\tau,$$
$$F_2 = \sum_{i=1}^{q} k_i\left\{\left(R^{N-l_i} - R^{N+l_i}\right)D\sum_{j=1}^{N-1}\left(R^{N-j-1} - R^{N+j-1}\right)g_j\tau - P^{-1}D\sum_{j=1}^{N-1}\left(R^{|l_i-j|-1} - R^{l_i+j-1}\right)g_j\tau\right\} + P^{-1}\eta.$$

Determinant operator of system (24) equals to Δ_1 which is defined by (7). According Lemma 2.1 it has bounded inverse. Then, solution of system (24) is obtained by

$$\begin{aligned} u_0 &= \Delta_1^{-1}\left[\left(I - R^{2N} - \sum_{i=1}^{q} k_i(R^{N-l_i} - R^{N+l_i})\right)F_1 + \left(R^{N-l_0} - R^{N+l_0}\right)F_2\right], \\ u_N &= \Delta_1^{-1}\left[\left(I - R^{2N} - R^{l_0} + R^{2N-l_0}\right)F_2 + \sum_{i=1}^{q} k_i\left(R^{l_i} - R^{2N-l_i}\right)F_1\right]. \end{aligned} \tag{25}$$

Thus, difference scheme (18) has a unique solution $\{u_k\}_{k=0}^{N}$ which is defined by (5) and (25). By using formulas (5), (25), estimates (6), (8), one can show that

$$\left\|\{u_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)} \le M\left(\delta,\lambda_1,\ldots,\lambda_q\right)\left[\left\|\phi\right\|_H + \left\|\eta\right\|_H + \|\zeta\|_H + \left\|g_\tau\right\|_{C_\tau(H)}\right], \tag{26}$$

$$\left\|\left\{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\right\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)} + \left\|\{Au_k\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)} \tag{27}$$
$$\le M\left(\delta,\lambda_1,\ldots,\lambda_q\right)\left[\left\|A\phi\right\|_H + \left\|A\eta\right\|_H + \|A\zeta\|_H + \frac{1}{\alpha(1-\alpha)}\left\|g_\tau\right\|_{C_\tau^{\alpha,\alpha}(H)}\right].$$

The proofs of estimates (22), (23) for solution of difference problem (15) are based on formula (17) and inequalities (26), (27). Finally, by using formula (17) and inequalities (26), (22), we can get estimate (21). □

Theorem 2.5. *Suppose that assumptions (1) are satisfied, $\phi, \eta, \zeta \in D(A)$ and $g_\tau \in C_\tau^{\alpha,\alpha}(H)$, then for the solution $\left(\{v_k\}_{k=1}^{N-1}, p\right)$ of problem (15) the coercive stability inequality*

$$\left\|\left\{\tau^{-2}(v_{k+1}-2v_k+v_{k-1})\right\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)} + \left\|\{Av_k\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)} + \left\|p\right\|_H \tag{28}$$
$$\leq M\left(\delta, \lambda_1, \ldots, \lambda_q\right)\left[\frac{1}{\alpha(1-\alpha)}\left\|g_\tau\right\|_{C_\tau^{\alpha,\alpha}(H)} + \left\|A\phi\right\|_H + \left\|A\eta\right\|_H + \|A\zeta\|_H\right]$$

is valid, where $M\left(\delta, \lambda_1, \ldots, \lambda_q\right)$ is independent of $\phi, \eta, \zeta, \tau, \alpha$, and g_τ.

The proof of inequality (28) is based on formula (17), estimates (27), (23).

Theorem 2.6. *Suppose that assumptions (1) are satisfied, $\phi, \eta, \zeta \in D(A)$ and $g_\tau \in C_\tau^{\alpha,\alpha}(H)$, then the solution $\left(\{v_k\}_{k=1}^{N-1}, p\right)$ of difference problem (15) obeys stability estimates (21), (22) and (23).*

Proof. Applying formula (5) for solving auxiliary difference problem (19) to corresponding NLBCs, we have the system of equations

$$\begin{cases} \left[I - R^{2N} + (\mu_0 - 1)\left(R^{l_0} - R^{2N-l_0}\right) - \mu_0\left(R^{l_0+1} - R^{2N-l_0-1}\right)\right]u_0 \\ +\left[(\mu_0-1)\left(R^{N-l_0} - R^{N+l_0}\right) - \mu_0\left(R^{N-l_0-1} - R^{N+l_0+1}\right)\right]u_N = F_3, \\ \left[\sum\limits_{i=1}^{q} k_i(\mu_i - 1)\left(R^{l_i} - R^{2N-l_i}\right) - \sum\limits_{i=1}^{q} k_i\mu_i\left(R^{l_i+1} - R^{2N-l_i-1}\right)\right]u_0 \\ +\left[I - R^{2N} + \sum\limits_{i=1}^{q} k_i(\mu_i-1)\left(R^{N-l_i} - R_i^{N+l}\right) - \sum\limits_{i=1}^{q} k_i\mu_i\left(R^{N-l_i-1} - R^{N+l_i+1}\right)\right]u_N = F_4. \end{cases} \tag{29}$$

Here,

$$F_3 = P^{-1}\left(\phi - \zeta\right) + \left[(\mu_0-1)\left(R^{N-l_0} - R^{N+l_0}\right) - \mu_0\left(R^{N-l_0-1} - R^{N+l_0+1}\right)\right]D\sum_{j=1}^{N-1}\left(R^{N-j} - R^{N+j}\right)g_j\tau$$
$$-P^{-1}D\sum_{j=1}^{N-1}\left[(\mu_0-1)\left(R^{|l_0-j|} - R^{l_0+j}\right) - \mu_0\left(R^{|l_0+1-j|} - R^{l_0+j+1}\right)\right]g_j\tau,$$
$$F_4 = P^{-1}\eta + \sum_{i=1}^{q} k_i\left[(\mu_i-1)\left(R^{N-l_i} - R^{N+l_i}\right) - \mu_i\left(R^{N-l_i-1} - R^{N+l_i+1}\right)\right]D\sum_{j=1}^{N-1}\left(R^{N-j} - R^{N+j}\right)g_j\tau$$
$$-P^{-1}D\sum_{j=1}^{N-1}\sum_{i=1}^{q} k_i\left[(\mu_i-1)\left(R^{|l_i-j|} - R^{l_i+j}\right) - \mu_0\left(R^{|l_i+1-j|} - R^{l_i+j+1}\right)\right]g_j\tau.$$

By Lemma 2.2, the determinat operator of system (29) has inverse Δ_2^{-1}. Therefore, solving it, we get

$$\begin{aligned} u_0 &= \Delta_2^{-1}\left\{\left[I - R^{2N} + \sum_{i=1}^{q} k_i(\mu_i-1)\left(R^{N-l_i} - R_i^{N+l}\right) - \sum_{i=1}^{q} k_i\mu_i\left(R^{N-l_i-1} - R^{N+l_i+1}\right)\right]F_3\right. \\ &-\left[(\mu_0-1)\left(R^{N-l_0} - R^{N+l_0}\right) - \mu_0\left(R^{N-l_0-1} - R^{N+l_0+1}\right)\right]F_4, \\ u_N &= \Delta_2^{-1}\left\{\left[I - R^{2N} + (\mu_0-1)\left(R^{l_0} - R^{2N-l_0}\right) - \mu_0\left(R^{l_0+1} - R^{2N-l_0-1}\right)\right]F_4\right. \\ &-\left[\sum_{i=1}^{q} k_i(\mu_i-1)\left(R^{l_i} - R^{2N-l_i}\right) - \sum_{i=1}^{q} k_i\mu_i\left(R^{l_i+1} - R^{2N-l_i-1}\right)\right]F_3. \end{aligned} \tag{30}$$

Thus, solution $\{u_k\}_{k=0}^{N}$ of difference problem (19) exists. Moreover, unique solution is defined by formulas (5) and (30). By using formulas (5), (30), estimates (6), (8), one can get estimates (26) and (27). Then, the proofs of estimates (22), (23) for solution of difference problem (16) are based on formula (17) and inequalities (26), (27). Applying formula (17) and inequalities (26), (22), we can obtain estimate (21) for solution of difference problem (16). □

Theorem 2.7. *Suppose that assumptions (1) are satisfied, $\phi, \eta, \zeta \in D(A)$ and $g_\tau \in C_\tau^{\alpha,\alpha}(H)$ $(0 < \alpha < 1)$, then for the solution $\left(\{v_k\}_{k=1}^{N-1}, p\right)$ of problem (16) the coercive stability inequality (28) holds.*

The proof of Theorem 2.7 is based on formulas (5) and (30) and inequalities (27), (23).

3. A First and a Second Order of ADSs for Problem (3)

Abstract Theorems 2.4-2.6 permit us to get stable difference schemes for problem (3). We will discretize problem (3) in two steps. In the first step, we define the grid spaces

$$\begin{aligned}\widetilde{\Omega}_h &= \{x = (h_1 m_1, \cdots, h_n m_n); m = (m_1, \cdots, m_n), m_r = 0, \cdots, M_r,\ h_r M_r = \ell, r = 1, \cdots, n\},\\ \Omega_h &= \widetilde{\Omega}_h \cap \Omega, S_h = \widetilde{\Omega}_h \cap S.\end{aligned}$$

To the differential operator A^x generated by problem (3), we assign the difference operator A_h^x defined by the formula

$$A_h^x v^h(x) = -\sum_{r=1}^{n} \left(a_r(x) v_{\overline{x}_r}^h\right)_{x_r, j_r}$$

acting in the space of grid functions $v^h(x)$, satisfying the condition $v^h(x) = 0$ for all $x \in S_h$. It is well-known that A_h^x is a self-adjoint positive definite operator.

By using A_h^x, for obtaining $v^h(t,x)$ functions we arrive at the following boundary value problem

$$\begin{cases} -\frac{d^2 v^h(t,x)}{dt^2} + A_h^x v^h(t,x) = f^h(t,x) + p^h(x),\ 0 < t < T,\ x \in \Omega_h, \\ v^h(0,x) = \phi(x),\ v^h(\lambda_0, x) = \zeta^h(x), v^h(T,x) - \sum\limits_{i=1}^{q} k_i v^h(\lambda_i, x) = \eta^h(x), x \in \widetilde{\Omega}_h \end{cases} \tag{31}$$

for a system of ordinary differential equations. In the second step, problem (31) is replaced by

$$\begin{cases} -\tau^{-2}\left[v_{k+1}^h(x) - 2v_k^h(x) + v_{k-1}^h(x)\right] + A_h^x v_k^h(x) = g_k^h(x) + p^h(x), \quad 1 \le k \le N-1,\ x \in \Omega_h, \\ v_0 = \phi^h(x), v_N^h(x) = \sum\limits_{i=1}^{q} k_i v_{l_i}^h(x) + \eta^h(x), v_{l_0}^h(x) = \zeta^h(x), x \in \widetilde{\Omega}_h, \end{cases} \tag{32}$$

and

$$\begin{cases} -\tau^{-2}\left[v_{k+1}^h(x) - 2v_k^h(x) + v_{k-1}^h(x)\right] + A_h^x v_k^h(x) = g_k^h(x) + p^h(x), \quad 1 \le k \le N-1,\ x \in \Omega_h, \\ v_0^h(x) = \phi^h(x),\ v_N^h(x) = \sum\limits_{i=1}^{q} k_i \left(v_{l_i}^h(x) + \mu_i \left(v_{l_i+1}^h(x) - v_{l_i}^h(x)\right)\right) + \eta^h(x), \\ v_{l_0}^h(x) + \mu_0\left(v_{l_0+1}^h(x) - v_{l_0}^h(x)\right) = \zeta^h(x), x \in \widetilde{\Omega}_h, \end{cases} \tag{33}$$

respectively.

For calculatation of $p^h(x)$ we have formula

$$p^h(x) = A_h^x \zeta^h(x) - A_h^x v^h(\lambda_0, x), x \in \widetilde{\Omega}_h. \tag{34}$$

Let $L_{2h} = L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2 = W_2^2(\widetilde{\Omega}_h)$ be Banach spaces of the grid functions $g^h(x) = \{g(h_1 m_1, \cdots, h_n m_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\begin{aligned}&\left\|g^h\right\|_{L_{2h}} = \left(\sum_{x\in\widetilde{\Omega}_h} |g^h(x)|^2 h_1 \cdots h_n\right)^{1/2},\\ &\left\|g^h\right\|_{W_{2h}^2} = \left\|g^h\right\|_{L_{2h}} + \left(\sum_{x\in\widetilde{\Omega}_h} \sum_{r=1}^{n} \left|(g^h)_{x_r}\right|^2 h_1 \cdots h_n\right)^{1/2} + \left(\sum_{x\in\widetilde{\Omega}_h} \sum_{r=1}^{n} \left|(g^h(x))_{x_r \overline{x}_r,\ m_r}\right|^2 h_1 \cdots h_n\right)^{1/2}.\end{aligned}$$

Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ be sufficiently small positive numbers.

Theorem 3.1. *Suppose that assumptions (1) are satisfied, then for the solution of difference problems (32) and (33) the next stability inequalities hold:*

$$\left\|\left\{v_k^h\right\}_1^{N-1}\right\|_{C_\tau(L_{2h})} \leq M\left(\delta,\lambda_1,\ldots,\lambda_q\right)[\left\|\phi^h\right\|_{L_{2h}} + \left\|\zeta^h\right\|_{L_{2h}} + \left\|\eta^h\right\|_{L_{2h}} + \left\|\left\{g_k^h\right\}_1^{N-1}\right\|_{C_\tau(L_{2h})}],$$
$$\left\|p^h\right\|_{L_{2h}} \leq M\left(\delta,\lambda_1,\ldots,\lambda_q\right)\left[\left\|\phi^h\right\|_{W_{2h}^2} + \left\|\zeta^h\right\|_{W_{2h}^2} + \left\|\eta^h\right\|_{W_{2h}^2} + \frac{1}{\alpha(1-\alpha)}\left\|\left\{g_k^h\right\}_1^{N-1}\right\|_{C_\tau(L_{2h})}\right],$$

where $M\left(\delta,\lambda_1,\ldots,\lambda_q\right)$ does not depend on $\tau,\alpha,h,\phi^h(x),\zeta^h(x)$, $\eta^h(x)$ and $\left\{g_k^h(x)\right\}_1^{N-1}$.

Theorem 3.2. *Suppose that assumptions (1) are satisfied, then for the solution of difference problems (32) and (33) the coercive stability inequality holds:*

$$\left\|\left\{\frac{v_{k+1}^h-2v_k^h+v_{k-1}^k}{\tau^2})\right\}_1^{N-1}\right\|_{C_\tau(L_{2h})} + \left\|\left\{v_k^h\right\}_1^{N-1}\right\|_{C_\tau(W_{2h}^2)} + \left\|p^h\right\|_{L_{2h}}$$
$$\leq M\left(\delta,\lambda_1,\ldots,\lambda_q\right)[\left\|\phi^h\right\|_{W_{2h}^2} + \left\|\zeta^h\right\|_{W_{2h}^2} + \left\|\eta^h\right\|_{W_{2h}^2} + \frac{1}{\alpha(1-\alpha)}\left\|\left\{g_k^h\right\}_1^{N}\right\|_{C_\tau(L_{2h})}],$$

where $M\left(\delta,\lambda_1,\ldots,\lambda_q\right)$ does not depend on $\tau,\alpha,h,\phi^h(x),\eta^h(x)$, $\zeta^h(x)$, and $\left\{g_k^h(x)\right\}_1^{N-1}$.

The proofs of Theorems 3.1 and 3.2 are based on the symmetry property of the operator A_h^x in L_{2h} and the theorem in [33] on the coercivity stability inequality for the solution of the elliptic difference problem in L_{2h} with Dirichlet type boundary condition.

4. Numerical Results

In this section, numerical results for overdetermined problem for two dimensional and three dimensional elliptic partial differential equations with explanation on the realization in computer are presented. Numerical calculations are carried out by MATLAB program.

4.1. Two dimensional case

Consider the following two dimensional elliptic overdetermined problem with three point nonlocal boundary codition

$$\begin{cases} -\frac{\partial^2 v(t,x)}{\partial t^2} - \frac{\partial}{\partial x}\left((1+x)\frac{\partial v(t,x)}{\partial x}\right) + v(t,x) = g(t,x) + p(x), x,t \in (0,1), \\ v(0,x) = \phi(x),\ v\left(\frac{3}{5},x\right) = \zeta(x), \\ v(1,x) - \frac{1}{5}v(0.6,x) - \frac{3}{10}v(0.7,x) - \frac{1}{2}v(0.8,x) = \eta(x), x \in [0,1], \\ v(t,0) = v(t,1) = 0,\ t \in [0,1], \end{cases} \tag{35}$$

where

$$g(t,x) = \left[\left(x\pi^2+1\right)e^{-\pi t} + \left((1+x)\pi^2+1\right)t\right]\sin(\pi x) - \pi\left(e^{-\pi x}+t\right)\cos(\pi x), \phi(x) = 2\sin(\pi x),$$
$$\zeta(x) = \left(e^{-\frac{\pi}{2}} + \frac{3}{2}\right)\sin(\pi x), \eta(x) = \left(e^{-\pi} - \frac{1}{5}e^{-\frac{3\pi}{5}} - \frac{3}{10}e^{-\frac{7\pi}{10}} + \frac{27}{100}\right)\sin(\pi x).$$

The pair of functions (v,p) such that $v(t,x) = \left(e^{-\pi t}+t+1\right)\sin(\pi x)$ and $p(x) = \left[(1+x)\pi^2+1\right]\times\sin(\pi x) - \pi\cos(\pi x)$ is exact solution of problem (35).

Denote by $[0,1]_\tau \times [0,1]_h$ set of grid points

$$[0,1]_\tau \times [0,1]_h = \{(t_k,x_n) : t_k = k\tau,\ k = \overline{0,N},\ x_n = nh,\ n = \overline{0,M}\}$$

with small parameters τ and h such that $N\tau = 1, Mh = 1$. In addition,

$$\lambda_0 = \tfrac{1}{2},\ \lambda_1 = \tfrac{3}{5},\ \lambda_2 = \tfrac{7}{10},\ \lambda_3 = \tfrac{4}{5};\ l_i = \left[\tfrac{\lambda_i}{\tau}\right],\ \mu_i = \tfrac{\lambda_i}{\tau} - l_i,\ i = 0,1,2,3;$$
$$\phi_n = \phi(x_n),\ \zeta_n = \zeta(x_n),\ \eta_n = \eta(x_n),\ p_n = p(x_n),\ n = \overline{0,M};\ g_n^k = g(t_k, x_n), k = \overline{0,N},\ n = \overline{0,M}.$$

Algorithm of solving (35) contains three stages. In the first stage, we find numerical solutions $\left\{u_n^k \mid n = \overline{1,M-1}, k = \overline{1,N-1}\right\}$ of corresponding the first and second order of accuracy auxiliary difference schemes

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} + (1+x_n)\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} + \frac{u_{n+1}^k-u_{n-1}^k}{2h} = g_n^k, n = \overline{1,M-1}, k = \overline{1,N-1}; \\ u_0^k = u_M^k = 0, k = \overline{0,N};\ u_n^0 - u_n^{l_0} = \phi_n - \zeta_n,\ \ u_n^N - \frac{1}{5}u_n^{l_1} - \frac{3}{10}u_n^{l_2} - \frac{1}{2}u_n^{l_3} = \eta_n,\ n = \overline{0,M}, \end{cases} \tag{36}$$

and

$$\begin{cases} \frac{u_n^{k+1}-2v_n^k+v_n^{k-1}}{\tau^2} + (1+x_n)\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2} + \frac{u_{n+1}^k-u_{n-1}^k}{2h} = g_n^k, n = \overline{1,M-1}, k = \overline{1,N-1}; \\ u_0^k = u_M^k = 0, k = \overline{0,N}; \\ u_n^0 + (\mu_0 - 1)u_n^{l_0} - \mu_0 u_n^{l_0+1} = \phi_n - \zeta_n,\ \ u_n^N + \frac{1}{5}\left[(\mu_1 - 1)u_n^{l_1} - \mu_1 u_n^{l_1+1}\right] + \frac{3}{10}\left[(\mu_2 - 1)u_n^{l_2} - \mu_2 u_n^{l_2+1}\right] \\ +\frac{1}{2}\left[(\mu_3 - 1)u_n^{l_3} - \mu_3 u_n^{l_3+1}\right] = \eta_n,\ n = \overline{0,M}. \end{cases} \tag{37}$$

In the second stage, we find $\{p_n\}$ by

$$p_n = -(1+x_n)\frac{(\zeta_{n+1} - u_{n+1}^{l_0}) - 2(\zeta_n - u_n^{l_0}) + (\zeta_{n-1} - u_{n-1}^{l_0})}{h^2} - \frac{(\zeta_{n+1} - u_{n+1}^{l_0}) - (\zeta_{n-1} - u_{n-1}^{l_0})}{2h} + u_n^{l_0},$$

and

$$\begin{aligned} p_n = &-\tfrac{1+x_n}{h^2}\left\{\left[\zeta_{n+1} + \left((\mu_0 - 1)u_{n+1}^{l_0} - \mu_0 u_{n+1}^{l_0+1}\right)\right] - 2\left[\zeta_n + \left((\mu_0 - 1)u_n^{l_0} - \mu_0 u_n^{l_0+1}\right)\right]\right. \\ &\left.+\left[\zeta_{n-1} + \left((\mu_0 - 1)u_{n-1}^{l_0} - \mu_0 u_{n-1}^{l_0+1}\right)\right]\right\} - \tfrac{1}{2h}\left\{\left[\zeta_{n+1} + \left((\mu_0 - 1)u_{n+1}^{l_0} - \mu_0 u_{n+1}^{l_0+1}\right)\right]\right. \\ &\left.-\left[\zeta_{n-1} + \left((\mu_0 - 1)v_{n-1}^{l_0} - \mu_0 v_{n-1}^{l_0+1}\right)\right]\right\} + \zeta_n - \left(\mu_0 v_n^{l_0+1} - (\mu_0 - 1)v_n^{l_0}\right), n = \overline{1,M-1} \end{aligned}$$

for the first and second order approximation, respectively.
Difference schemes (36) and (37) can be rewritten in the next matrix form

$$\begin{cases} A^{(n)}u_{n+1} + B^{(n)}u_n + C^{(n)}u_{n-1} = Ig^{(n)},\ n = \overline{1,M-1}, \\ u_0 = \overrightarrow{0}, u_M = \overrightarrow{0}. \end{cases} \tag{38}$$

Here, $A^{(n)}, B^{(n)}, C^{(n)}$ are $(N+1)\times(N+1)$ matrices, I is the $(N+1)\times(N+1)$ identity matrix, $g^{(n)}$ and $u_s = \left[\begin{array}{ccc} u_s^0 & \ldots & u_s^N \end{array}\right]^t, s = n-1, n, n+1$ are $(N+1)\times 1$ matrices . Let us

$$a^{(n)} = (1+x_n)h^{-2} + h^{-1}/2, c^{(n)} = (1+x_n)h^{-2} - h^{-1}/2,\ z^{(n)} = -2\tau^{-2} - 2(1+x_n)h^{-2}, d = \tau^{-2}.$$

Then, we have

$$A^{(n)} = diag\left\{0, a^{(n)}, a^{(n)}, \ldots, a^{(n)}, 0\right\}, C^{(n)} = diag\left\{0, c^{(n)}, c^{(n)}, \ldots, c^{(n)}, 0\right\},$$
$$g_n^0 = \phi_n - \zeta_n,\ g_n^N = \eta_n,\ n = \overline{1,M-1}$$

for both schemes (36) and (37). Elements $b_{i,j}^{(n)}$ of matrix $B^{(n)}$ are defined by

$$b_{i,i}^{(n)} = z^{(n)}, b_{i-1,i}^{(n)} = b_{i,i-1}^{(n)} = d, i = \overline{2,N}; b_{1,1}^{(n)} = 1, b_{1,l_0}^{(n)} = -1, b_{N+1,N+1}^{(n)} = 1, b_{N+1,l_1}^{(n)} = -\tfrac{1}{5}, b_{N+1,l_2}^{(n)} = -\tfrac{3}{10},$$
$$b_{N+1,l_3}^{(n)} = -\tfrac{1}{2}, b_{N+1,l_3+1}^{(n)} = \tfrac{1}{4}, b_{i,j}^{(n)} = 0 \text{ in other cases},$$

for problem (36), and

$$
\begin{aligned}
&b_{i,i}^{(n)} = z^{(n)},\ b_{i-1,i}^{(n)} = b_{i,i-1}^{(n)} = d,\ i = \overline{2,N}; b_{1,1}^{(n)} = 1, b_{1,l_0}^{(n)} = \mu_0 - 1, b_{1,l_0+1}^{(n)} = -\mu_0, b_{N+1,N+1}^{(n)} = 1,\\
&b_{N+1,l_1+1}^{(n)} = -\tfrac{\mu_1}{5}, b_{N+1,l_1}^{(n)} = \tfrac{\mu_1-1}{5}, b_{N+1,l_2+1}^{(n)} = -\tfrac{3\mu_2}{10}, b_{N+1,l_2}^{(n)} = \tfrac{3(\mu_2-1)}{10}, b_{N+1,l_3+1}^{(n)} = -\tfrac{\mu_3}{2}, b_{N+1,l_3}^{(n)} = \tfrac{\mu_3-1}{2},\\
&b_{i,j}^{(n)} = 0 \text{ in other cases,}
\end{aligned}
$$

for problem (37).

Finally, in the third stage, $\left\{v_n^k\right\}$ are calculated by $v_n^k = u_n^k + \zeta_n - v_n^{l_0}$, and $v_n^k = v_n^k + \zeta_n - \left(\mu_0 u_n^{l_0+1} - (\mu_0 - 1)\, u_n^{l_0}\right)$, for the first and second order of accuracy approximately solutions of problems (37) and (37), respectively.

Let $u_M = \overrightarrow{0}$, α_n $(n = 1, \cdots, M-1)$ be $(N+1)\times(N+1)$ matrices and β_n $(n = 1, \cdots, M-1)$ be $(N+1)\times 1$ column vectors, α_1 be the zero matrix and β_1 be zero column vector. Then, solution of (38) is defined by ([31])

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, n = M-1, \cdots, 1,$$

where $\alpha_{n+1}, \beta_{n+1}$ are equal to

$$\alpha_{n+1} = -(B^{(n)} + C^{(n)}\alpha_n)^{-1} A_n, \beta_{n+1} = -(B^{(n)} + C^{(n)}\alpha_n)^{-1}(Ig^{(n)} - C^{(n)}\beta_n), n = 1, \cdots, M-1.$$

Numerical calculations by using MATLAB program are carried out for N=M=20, 40, 80 and 160. In the Tables 1-3, we give error of numerical solution for inverse problem (35) and auxiliary NBVP. Table 1 contains error between exact solution of NBVP and solutions derived by difference schemes (36) and (37). Table 2 and Table 3 contain error between exact and approximately solution of overdetermined problem (35) for p and u, respectively. Tables 1-3 show that the second order of ADS is more accurate comparing with the first order of ADS.

Table 1. Error for NBVP

	N=M=20	N=M=40	N=M=80	N=M=160
First order of ADS	0.014125	7.72×10^{-3}	4.03×10^{-3}	2.05×10^{-3}
Second order of ADS	1.86×10^{-3}	4.65×10^{-4}	1.16×10^{-4}	2.91×10^{-5}

Table2. Error of p for problem (35)

	N=M=20	N=M=40	N=M=80	N=M=160
First order of ADS	0.0216	0.0112	7.72×10^{-3}	4.49×10^{-3}
Second order of ADS	4.53×10^{-2}	1.13×10^{-2}	2.84×10^{-3}	7.12×10^{-4}

Table 3. Error of v for problem (35)

	N=M=20	N=M=40	N=M=80	N=M=160
First order of ADS	0.0128	6.74×10^{-3}	3.45×10^{-3}	1.74×10^{-3}
Second order of ADS	8.38×10^{-5}	2.10×10^{-5}	5.27×10^{-6}	1.31×10^{-6}

4.2. Three dimensional case

Now, consider the three dimensional Bitsadze-Samarskii type inverse elliptic problem

$$
\begin{cases}
-\frac{\partial^2 u}{\partial t^2}(t,x,y) - \frac{\partial^2 u}{\partial x^2}(t,x,y) - \frac{\partial^2 u}{\partial y^2}(t,x,y) + u(t,x,y) = f(t,x,y) + p(x,y),\\
0 < x < 1, 0 < y < 1, 0 < t < 1,\\
u(0,x,y) = \phi(x,y),\ u(1,x,y) - u(0.588,x,y) = \eta(x,y), u(0.788,x,y) = \zeta(x,y),\\
u(t,0,y) = u(t,1,y) = u(t,x,0) = u(t,x,1) = 0, 0 \le x \le 1, 0 \le y \le 1, 0 \le t \le 1,
\end{cases}
\tag{39}
$$

where

$$f(t,x,y)=\left[\left(1+2\pi^2\right)\left(e^{-t}+t\right)-e^{-t}\right]\sin(\pi x)\sin(\pi y),\varphi(x,y)=2\sin(\pi x)\cos(\pi y),$$
$$\eta(x,y)=\left[e^{-1}-e^{-0.588}+0.412\right]\sin(\pi x)\cos(\pi y),\zeta(x,y)=\left(e^{-0.788}+1.788\right)\sin(\pi x)\cos(\pi y).$$

Pair of functions $u(t,x,y)=(e^{-t}+t+1)\sin(\pi x)\sin(\pi y)$ and $p(x,y)=\left(2\pi^2+1\right)\sin(\pi x)\sin(\pi y)$ is exact solution of (39).

Denote by $[0,1]_\tau\times[0,1]_h\times[0,1]_h$ set of grid points

$$[0,1]_\tau\times[0,1]_h\times[0,1]_h=\{(t_k,x_n,y_m):t_k=k\tau,\ k=\overline{0,N},x_n=nh,\ n=\overline{0,M},$$
$$y_m=mh,\ m=\overline{0,M},\ N\tau=1,Mh=1\}.$$

In addition,

$$\lambda_0=0.788,\lambda_1=0.588,l_i=\left[\tfrac{\lambda_i}{\tau}\right],\mu_i=\tfrac{\lambda_i}{\tau}-l_i\ ,i=0,1;\phi_{m,n}=\phi\left(x_n,y_m\right),\eta_{m,n}=\eta\left(x_n,y_m\right),$$
$$\zeta_{m,n}=\zeta\left(x_n,y_m\right),\ n=\overline{0,M},\ m=\overline{0,M};g^k_{m,n}=g(t_k,x_n,y_m),\ k=\overline{0,N},\ n=\overline{0,M},m=\overline{0,M}.$$

In the first stage, the first and second order of ADSs for approximately solution of NBVP can be written in the following forms:

$$\begin{cases}-\frac{u^{k+1}_{m,n}-2u^k_{m,n}+v^{k-1}_{m,n}}{\tau^2}-\frac{u^k_{m,n+1}-2u^k_{m,n}+u^k_{m,n-1}}{h^2}-\frac{u^k_{m+1,n}-2u^k_{m,n}+u^k_{m-1,n}}{\tau^2}+u^k_{m,n}=g^k_{m,n},\\ k=\overline{1,N-1},\ m=\overline{1,M-1},n=\overline{1,M-1};\\ u^k_{0,n}=0,u^k_{M,n}=0,k=\overline{0,N},\ \ n=\overline{1,M-1};\\ u^k_{m,0}=0,u^k_{m,M}=0,\ k=\overline{0,N},m=\overline{1,M-1};\\ u^0_{m,n}-u^{l_0}_{m,n}=\phi_{m,n}-\zeta_{m,n},\ \ u^N_{m,n}-u^{l_1}_{m,n}=\eta_{m,n},\ n=\overline{1,M-1},n=\overline{1,M-1},\end{cases}\tag{40}$$

and

$$\begin{cases}-\frac{u^{k+1}_{m,n}-2u^k_{m,n}+v^{k-1}_{m,n}}{\tau^2}-\frac{u^k_{m,n+1}-2u^k_{m,n}+u^k_{m,n-1}}{h^2}-\frac{u^k_{m+1,n}-2u^k_{m,n}+u^k_{m-1,n}}{\tau^2}+u^k_{m,n}=g^k_{m,n},\\ k=\overline{1,N-1},\ n=\overline{1,M-1},\ m=\overline{1,M-1};\\ u^k_{0,n}=0,\ u^k_{M,n}=0,k=\overline{0,N},\ \ n=\overline{1,M-1};\\ u^k_{m,0}=0,\ u^k_{m,M}=0,k=\overline{0,N},\ m=\overline{1,M-1};\\ u^0_{m,n}+\left(\mu_0-1\right)u^{l_0}_{m,n}-\mu_0u^{l_0+1}_{m,n}=\phi_{m,n}-\zeta_{m,n},\\ u^N_{m,n}+\left(\mu_1-1\right)u^{l_1}_{m,n}-\mu_1u^{l_1+1}_{m,n}=\eta_{m,n},\ m=\overline{1,M-1},n=\overline{1,M-1},\end{cases}\tag{41}$$

respectively.

In the second stage, calculation of $p_{n,m}$ $\left(n=\overline{1,M-1},m=\overline{1,M-1}\right)$ is caried out by

$$p_{m,n}=-\frac{(\zeta_{m,n+1}-u^{l_0}_{m,n+1})-2(\zeta_{m,n}-u^{l_0}_{m,n})+(\zeta_{m,n-1}-u^{l_0}_{m,n-1})}{h^2}$$
$$-\frac{(\zeta_{m+1,n}-u^{l_0}_{m+1,n})-2(\zeta_{m,n}-u^{l_0}_{m,n})+(\zeta_{m-1,n}-u^{l_0}_{m-1,n})}{h^2}+\zeta_{m,n}-u^{l_0}_{m,n}$$

for the first order approximation, and

$$p_{m,n}=-\tfrac{1}{h^2}\left\{\left[\zeta_{m,n+1}-\left(\mu_0u^{l_0+1}_{m,n+1}-\left(\mu_0-1\right)u^{l_0}_{m,n+1}\right)\right]-2\left[\zeta_{m,n}-\left(\mu_0u^{l_0+1}_{m,n}-\left(\mu_0-1\right)u^{l_0}_{m,n}\right)\right]\right.$$
$$\left.+\left[\zeta_{m,n-1}-\left(\mu_0u^{l_0+1}_{m,n-1}-\left(\mu_0-1\right)u^{l_0}_{m,n-1}\right)\right]\right\}-\tfrac{1}{h^2}\left\{\left[\zeta_{m+1,n}-\left(\mu_0v^{l_0+1}_{m+1,n}-\left(\mu_0-1\right)v^{l_0}_{m+1,n}\right)\right]\right.$$
$$\left.-2\left[\zeta_{m,n}-\left(\mu_0v^{l_0+1}_{m,n}-\left(\mu_0-1\right)v^{l_0}_{m,n}\right)\right]\right\}+\left[\zeta_{m-1,n}-\left(\mu_0v^{l_0+1}_{m-1,n}-\left(\mu_0-1\right)v^{l_0}_{m-1,n}\right)\right]$$

for the second order approximation.

In the third stage, $\left\{v_n^k\right\}$ are calculated by

$$u_{m,n}^k = v_{m,n}^k + \zeta_n - v_{m,n}^{l_0}, \text{ and } u_{m,n}^k = v_{m,n}^k + \zeta_{m,n} - \left(\mu_0 v_{m,n}^{l_0+1} - (\mu_0 - 1)\, v_{m,n}^{l_0}\right)$$

for the first and second order of accuracy approximately solutions of problems (40) and (41), respectively.

Difference problems (40) and (41) can be rewritten in the matrix form (38). In this case, g_n is $(N+1)(M+1) \times 1$ a column matrix, $A^{(n)}$, $B^{(n)}$, $C^{(n)}, I$ are $(N+1)(M+1) \times (N+1)(M+1)$ square matrices, and I is the identity matrix, v_s is the $(N+1)(M+1) \times 1$ column matrix such that

$$v_s = \left[\begin{array}{cccccccccccccccc} v_{0,s}^0 & v_{0,s}^1 & \cdots & v_{0,s}^N & v_{1,s}^0 & v_{1,s}^1 & \cdots & v_{1,s}^N & \cdots & v_{m,s}^0 & v_{m,s}^1 & \cdots & v_{m,s}^N & \cdots & v_{M,s}^0 & v_{M,s}^1 & \cdots & v_{M,s}^N \end{array}\right]^t$$
$s = n-1, n, n+1$.

Let us

$$a = \frac{1}{h^2}, z = 1 + \frac{2}{\tau^2} + \frac{4}{h^2}, d = \frac{1}{\tau^2}.$$

Then,

$$A = C = \left[\begin{array}{ccccc} O & O & \cdots & O & O \\ O & E & \cdots & O & O \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ O & O & \cdots & E & \\ O & O & \cdots & O & O \end{array}\right], \quad B = \left[\begin{array}{ccccc} Q & O & \cdots & O & O \\ O & W & \cdots & O & O \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ O & O & \cdots & W & \\ O & O & \cdots & O & Q \end{array}\right],$$

$$A^{(n)} = A = C^{(n)} = C,$$

$$E = diag\{0, a, a, \ldots, a, 0\}, O = O_{(N+1)\times(N+1)}, Q = I_{(N+1)\times(N+1)}, \; g_{m,n}^k = g(t_k, x_n, y_m), n = \overline{1, M-1},$$
$$m = \overline{1, M-1}, \; k = \overline{1, N-1}; g_{m,n}^0 = \phi_{m,n} - \zeta_{m,n}, \; g_{m,n}^N = \eta_{m,n}, \; n = \overline{1, M-1}\,, \; m = \overline{1, M-1}$$

for both schemes (36) and (37). Elements $w_{i,j}$ of matrix W are defined by

$$w_{i,i} = z, w_{i-1,i} = d, w_{i,i-1} = d, i = \overline{2, N}, w_{1,1} = 1, w_{1,l_0} = -1, w_{N+1,N+1} = 1, w_{N+1,l_1} = -1,$$
$$w_{i,j} = 0, \text{ in other cases}$$

for first order approximation, and

$$w_{i,i} = z, w_{i-1,i} = d, w_{i,i-1} = d, i = \overline{2, N}, w_{1,1} = 1, w_{1,l_0} = \mu_0 - 1, w_{1,l_0+1} = -\mu_0,$$
$$w_{N+1,N+1} = 1, w_{N+1,l_1} = \mu_1 - 1, w_{N+1,l_1+1} = -\mu_1,$$
$$w_{i,j} = 0, \text{ in other cases}$$

for second order approximation.

In Tables 4-6, we give results of numerical calculations for both first and second order approximations in case N=M=10, 20, 40. Table 4 presents error for NBVP . Table 5 includes error for p. Tables 3 gives error for u. It can be seen from Tables 4-6 that the second order of ADS is more accurate comparing to the first order of ADS.

Table 4. Error for NBVP

	N=M=10	N=M=20	N=M=40
First order of ADS	0.024712	0.01151	4.11×10^{-3}
Second order of ADS	4.30×10^{-4}	1.34×10^{-4}	3.93×10^{-5}

Table 5. Error of p for problem (35)

	N=M=10	N=M=20	N=M=40
First order of ADS	0.6356	0.2927	0.10408
Second order of ADS	1.11×10^{-3}	4.70×10^{-4}	2.77×10^{-4}

Table6. Error of u for problem (35)

	N=M=10	N=M=20	N=M=40
First order of ADS	0.026002	0.012923	4.71×10^{-3}
Second order of ADS	4.83×10^{-3}	1.24×10^{-3}	3.17×10^{-4}

5. Conclusion

In the present paper, finite difference method is applied to Bitsadze-Samarskii type overdetermined elliptic problem with Dirichlet condition. For the approximate solution of this problem a first and a second order of ADSs are presented. Stability and coercive stability estimates for solutions of both difference schemes are established. Abstract results are applied to the investigation of overdetermined multidimensional elliptic problem with multipoint Dirichlet type boundary conditions. Finally, we give some numerical results for both difference schemes in two- and three-dimensional cases.

Moreover, applying the results of works [11, 15, 23] the high order of accuracy stable difference schemes for the numerical solution of the Bitsadze-Samarskii type overdetermined elliptic problem with Dirichlet condition can be presented.

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