Filomat 32:3 (2018), 873–883
https://doi.org/10.2298/FIL1803873K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: `http://www.pmf.ni.ac.rs/filomat`

On Some Generalizations of Properties of the Lowndes Operator and their Applications to Partial Differential Equations of High Order

Shakhobiddin T. Karimov[a]

[a]*Faculty of Mathematics and Physics, Ferghana State University, Ferghana, 150100 Uzbekistan*

Abstract. In this work we proved a composition of Lowndes' operator with differential operators of the high order, particularly, with iterated Bessel differential singular operator. Examples on applications of the proved properties to partial differential equations of the fourth and high order with singular coefficients were showed. By applying the proved theorems, explicit formulas of the solutions of considered problems were constructed.

1. Introduction

Various modifications and generalizations of the classical fractional integration operators are known and are widely used both in theory and applications. The Erdélyi-Kober operators concern such modifications in particular, [1, 12, 16]. Their various modifications, generalizations and applications can be found in works by Erdélyi [3–5], Sneddon [18, 19], Lowndes [13–15] and Kiryakova [12].

In the work [13] of Lowndes the following generalized Erdélyi-Kober operator with the Bessel function in the kernel was introduced and investigated.

$$J_\lambda(\eta,\ \alpha)f(x) = 2^\alpha \lambda^{1-\alpha} x^{-2\alpha-2\eta} \int\limits_0^x \frac{J_{\alpha-1}\left(\lambda\sqrt{x^2-t^2}\right)}{(x^2-t^2)^{(1-\alpha)/2}} t^{2\eta+1} f(t)dt, \tag{1}$$

where $\alpha, \eta, \in R,\ \ \lambda \in C$, such that $\alpha > 0,\ \ \eta \geqslant -(1/2)$, and $J_\nu(z)$ is the Bessel function of the first kind, [6]. It is obvious that if $\lambda \to 0$, then the operator in (1) coincides with the Erdélyi-Kober operator [1]:

$$I_{\eta,\alpha} f(x) = \frac{2x^{-2(\eta+\alpha)}}{\Gamma(\alpha)} \int\limits_0^x (x^2-t^2)^{\alpha-1} t^{2\eta+1} f(t)dt, \tag{2}$$

where $\Gamma(\alpha)$ is the Euler gamma-function [6].

The basic properties of these operators can be found in the works [1–5, 7, 8, 12–16, 18, 19]

2010 *Mathematics Subject Classification*. Primary 26A33; Secondary 35L10, 35L80, 35L81

Keywords. Generalized Erdélyi-Kober operator, Lowndes operator, Cauchy problem, Bessel differential operator, partial differential equations of the high order

Received: 23 December 2016; Revised: 25 May 2017; Accepted: 25 May 2017

Communicated by Allaberen Ashyralyev

Email address: `shaxkarimov@gmail.com` (Shakhobiddin T. Karimov)

Properties of operator (1) in weight spaces $L_p(0,\infty)$ were studied in works by Heywood [7] and by Heywood and Rooney [8]. In these works, the generalized Erdélyi-Kober operators are named by Lowndes operator.

Further, we need the following modified form of operator (1):

$$J_\lambda(\eta,\,\alpha)f(x)=\frac{2x^{-2(\alpha+\eta)}}{\Gamma(\alpha)}\int\limits_0^x t^{2\eta+1}(x^2-t^2)^{\alpha-1}\bar{J}_{\alpha-1}\left(\lambda\sqrt{x^2-t^2}\right)f(t))dt, \tag{3}$$

where $\bar{J}_\nu(z)$ is the Bessel-Clifford function [12, 16] defined as

$$\bar{J}_\nu(z)=\Gamma(\nu+1)(z/2)^{-\nu}J_\nu(z)={}_0F_1(\nu+1;-z^2/4)=\sum_{k=0}^{\infty}\frac{(-z^2/4)^k}{(\nu+1)_k k!}. \tag{4}$$

Further, let $B^x_\eta=x^{-2\eta-1}\dfrac{\partial}{\partial x}x^{2\eta+1}\dfrac{\partial}{\partial x}=\dfrac{\partial^2}{\partial x^2}+\dfrac{2\eta+1}{x}\dfrac{\partial}{\partial x}$ be the Bessel differential operator on the variable x.

For operators (1) and (2) holds the following theorem [16, lemma 40.2], [4], [14].

Theorem 1.1. *Let $\alpha>0,\ f(x)\in C^2(0,b),\ b>0,$ function $x^{2\eta+1}B^x_\eta f(x)$ is integrable at zero, and $\lim\limits_{x\to 0}x^{2\eta+1}f'(x)=0$. Then*

$$(B^x_{\eta+\alpha}+\lambda^2)J_\lambda(\eta,\alpha)f(x)=J_\lambda(\eta,\alpha)B^x_\eta f(x) \tag{5}$$

and, in particular, if $\lambda=0$, then

$$B^x_{\eta+\alpha}I_{\eta,\alpha}f(x)=I_{\eta,\alpha}B^x_\eta f(x).$$

In the given work these properties are generalized for iterated Bessel differential operator of the high order. The obtained results are applied to the investigation of problems for partial differential equations of the high order with singular coefficients.

2. Generalzaton of Propertes of Lowndes Operator

2.1. *Composition of an Operator* (3) *with Iterated Bessel Differential Operator of the High Order*

Let $[B^x_\eta]^0=E$, E is identity operator, $[B^x_\eta]^m=[B^x_\eta]^{m-1}[B^x_\eta]=[B^x_\eta][B^x_\eta]...[B^x_\eta]$ be m-th power of Bessel operator. Further m means natural number.

Theorem 2.1. *Let $\alpha>0,\ \eta\geqslant-(1/2),\ f(x)\in C^{2m}(0,b),\ b>0,$ functions $x^{2\eta+1}[B^x_\eta]^{k+1}f(x)$ is integrable at zero, and $\lim\limits_{x\to 0}x^{2\eta+1}\dfrac{d}{dx}[B^x_\eta]^k f(x)=0$, $k=\overline{0,m-1}$. Then*

$$[B^x_{\eta+\alpha}+\lambda^2]^m J_\lambda(\eta,\alpha)f(x)=J_\lambda(\eta,\alpha)[B^x_\eta]^m f(x) \tag{6}$$

and, in particular, if $\lambda=0$, then

$$[B^x_{\eta+\alpha}]^m I_{\eta,\alpha}f(x)=I_{\eta,\alpha}[B^x_\eta]^m f(x).$$

Proof. We use a method of a mathematical induction on m. For $m=1$ it is proved in the Theorem 1.1. We assume that the equality (6) is true for $m=k$ and we prove that the equality is true for $m=k+1$.

The left hand side of (6) can be written as

$$(B^x_{\eta+\alpha}+\lambda^2)^{k+1}J_\lambda(\eta,\alpha)f(x)=(B^x_{\eta+\alpha}+\lambda^2)(B^x_{\eta+\alpha}+\lambda^2)^k J_\lambda(\eta,\alpha)f(x),$$

but by the inductive hypothesis,

$$(B^x_{\eta+\alpha}+\lambda^2)(B^x_{\eta+\alpha}+\lambda^2)^k J_\lambda(\eta,\alpha)f(x)=(B^x_{\eta+\alpha}+\lambda^2)J_\lambda(\eta,\alpha)[B^x_\eta]^k f(x).$$

In last equality applying the Theorem 1.1 for function $[B^x_\eta]^k f(x)$ at realization of conditions $\lim\limits_{x\to0} x^{2\eta+1}\dfrac{d}{dx}[B^x_\eta]^j f(x)=0,\ j=\overline{0,k-1}$, we obtain equality (6) for $m=k+1$. □

Let function $u(x,y)=u(x_1,\,x_2,\,\ldots,x_n,\,y)$ be continuously differentiable up to the order $2m$ on a variable y and the order is not less than m on x. L_x is a linear differential operator of any order on a variable $x\in R^n$ and it does not depend on y.

Theorem 2.2. *Let $\alpha>0,\ \eta\geqslant-(1/2)$, functions $y^{2\eta+1}[B^y_\eta]^k u(x,y)$ be integrable at $y\to0$ and $\lim\limits_{y\to0} y^{2\eta+1}\dfrac{\partial}{\partial y}[B^y_\eta]^k u(x,y)=0,\ \ k=\overline{0,m-1}$. Then*

$$(B^y_{\eta+\alpha}+\lambda^2\pm L_x)^m J^y_\lambda(\eta,\alpha)u(x,y)=J^y_\lambda(\eta,\alpha)(B^y_\eta\pm L_x)^m u(x,y) \tag{7}$$

and, in particular, if $\lambda=0$, then

$$(B^y_{\eta+\alpha}\pm L_x)^m I^y_{\eta,\alpha}u(x,y)=I^y_{\eta,\alpha}(B^y_\eta\pm L_x)^m u(x,y),$$

where the indices y in the above operators imply a variable to which these operators are applied.

Theorem 2.2 is proved by means of formal expansion of the operator $[(B^y_\eta+\lambda^2)\pm L_x]^m$ by a binominal formula $[(B^y_\eta+\lambda^2)\pm L_x]^m=\sum\limits_{k=1}^{m}\begin{pmatrix} m\\ k\end{pmatrix}(\pm L_x)^{m-k}\left(B^y_\eta+\lambda^2\right)^k$ and by applying Theorem 2.1.

Corollary 2.1. *Let $\eta=-1/2,\ \alpha>0$, functions $\dfrac{\partial^{2k}u(x,y)}{\partial y^{2k}}$ be integrable at $y\to0$ and $\lim\limits_{y\to0}\dfrac{\partial^{2k+1}u(x,y)}{\partial y^{2k+1}}=0,\ \ k=\overline{0,m-1}$. Then*

$$\left(\frac{\partial^2}{\partial y^2}+\frac{2\alpha}{y}\frac{\partial}{\partial y}+\lambda^2\pm L_x\right)^m J^y_\lambda\left(-\frac12,\alpha\right)u(x,y)=J^y_\lambda\left(-\frac12,\alpha\right)\left(\frac{\partial^2}{\partial y^2}\pm L_x\right)^m u(x,y).$$

2.2. Derivatives of Higher Order Lowndes Operator (3)

Let $D^0_\eta=E,\ D_\eta=x^{-2\eta}\left(\dfrac1x\dfrac{d}{dx}\right)x^{2\eta},\ D^m_\eta=D^{m-1}_\eta D_\eta=D_\eta D_\eta ... D_\eta$ is m -th power of operator D_η which is represented in the form $D^m_\eta=x^{-2\eta}\left(\dfrac1x\dfrac{d}{dx}\right)^m x^{2\eta}$.

Theorem 2.3. *If $\alpha>0,\ \eta\geqslant-(1/2),\ \ f(x)\in C^m(0,b),\ b>0,$ functions $x^{2\eta+1}D^{k+1}_\eta f(x)$ are integrated in zero and $\lim\limits_{x\to0}x^{2\eta}D^k_\eta f(x)=0,\ k=\overline{0,m-1}$, then an equality*

$$D^m_{\eta+\alpha}J_\lambda(\eta,\alpha)f(x)=J_\lambda(\eta,\alpha)D^m_\eta f(x) \tag{8}$$

is true.

Proof. This theorem is proved by the method of mathematical induction on m as well. We show that the equality (8) is true for $m=1$:

$$D_{\eta+\alpha}J_\lambda(\eta,\alpha)f(x)=J_\lambda(\eta,\alpha)D_\eta f(x). \tag{9}$$

Let's consider function

$$D_{\eta+\alpha}J_\lambda(\eta,\alpha)f(x)=\frac{2x^{-2(\eta+\alpha)}}{\Gamma(\alpha)}\lim_{\varepsilon\to 0}F_\varepsilon(x),$$

where ε is enough small positive real number and

$$F_\varepsilon(x)=\left(\frac{1}{x}\frac{d}{dx}\right)\int\limits_0^{x-\varepsilon}(x^2-t^2)^{\alpha-1}\bar{J}_{\alpha-1}\left(\lambda\sqrt{x^2-t^2}\right)t^{2\eta+1}f(t)dt$$

by applying the rule of derivation of integral, we derive

$$F_\varepsilon(x)=\frac{\varepsilon^{\alpha-1}}{x}(2x-\varepsilon)^{\alpha-1}\bar{J}_{\alpha-1}\left(\lambda\sqrt{\varepsilon(2x-\varepsilon)}\right)(x-\varepsilon)^{2\eta+1}f(x-\varepsilon)$$

$$+\int\limits_0^{x-\varepsilon}\left(\frac{1}{x}\frac{d}{dx}\right)\left[(x^2-t^2)^{\alpha-1}\bar{J}_{\alpha-1}\left(\lambda\sqrt{x^2-t^2}\right)\right]t^{2\eta+1}f(t)dt.$$

Further, considering easily checked equality

$$\left(\frac{1}{x}\frac{d}{dx}\right)\left[(x^2-t^2)^{\alpha-1}\bar{J}_{\alpha-1}\left(\lambda\sqrt{x^2-t^2}\right)\right]=-\left(\frac{1}{t}\frac{d}{dt}\right)\left[(x^2-t^2)^{\alpha-1}\bar{J}_{\alpha-1}\left(\lambda\sqrt{x^2-t^2}\right)\right],$$

we have

$$F_\varepsilon(x)=\frac{\varepsilon^{\alpha-1}}{x}(2x-\varepsilon)^{\alpha-1}\bar{J}_{\alpha-1}\left(\lambda\sqrt{\varepsilon(2x-\varepsilon)}\right)(x-\varepsilon)^{2\eta+1}f(x-\varepsilon)$$

$$-\int\limits_0^{x-\varepsilon}\left(\frac{d}{dt}\right)\left[(x^2-t^2)^{\alpha-1}\bar{J}_{\alpha-1}\left(\lambda\sqrt{x^2-t^2}\right)\right]t^{2\eta}f(t)dt.$$

Applying to the last integral a rule of integration by parts and considering the condition of the Theorem 2.3, after canceling of the terms, we have

$$F_\varepsilon(x)=\frac{-\varepsilon^{\alpha}}{x(x-\varepsilon)}(2x-\varepsilon)^{\alpha-1}\bar{J}_{\alpha-1}\left(\lambda\sqrt{\varepsilon(2x-\varepsilon)}\right)(x-\varepsilon)^{2\eta+1}f(x-\varepsilon)$$

$$+\int\limits_0^{x-\varepsilon}\left[(x^2-t^2)^{\alpha-1}\bar{J}_{\alpha-1}\left(\lambda\sqrt{x^2-t^2}\right)\right]t^{2\eta+1}D_\eta f(t)dt.$$

From here, by virtue of $\alpha>0$, for $\varepsilon\to 0$ we obtain equality (9).

Let's assume that the equality (8) is true for $m=k$. We prove that it is true for $m=k+1$.

Consider left hand side of (8)

$$D^{k+1}_{\eta+\alpha}J_\lambda(\eta,\alpha)f(x)=D_{\eta+\alpha}D^k_{\eta+\alpha}J_\lambda(\eta,\alpha)f(x).$$

On the other hand, by the inductive hypothesis,

$$D_{\eta+\alpha}D^k_{\eta+\alpha}J_\lambda(\eta,\alpha)f(x)=D_{\eta+\alpha}J_\lambda(\eta,\alpha)D^k_\eta f(x).$$

By applying the (9) for $D^k_\eta f(x)$ function in last equality, we get equality (8) for $m=k+1$. The proof of the Theorem 2.3 is finished. □

Corollary 2.2. *Let $\eta = 0$, $\alpha > 0$, $f(x) \in C^m(0,b)$, $b > 0$, functions* $\dfrac{d}{dx}\left(\dfrac{1}{x}\dfrac{d}{dx}\right)^k f(x)$ *be integrable in zero and* $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\dfrac{d}{dx}\right)^k f(x) = 0$, $k = \overline{0, m-1}$. *Then*

$$\left(\frac{1}{x}\frac{d}{dx}\right)^m \int_0^x (x^2-t^2)^{\alpha-1}\bar{J}_{\alpha-1}\left(\lambda\sqrt{x^2-t^2}\right)f(t)tdt = \int_0^x (x^2-t^2)^{\alpha-1}\bar{J}_{\alpha-1}\left(\lambda\sqrt{x^2-t^2}\right)\left[\left(\frac{1}{t}\frac{d}{dt}\right)^m f(t)\right]tdt.$$

Considering easily checked equalities $D_0 B_\eta = B_{\eta+1}D_0$, $B_\eta = D_\eta x^2 D_0$, $x^2 D_\eta = x\dfrac{d}{dx} + 2\eta$, $D_\eta = D_0 + \dfrac{2\eta}{x^2}$, $D_\eta B_\eta = D_\eta^2\left(x\dfrac{d}{dx}\right)$ where $D_0 = \dfrac{1}{x}\dfrac{d}{dx}$, the other properties of operator 1 can be proved.

The proved theorems allow to reduce singular (or degenerated) equations of high order to equations without singularity and thus, to formulate and investigate correct initial and boundary problems for such equations.

3. Applications

The method of fractional integro-differentiation for the differential equations in generalized axially symmetric potential theory was first considered by Weinstein [21–23]. In the paper [23] and [24], he proved relations connecting the solutions of equation

$$L^\lambda_{\alpha,\beta}(u) \equiv \frac{\partial^2 u}{\partial t^2} + \frac{2\beta}{t}\frac{\partial u}{\partial t} - \sum_{k=1}^n\left(\frac{\partial^2 u}{\partial x_k^2} + \frac{2\alpha_k}{x_k}\frac{\partial u}{\partial x_k}\right) + \lambda^2 u = 0, \tag{10}$$

for $n = 1$, $\alpha_1 = \lambda = 0$ and various values of parameter β by means of an integral of the fractional order. This idea was developed by Erdélyi [3–5] who investigated the properties of the Bessel differential operator. In particular, in the papers [3] and [4] he proved Theorem 1.1 in the case $\lambda = 0$.

The results of Erdélyi were generalized by Lowndes [14] who proved Theorem 1.1. Lowndes obtained results applied to a solution of some boundary value problems for the equation of Laplace with the mixed boundary conditions. Proved Theorem 1.1 enables to receive accordingly a fundamental solution of more common of Helmholtz type equations from fundamental solutions of the Laplace equation accordingly. Besides that in the work [15], applying the Theorem 1.1 he has solved a Cauchy problem for the equation (10) for $n \geqslant 1$, $\beta = 0$, $\alpha_k = 0$, $k = \overline{1,n}$; $\lambda \neq 0$.

In this direction also it is necessary to note the work [20] in which, applying the Theorem 1.1, the Cauchy problem for the equation (10) is investigated for $n \geqslant 1$, $\beta \neq 0$, $\lambda \neq 0$, $\alpha_k = 0$, $k = \overline{1,n}$. In this work, the explicit formula of a solution of the studied problem is obtained at various values of parameter β.

In the works [9] and [10] the properties of many-dimensional generalized Erdélyi-Kober operator (Lowndes) are investigated and the received results are applied to a solution of a Cauchy problem to the equation (10) for $n = 1, 2$, and in work [11] it is solved for $n = 3$, $\beta = 0$, $\lambda = 0$ and $\alpha_k \neq 0$, $k = 1, 2, 3$.

Further, in this work, by examples, we will show application of the proved theorems to construction of explicit formulas of a solution of problems for the equations of the fourth and high orders.

3.1. Application Lowndes Operator to Partial Differential Equation of the Fourth Order

In the domain $\Omega = \{(x,y) : -\infty < x < +\infty,\ 0 < y < +\infty\}$ for the equation of the fourth order

$$L^\lambda_\beta(u) \equiv \frac{\partial^2 u}{\partial y^2} + \frac{2\beta}{y}\frac{\partial u}{\partial y} + \frac{\partial^4 u}{\partial x^4} + \lambda^2 u = 0 \tag{11}$$

it is possible to formulate and investigate a problem with initial conditions

$$u(x,0) = f(x), \ \lim_{y\to 0} y^{2\beta} u_y(x,y) = g(x), \ -\infty < x < +\infty, \tag{12}$$

where $\beta,\ \lambda \in R$, and $0 < \beta < (1/2)$, $f(x)$, $g(x)$ are given smooth functions.

The given problem is not investigated earlier. First, we find a solution of the equation (11), satisfying to homogeneous initial conditions

$$u(x,0) = f(x), \ u_y(x,0) = 0, \ -\infty < x < +\infty. \tag{13}$$

Let's assume that a solution of a problem (11), (13) exists. For this solution we search in the form of

$$u(x,y) = J_\lambda^y(-1/2,\ \beta)U(x,y), \tag{14}$$

where $U(x,y)$ is unknown smooth function.

Substituting (14) in the equation (11) and initial conditions (13), and then, using the Theorem 1.1, we get the following problem of a determination of a solution $U(x,y)$ of the equation

$$\frac{\partial^2 U}{\partial y^2} + \frac{\partial^4 U}{\partial x^4} = 0 \tag{15}$$

satisfying to initial conditions

$$U(x,0) = k_0 f(x), \ \ U_y(x,0) = 0, \ \ x \in R, \tag{16}$$

where $k_0 = \Gamma(\beta + (1/2))/\sqrt{\pi}$.

The solution of a problem (11), (16) has a form [17]

$$U(x,y) = \frac{k_0}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} f(x - 2\xi\sqrt{y})\left(\cos\xi^2 + \sin\xi^2\right) d\xi. \tag{17}$$

Substituting (17) in (14), after change of the order of an integration and having calculated an interior integral, we receive

$$u(x,y) = \gamma_1 \int\limits_{-\infty}^{+\infty} f(x + 2\xi\sqrt{y})G(\xi,\ y;\beta)d\xi, \tag{18}$$

where $\gamma_1 = k_0/\sqrt{2\pi}$,

$$G(\xi,y;\beta) = \frac{\Gamma(1/4)}{\Gamma(\beta+(1/4))} K_1\left(\beta+\frac{1}{4};\ \frac{3}{4},\ \frac{1}{2};\ -\frac{\xi^4}{4},\ -\frac{1}{4}\lambda^2 y^2\right) + \frac{\Gamma(-1/4)}{\Gamma(\beta-(1/4))}\xi^2 K_1\left(\beta-\frac{1}{4};\ \frac{5}{4},\ \frac{3}{2};\ -\frac{\xi^4}{4},\ -\frac{1}{4}\lambda^2 y^2\right).$$

Here $K_1(a,b,c;\ x,y) = \sum\limits_{m=0}^{\infty} \frac{y^m}{(a)_m m!}\, {}_1F_2\left(1-a-m;\ b,\ c;\ x\right)$, ${}_1F_2\left(a;\ b,\ c;\ z\right)$ is the generalized hypergeometric function [6].

For construction of a solution of the equation (11), satisfying to a half homogeneous initial conditions

$$u(x,0) = 0, \ \lim_{y\to 0} y^{2\beta} u_y(x,y) = g(x), \ -\infty < x < +\infty. \tag{19}$$

Let's take advantage of the following property of the equation (11): If $u(x,y;\ 1-\beta)$ is a solution of the equation $L_{1-\beta}^\lambda(u) = 0$, satisfying to conditions (13), function $w(x,y;\beta) = y^{1-2\beta}u(x,y;1-\beta)$ at $0 < \beta < 1/2$ will be a solution of the equation $L_\beta^\lambda(u) = 0$, satisfying to conditions

$$w(x,0;\beta) = 0, \ \lim_{y\to 0} y^{2\beta} w_y(x,y;\beta) = (1-2\beta)f(x), \ \ -\infty < x < +\infty.$$

This property is proved by an immediate evaluation.

Considering this property and having substituted $(1-2\beta)f(x)$ on $g(x)$, from equality (18), we receive

$$w(x,y;\beta) = \gamma_2 y^{1-2\beta} \int\limits_{-\infty}^{+\infty} g(x+2\xi\sqrt{y}) G(\xi,\ y;\ 1-\beta) d\xi, \tag{20}$$

where $\gamma_2 = \Gamma((1/2)-\beta)/(\pi 2\sqrt{2})$.

Thus, the solution of a problem (11), (12) by virtue of formulas (18), (20) and a principle of linear superposition look like

$$u(x,y) = \gamma_1 \int\limits_{-\infty}^{+\infty} f(x+2\xi\sqrt{y}) G(\xi,\ y;\beta) d\xi + \gamma_2 y^{1-2\beta} \int\limits_{-\infty}^{+\infty} g(x+2\xi\sqrt{y}) G(\xi,\ y;\ 1-\beta) d\xi.$$

3.2. Application Lowndes operator to Partial Differential Equation with the Square of the Bessel Operator

In the domain $\Omega^+ = \{(x,y):\ 0<x<+\infty,\ 0<y<+\infty\}$ for the equation of the fourth order

$$\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial^2}{\partial x^2} + \frac{2\alpha}{x}\frac{\partial}{\partial x}\right)^2 u = 0 \tag{21}$$

it is possible to formulate and investigate a problem with initial

$$u(x,0) = f(x),\ 0 \leqslant x < +\infty,\ \ u_y(x,0) = 0,\ 0 < x < +\infty, \tag{22}$$

and boundary conditions

$$u_x(0,y) = 0, \quad u_{xxx}(0,y) = 0, \quad 0 < y < +\infty, \tag{23}$$

where $\alpha \in R$, and $0 < \alpha < (1/2)$, $f(x)-$ the set smooth function.

Equation (21), in particular, arises at study of the equation of a many-dimensional free transverse vibration of a thin elastic plate $u_{yy} + \Delta^2 u = 0$ [17] at a rotational symmetry in a spherical frame, where $\Delta^2 = \Delta\Delta$ is a biharmonic operator, and Δ is a many-dimensional Laplace operator.

As well as in the previous example, for a solution of the equation (21) we search in the form of

$$u(x,y) = J_0^x(-1/2,\ \alpha) U(x,y) = I^x_{-1/2,\alpha} U(x,y), \tag{24}$$

where $I^x_{-1/2,\alpha}$ is Erdéélyi-Kober operator (2), and $U(x,y)$ is unknown smooth function.

Substituting (24) in the equation (21), initial conditions (22), and then, using the Theorem 2.1 for $\lambda = 0,\ m = 2$ and considering boundary conditions (23), we obtain the following problem of a determination of a solution $U(x,y)$ of the equation (15) satisfying to initial conditions

$$U(x,0) = F(x),\ 0 \leqslant x < +\infty,\ \ U_y(x,0) = 0,\ 0 < x < +\infty \tag{25}$$

and to homogeneous boundary conditions

$$U_x(0,y) = 0, \quad U_{xxx}(0,y) = 0, \quad 0 < y < +\infty, \tag{26}$$

where

$$F(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int\limits_0^x (x^2 - s^2)^{-\alpha} s^{2\alpha} f(s) ds.$$

To find a solution of the problem {(15), (25), (26)} it is impossible to take advantage immediately of the formula (17), since for negative values of arguments the initial function $F(x)$ is not defined.

Let us extend $F(x)$ to $x < 0$ by even reflection, and consider $F_0(x)$ as such extension. Thus, we ensure realization of the conditions (26) and now we can use the formula (17) which looks like

$$U(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F_0(x - 2\xi\sqrt{y}) \left(\cos\xi^2 + \sin\xi^2\right) d\xi. \tag{27}$$

Substituting (27) in (24) after simple transformations and change of the order of an integration, and also having calculated an interior integral, we receive the explicit formula of a solution of a problem {(21) - (23)} in the form of

$$u(x,y) = \frac{x^{(1/2)-\alpha}}{2y} \int_0^{+\infty} f(\xi)\xi^{(1/2)+\alpha} J_{\alpha-(1/2)}\left(\frac{x\xi}{2y}\right) \sin\left[\frac{x^2+\xi^2}{4y} + \frac{\pi}{2}\left(\frac{1}{2} - \alpha\right)\right] d\xi.$$

Following example shows application of the Theorem 2.2.

3.3. Application Lowndes operator to the Polywave Equation with Bessel Operator

In the domain $\Omega = \{(x,y) : -\infty < x < +\infty,\ 0 < y < +\infty\}$ it is required to discover a classical solution of the iterated equation

$$\left(\frac{\partial^2}{\partial y^2} + \frac{2\beta}{y}\frac{\partial}{\partial y} + \lambda^2 - \frac{\partial^2}{\partial x^2}\right)^m u(x,y) = 0,\ (x,y) \in \Omega, \tag{28}$$

satisfying to initial conditions

$$\left.\frac{\partial^{2k} u}{\partial y^{2k}}\right|_{y=0} = \varphi_k(x)\ \left.\frac{\partial^{2k+1} u}{\partial y^{2k+1}}\right|_{y=0} = 0,\ x \in R,\ k = \overline{0, m-1}, \tag{29}$$

where β, $\lambda \in R$, and $0 < \beta < (1/2)$, and $\varphi_k(x)$ $(k = \overline{0, m-1})$ is the set smooth functions.

Similarly as in the subsection 3.1 we assume, that a solution of a problem (28), (29) exists. We search for this solution in the form of

$$u(x,y) = J_\lambda^y(-1/2, \beta)U(x,y), \tag{30}$$

where $U(x,y)$ is unknown smooth function.

Substituting (30) in the equation (28) and initial conditions (29), and then, using a corollary 2.1 at $L_x = \dfrac{\partial^2}{\partial x^2}$, we receive a following problem of a determination of a solution $U(x,y)$ of the equation

$$\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right)^m U(x,y) = 0,\ x \in R,\ y > 0 \tag{31}$$

satisfying to initial conditions

$$\left.\frac{\partial^{2k} U}{\partial y^{2k}}\right|_{y=0} = \Phi_k(x),\quad \frac{\partial^{2k+1} U}{\partial y^{2k+1}}\,|_{y=0} = 0,\ x \in R,\ k = \overline{0, m-1}, \tag{32}$$

where $\Phi_k(x) = \sum\limits_{j=0}^{k} \gamma_j C_k^j \lambda^{2(k-j)} \varphi_j(x)$, $\gamma_j = \Gamma\left(\dfrac{2j+1}{2} + \beta\right) / \Gamma\left(\dfrac{2j+1}{2}\right)$, $C_k^j = \dfrac{k!}{j!(k-j)!}$ is binomial coefficient, $k! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (k-1) \cdot k$.

Let $V_0(x,y)=U(x,y)$ and $V_n(x,y)=\left(\dfrac{\partial^2}{\partial y^2}-\dfrac{\partial^2}{\partial x^2}\right)^n V_0(x,y)$. Then the problem {(31), (32)} is reduced to an equivalent problem of a determination of solutions $V_n(x,y),\ n=\overline{0,m-1}$ of a set of equations

$$\begin{cases}\dfrac{\partial^2 V_n}{\partial y^2}-\dfrac{\partial^2 V_n}{\partial x^2}=V_{n+1},\ \ n=\overline{0,m-2},\\ \dfrac{\partial^2 V_{m-1}}{\partial y^2}-\dfrac{\partial^2 V_{m-1}}{\partial x^2}=0\end{cases} \tag{33}$$

satisfying to initial conditions

$$V_n(x,0)=p_n(x),\ \ \frac{\partial V_n(x,0)}{\partial y}=0,\ \ x\in R,\ n=\overline{0,m-1}, \tag{34}$$

where $p_n(x)=\sum\limits_{k=0}^{n}(-1)^k C_n^k \Phi_{n-k}^{(2k)}(x)$, $\quad \Phi_{n-k}^{(2k)}(x)=\dfrac{d^{2k}}{dx^{2k}}\Phi_{n-k}(x)$.

At a solution of a problem (33), (34) we take advantage of the following lemma.

Lemma 3.1. *If $g(x)\in L_1(\Omega)$, $\Omega=(a,b)$, $-\infty\leqslant a<b\leqslant+\infty$, equalities take place*

$$\int_0^y d\eta\int_{x-y+\eta}^{x+y-\eta}[g(\xi+\eta)+g(\xi-\eta)]d\xi=y\int_{x-y}^{x+y}g(\xi)d\xi, \tag{35}$$

$$\int_0^y \eta d\eta\int_{x-y+\eta}^{x+y-\eta}d\xi\int_{\xi-\eta}^{\xi+\eta}[\eta^2-(\xi-s)^2]^n g(s)ds=\frac{y}{2(n+1)(n+2)}\int_{x-y}^{x+y}[y^2-(x-s)^2]^{n+1}g(s)ds,\quad n=0,1,2,\ldots. \tag{36}$$

Proof. In a left member of equality (35) in interior integrals accordingly having made a change of variables $s=\xi+\eta$ and $s=\xi-\eta$, we have

$$\int_0^y\left[\int_{x-y+2\eta}^{x+y}g(s)ds+\int_{x-y}^{x+y-2\eta}g(s)ds\right]d\eta=\int_0^y\left[\int_{x-y}^{x+y}g(s)ds+\int_{x-y+2\eta}^{x+y-2\eta}g(s)ds\right]d\eta$$

$$=y\int_{x-y}^{x+y}g(s)ds+\int_0^y d\eta\int_{x-y+2\eta}^{x+y-2\eta}g(s)ds. \tag{37}$$

We calculate the second integral. Let $G(z)=\int\limits_0^z g(s)ds$, then $G'(z)=g(z)$ and

$$\int_{x-y+2\eta}^{x+y-2\eta}g(s)ds=G(x+y-2\eta)-G(x-y+2\eta).$$

Considering the last, we have

$$\int_0^y d\eta\int_{x-y+2\eta}^{x+y-2\eta}g(s)ds=\int_0^y[G(x+y-2\eta)-G(x-y+2\eta)]d\eta.$$

In last integral having made a change of variables of integration, we get

$$\int_0^y d\eta \int_{x-y+2\eta}^{x+y-2\eta} g(s)ds = \frac{1}{2}\int_{x+y}^{x-y} G(s)ds - \frac{1}{2}\int_{x+y}^{x-y} G(s)ds = 0.$$

Then from (37), validity of equality (36) follows.

Now we prove equality (36). In a left member of equality (36) sequentially having made permutation of the order of an integration all over again on η and on ξ, and then on η and on s and having calculated interior integrals on η, we have

$$J = \int_0^y \eta d\eta \int_{x-y+\eta}^{x+y-\eta} d\xi \int_{\xi-\eta}^{\xi+\eta} [\eta^2 - (\xi - s)^2]^n g(s)ds$$

$$= \frac{1}{2(n+1)} \int_{x-y}^{x} d\xi \int_{x-y}^{2\xi-x+y} [(s-x+y)(2\xi - s - x + y)]^{n+1} g(s)ds$$

$$+\frac{1}{2(n+1)} \int_{x}^{x+y} d\xi \int_{2\xi-x-y}^{x+y} [(x+y-s)(s+x+y-2\xi)]^{n+1} g(s)ds.$$

In both last integrals we make permutation of the order of an integration on ξ and on s, and then, having calculated interior integrals on ξ, we have

$$J = \frac{1}{4(n+1)(n+2)} \int_{x-y}^{x+y} (s-x+y)^{n+1}(x+y-s)^{n+2} g(s)ds$$

$$+\frac{1}{4(n+1)(n+2)} \int_{x-y}^{x+y} (s-x+y)^{n+2}(x+y-s)^{n+1} g(s)ds$$

$$= \frac{y}{2(n+1)(n+2)} \int_{x-y}^{x+y} \left[y^2 - (x-s)^2\right]^{n+1} g(s)ds.$$

Last proves validity of equality (36). The proof of the Lemma 3.1 is finished. □

Sequentially solving each equation of system (33) in view of initial conditions (34) and Lemma 3.1, we find a solution of the given system. Then, considering $V_0(x, y) = U(x, y)$, we get a solution of a problem (31), (34) in the form of

$$U(x,y) = \frac{1}{2}\left[p_0(x+y) + p_0(x-y)\right] + \sum_{n=1}^{m-1} \frac{y}{2^{2n}(n-1)!n!} \int_{x-y}^{x+y} \left[y^2 - (x-s)^2\right]^{n-1} p_n(s)ds, \tag{38}$$

where $p_n(s) = \sum\limits_{k=0}^{n} (-1)^k C_n^k \Phi_{n-k}^{(2k)}(s),\ n = \overline{0, m-1}.$

Substituting (38) in (30) after simple transformations and change of the order of an integration, and also having calculated an interior integral, we obtain the explicit formula a solution of a problem {(28) - (29)} in the form of

$$u(x,y)=\sum_{n=0}^{m-1}\frac{y^{1-2\beta}}{2^{2n}n!\Gamma(\beta+n)}\int_{x-y}^{x+y}\frac{\bar{J}_{\beta+n-1}\left(\lambda\sqrt{y^2-(x-s)^2}\right)}{\left[y^2-(x-s)^2\right]^{1-\beta-n}}p_n(s)ds.$$

Notice that except for the subsection 3.3 application of the Theorem 2.2 allows to reduce the equations of the high order with singular coefficients to polyharmonic, polycaloric and to polywave equations and by that to put and investigate correct initial and boundary problems for such equations.

4. Conclusion

In work generalized properties of Lowndes operator. It is proved a composition of this operator with differential operators of the high order, in particular with degrees of Bessel operator. The received outcomes are applied to a solution of boundary value problems to partial differential equations of the fourth and high order. The offered approach is very effective and allows constructing an exact solution of the formulated problems. These exact solutions allow understanding more deeply qualitative singularities of described processes and appearances, properties of mathematical models, and also can be used as test examples for the asymptotic, approximated and numerical methods.

References

[1] A. Erdélyi, On fractional integration and its application to the theory of Hankel transforms, Quart. J. Math. Oxford ser. 11:44 (1940) 293–303.
[2] A. Erdélyi, H. Kober, Some remarks on Hankel transforms, Quart. J. Math. Oxford ser. II:43 (1940) 212–221.
[3] A. Erdélyi, Some applications of fractional integration, Boeing Sci. Res. Labor. Docum. Math. Note No 316, Dl-82-0286, (1963) 23 p.
[4] A. Erdélyi, An application of fractional integrals, J. Analyse Math. 14 (1965) 113–126.
[5] A. Erdélyi, Axially symmetric potentials and fractional integration, J. Soc. Industr. Appl. Math. 13 (1965) 216–228.
[6] A. Erdélyi, et al. (Eds.), Higher Transcendental Functions, Vols. 1-3. McGraw-Hill, New York, 1953.
[7] P.Heywood, Improved boundedness conditions for Lowndes' operators, Proc. Roy. Soc. Edinburgh, A 73:19 (1975) 291–299.
[8] P.Heywood, P.G. Rooney, On the boundedness of Lowndes' operators, J. London Math. Soc. 10 (1975) 241–248.
[9] Sh.T. Karimov, On a method of solution of Cauchy problem for the generalized Euler-Poisson-Darboux equation, Uzbek Math. J. 3 (2013) 57–69 (in Russian).
[10] Sh.T. Karimov, Solution of a Cauchy problem for multidimensional hyperbolic equations with singular coefficients by the method of fractional integrals, Dokl. Akad. Nauk R. Uz. 1 (2013) 11–13 (in Russian).
[11] Sh.T. Karimov, Multidimensional generalized Erdélyi-Kober operator and its application to solving Cauchy problems for differential equations with singular coefficients, Fract. Calc. Appl. Anal. 18 (2015) 845–861.
[12] V. Kiryakova, Generalized Fractional Calculus and Applications, Longman Sci. & Technical and J. Wiley & Sons, Harlow and N. York, 1994.
[13] J.S. Lowndes, A generalization of the Erdélyi-Kober operators, Proc. Edinb. Math. Soc. 17 (1970) 139–148.
[14] J.S. Lowndes, An application of some fractional integrals. Proc. Edinb. Math. Soc. 20 (1979) 35–41.
[15] J.S. Lowndes, Cauchy problems for second order hyperbolic differential equations with constant coefficients, Proc. Edinb. Math. Soc. 26 (1983) 307–311.
[16] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives (Theory and Applications), Gordon & Breach Sci. Publ., N. York etc. 1993.
[17] I. Sneddon, Fourier Transformations, McGrawHill, New York, 1951.
[18] I.N. Sneddon, Mixed Boundary Value Problems in Potential Theory. North-Holland Publ., Amsterdam 1966.
[19] I.N. Sneddon, The use in mathematical analysis of Erdélyi-Kober′ operators and of some of their applications, In: Fractional Calculus and Its Applications, Proc. Publ. as: Lecture Notes in Mathematics 457, Springer-Verlag, New York (1975) 37–79
[20] A.K. Urinov, S.T. Karimov, Solution of the Cauchy problem for generalized Euler-Poisson-Darboux equation by the method of fractional integrals, Progress in Partial Differential Equations, Springer International Publishing (2013) 321–337.
[21] A. Weinstein, Generalized axially symmetric potential theory, Bull. Amer. Math. Soc. 59 (1953) 20–38.
[22] A. Weinstein, On the wave equation and the equation of Euler-Poisson//Wave motion and vibration theory, Proc. Sympos. Appl. Math. N. Y.: McGraw-Hill, Vol. 5 (1954) 137–147.
[23] A. Weinstein, The generalized radiatiion problem and the Euler-Poisson-Darboux equation, Summa Brasil Math. 3 (1955) 125–147.
[24] A. Weinstein, On a singular differential operator, Ann. Mat. Pura Appl. Ser. 4 49 (1960) 359–365.