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Isoperimetric Inequalities for the Heat Potential Operator

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Abstract. In this paper we prove that the circular cylinder is a maximizer of the Schatten p-norm of heat potential operator among all Euclidean cylindric domains of a given measure. We also give analogues of a Rayleigh-Faber-Krahn and a Hong-Krahn-Szegö type inequalities.

1. Introduction

Specral geometric extremum problems of compact operators is one of most popular research areas of modern mathematics and history of its scientific literature goes back to Rayleighs famous book The Theory of Sound (see e.g. [3]), in which it was stated that a disk minimizes (among all domains of the same area) the first eigenvalue of the Dirichlet Laplacian. This conjecture was proved after about a half century later, simultaneously (and independently) by G. Faber and E. Krahn. Nowadays, the Rayleigh-Faber-Krahn inequality has been generalised to many different operators; see e.g. [5–8] for further references. In the present paper we present an analogue of the Rayleigh-Faber-Krahn theorem for the heat potential operator H, i.e. it is showed that the first s-number of the integral operator H is maximized in a ball among all cylindric domains of a given measure in Rd and corresponding Hong-Krahn-Szeg?o type inequality, that is, the maximum of the second s-number of the heat operator among all cylindric domains with a given measure is approached by the union of two identical circular cylinders with mutual distance going to infinity. Moreover, it is proved that the p-Schatten norm of the heat operator is maximized on the circular cylinder among all domains of a fixed measure. This paper is mainly inspired by recent works of Rozenblum, Ruzhansky and Suragan (see e.g. [5–8]) in which analogues of Rayleigh-Faber-Krahn type inequalities were studied for self-adjoint convolution type integral operators. Thus, in this paper we proved:

- Rayleigh-Faber-Krahn type inequality: the first s-number of $\mathcal{H}_\Omega$ is maximized on the circular cylinder among all Euclidean cylindric domains of a given measure;

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- Hong-Krahn-Szegö type inequality: the maximum of the second s-number of $\mathcal{H}_\Omega$ among cylindric bounded open sets with a given measure is achieved by the union of two identical circular cylinders with mutual distance going to infinity;
- the p-Schatten norm of $\mathcal{H}_\Omega$ is maximized on the circular cylinder among all domains of a given measure;

In Section 2 we discuss some necessary tools. In Section 3 we present main results of this paper and their proofs.

2. Preliminaries

Let $D = \Omega \times (0,T)$ be a cylindrical domain, where $\Omega \subset \mathbb{R}^d$ is a simply-connected set with smooth boundary $\partial\Omega$. We consider the heat potential operator (see, for example, [9]) $\mathcal{H} : L^2(D) \to L^2(D)$ in the form

$$\mathcal{H}f(x,t) := \int_0^t \int_\Omega K(|x-\xi|, t-\tau) f(\xi,\tau) d\xi d\tau, \ \forall f \in L^2(D), \ \ t \in (0,T), \tag{1}$$

where $K(|x|,t) = \frac{\theta(t)}{(2\sqrt{\pi t})^d} e^{\frac{-|x|^2}{4t}}$, here $\theta(t)$ is the Heaviside's function and $K(|x|,t)$ is the fundamental solution of the Cauchy problem for the heat equation, that is

$$\left(\frac{\partial}{\partial t} - \Delta_x\right) K(|x-\xi|, t-\tau) = 0,$$

its adjoint

$$\left(-\frac{\partial}{\partial \tau} - \Delta_\xi\right) K(|x-\xi|, t-\tau) = 0$$

and

$$\lim_{t\to\tau} K(|x-\xi|, t-\tau) = \delta(|x-\xi|),$$

for all $x, \xi \in \mathbb{R}^d$, where δ is the Dirac delta 'function'.

The operator $\mathcal{H}$ is a non-selfadjoint operator in $L^2(D)$. We introduce $P : L^2(D) \to L^2(D)$ by the following formula

$$Pu = u(x, T-t), \ \ t \in (0,T).$$

This is also called "involution" operator and it has the properties

$$P^2 = E, \ P = P^*, \ \ P = P^{-1},$$

where E is the unit operator, P^* is the adjoint operator to the operator P and P^{-1} is the inverse operator to the operator P.

Definition 2.1. Let A be a compact operator. s-numbers are the eigenvalues of the operator $(A^*A)^{1/2}$, where A^* is the adjoint operator to A.

Definition 2.2. A Schatten p-norm of a compact operator A in Schatten class $\mathcal{S}_p$ is defined as

$$\|A\|_p = \left(\sum_{i=1}^{\infty} s_i^p(A)\right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty, \tag{2}$$

for $s_1 \geq s_2 \geq ... > 0$ being the s-numbers of A. For $p = \infty$, we understand

$$\|A\|_\infty := \|A\|,$$

i.e. the operator norm of A in $L^2(D)$.

The operator P acts to the operator $\mathcal{H}$ in $L^2(D)$ by the formula

$$P\mathcal{H}u = \int_0^{T-t}\int_\Omega K(|x-\xi|, T-t-\tau)u(\xi,\tau)d\xi d\tau. \tag{3}$$

Lemma 2.3. *The operator $P\mathcal{H}$ is a self-adjoint operator in $L^2(D)$.*

Proof. A direct computation shows

$$\begin{aligned}
&\langle P\mathcal{H}u, v\rangle_{L^2(D)} = \\
&\int_0^T\int_\Omega\left(\int_0^{T-t}\int_\Omega K(|x-\xi|, T-t-\tau)u(\xi,\tau)d\xi d\tau\right)v(x,t)dxdt \\
&= \int_0^T\int_0^{T-t}\int_\Omega\int_\Omega K(|x-\xi|, T-t-\tau)u(\xi,\tau)v(x,t)d\xi dxd\tau dt \\
&= \int_0^T\int_0^{T-\tau}\int_\Omega\int_\Omega K(|x-\xi|, T-t-\tau)u(\xi,\tau)v(x,t)d\xi dxdtd\tau \\
&= \int_0^T\int_\Omega\left(\int_0^{T-\tau}\int_\Omega K(|x-\xi|, T-t-\tau)v(x,t)dxdt\right)u(\xi,\tau)d\xi d\tau \\
&= \int_0^T\int_\Omega\left(\int_0^{T-\tau}\int_\Omega K(|\xi-x|, T-\tau-t)v(x,t)dxdt\right)u(\xi,\tau)d\xi d\tau \\
&= \langle u, P\mathcal{H}v\rangle_{L^2(D)}.
\end{aligned}$$

Thus, we arrive at $P\mathcal{H} = (P\mathcal{H})^*$ in $L^2(D)$, i.e., $P\mathcal{H}$ is a self-adjoint operator. □

Lemma 2.4. *s-numbers of the operator $\mathcal{H}$ coincide with eigenvalues of the operator $P\mathcal{H}$.*

Proof. We have $(P\mathcal{H})^* = \mathcal{H}^*P^*$ and

$$(P\mathcal{H})^*(P\mathcal{H}) = \mathcal{H}^*P^*P\mathcal{H}.$$

By the properties of the operator P, we obtain

$$(P\mathcal{H})^*(P\mathcal{H}) = \mathcal{H}^*P^*P\mathcal{H} = \mathcal{H}^*P^2\mathcal{H} = \mathcal{H}^*\mathcal{H}.$$

□

Remark 2.5. As a consequence of Lemma 2.4 we obtain $\|\mathcal{H}\|_p = \|P\mathcal{H}\|_p$, for $\mathcal{H} \in \mathcal{S}_p$.

3. Main Results and Their Proofs

We consider a (circular) cylinder $C = B\times(0,T)$ where $B \subset \mathbb{R}^d$ is an open ball. Let Ω be a simply-connected set with smooth boundary $\partial\Omega$ with $|B| = |\Omega|$, where $|\Omega|$ is the Lebesgue measure of the domain Ω.

Theorem 3.1. *The first eigenvalue of the operator $P\mathcal{H}$ is maximized in the circular cylinder C, that is,*

$$0 < \lambda_1(D) \le \lambda_1(C),$$

with $|\Omega| = |B|$.

Proof. Recall that $D = \Omega \times (0, T)$ is a bounded measurable set in $\mathbb{R}^{d+1}$. Its symmetric rearrangement $C = B \times (0, T)$ is the circular cylinder with the measure equals to the measure of D, i.e. $|D| = |C|$. Let u be a nonnegative measurable function in D, such that all its positive level sets have finite measure. With the definition of the symmetric-decreasing rearrangement of u we can use the layer-cake decomposition [4], which expresses a nonnegative function u in terms of its level sets with respect to the variable x for each $t \in (0, T)$ as

$$u(x,t) = \int_0^\infty \chi_{\{u(x,t)>z\}} dz, \quad \forall t \in (0,T), \tag{4}$$

where χ is the characteristic function of the domain. The function

$$u^*(x,t) = \int_0^\infty \chi_{\{u(x,t)>z\}^*} dz, \quad \forall t \in (0,T), \tag{5}$$

is called the (radially) symmetric-decreasing rearrangement of a nonnegative measurable function u.

Consider the following spectral problem

$$P\mathcal{H}u = \lambda u.$$

By the variational principle for the operator $P\mathcal{H}$, we have

$$\lambda_1(D) = \frac{\int_0^T \int_0^{T-t} \int_\Omega \int_\Omega K(|x-\xi|, T-t-\tau) u_1(\xi,\tau) u_1(x,t) d\xi dx d\tau dt}{\|u_1\|^2_{L^2(D)}},$$

where $u_1(x,t)$ is the first eigenfunction of the operator $P\mathcal{H}$. For each non-negative function $u \in L^2(D)$, we obtain

$$\|u\|_{L^2(D)} = \|u^*\|_{L^2(C)}. \tag{6}$$

By the Riesz inequality [4], we establish

$$\begin{aligned}\int_0^T \int_0^{T-t} \int_\Omega \int_\Omega & K(|x-\xi|, T-t-\tau) u_1(\xi,\tau) u_1(x,t) d\xi dx d\tau dt \\ &\leq \int_0^T \int_0^{T-t} \int_B \int_B K(|x-\xi|, T-t-\tau) u_1^*(\xi,\tau) u_1^*(x,t) d\xi dx d\tau dt.\end{aligned} \tag{7}$$

Applying (6) and (7), we get

$$\begin{aligned}\lambda_1(D) &= \frac{\int_0^T \int_0^{T-t} \int_\Omega \int_\Omega K(|x-\xi|, T-t-\tau) u_1(\xi,\tau) u_1(x,t) d\xi dx d\tau dt}{\|u_1\|^2_{L^2(D)}} \\ &\leq \frac{\int_0^T \int_0^{T-t} \int_B \int_B K(|x-\xi|, T-t-\tau) u_1^*(\xi,\tau) u_1^*(x,t) d\xi dx d\tau dt}{\|u_1\|^2_{L^2(C)}} \\ &\leq \sup_{v \in L^2(C), v \neq 0} \frac{\int_0^T \int_0^{T-t} \int_B \int_B K(|x-\xi|, T-t-\tau) v(\xi,\tau) v(x,t) d\xi dx d\tau dt}{\|v\|^2_{L^2(C)}} = \lambda_1(C).\end{aligned}$$

The proof is complete. □

Corollary 3.2. *According to Lemma 2.4 we have the eigenvalues of the operator $P\mathcal{H}$ coincide with the s-numbers of the operator $\mathcal{H}$, in particular, $\lambda_1(P\mathcal{H}) = s_1(\mathcal{H})$. This means $\|\mathcal{H}\| = \|P\mathcal{H}\|$, that is, the norm of the operator $P\mathcal{H}$ is maximized in the cylinder C, i.e. $\|\mathcal{H}_\Omega\| \leq \|\mathcal{H}_B\|$.*

Theorem 3.3. *Let $P\mathcal{H} \in \mathcal{S}_{p_0}$. For each integer p, $p_0 \leq p < \infty$, we have*

$$\|P\mathcal{H}_D\|_p \leq \|P\mathcal{H}_C\|_p,$$

for all Ω such that $|\Omega| = |B|$.

Proof. For all $p_0 \leq p < \infty$, we have

$$\sum_{j=1}^{\infty} \lambda_j^p(P\mathcal{H}) =$$

$$\int_0^T \int_0^{T-\tau_1} \dots \int^{T-\tau_{p-1}} \int_\Omega \dots \int_\Omega \prod_{k=1}^{p} K(|\xi_k - \xi_{k+1}|, T - \tau_k - \tau_{k+1}) d\tau_1 \dots d\tau_p d\xi_1 \dots d\xi_p. \tag{8}$$

where $\xi_1 = \xi_{p+1}$ and $\tau_1 = \tau_{p+1}$. Using the Brascamp-Lieb-Luttinger inequality [1], we obtain that

$$\sum_{i=1}^{\infty} \lambda_i^p(D) = \int_0^T \int_0^{T-\tau_1} \dots \int^{T-\tau_{p-1}} \int_\Omega \dots \int_\Omega \prod_{k=1}^{p} K(|\xi_k - \xi_{k+1}|, T - \tau_k - \tau_{k+1}) dz$$

$$\leq \int_0^T \int_0^{T-\tau_1} \dots \int^{T-\tau_{p-1}} \int_B \dots \int_B \prod_{k=1}^{p} K^*(|\xi_k - \xi_{k+1}|, T - \tau_k - \tau_{k+1}) dz$$

$$= \sum_{i=1}^{\infty} \lambda_i^p(C),$$

where $\xi_1 = \xi_{p+1}$, $\tau_1 = \tau_{p+1}$ and $dz = d\tau_1 \dots d\tau_p d\xi_1 \dots d\xi_p$. Here we have used that

$$K(|x-y|, T-t-\tau) = K^*(|x-y|, T-t-\tau).$$

Thus

$$\sum_{i=1}^{\infty} \lambda_i^p(D) \leq \sum_{i=1}^{\infty} \lambda_i^p(C), \quad p_0 \leq p < \infty.$$

Therefore,

$$\|P\mathcal{H}_D\|_p \leq \|P\mathcal{H}_C\|_p, \quad p_0 \leq p < \infty,$$

completes the proof. □

Theorem 3.4. *The maximum of the second eigenvalue λ_2 of $P\mathcal{H}$ among all cylindric domains D with a given measure is approached by the union of two identical circular cylinders with mutual distance going to infinity.*

Proof. Let $D^+ = \{(x,t) : u(x,t) > 0\}$, $D^- = \{(x,t) : u(x,t) < 0\}$, Thus

$$u_2(x,t) > 0, \quad t \in (0,T), \forall x \in \Omega^+ \subset \Omega, \ \Omega^+ \neq \{\emptyset\},$$

$$u_2(x,t) < 0, \quad t \in (0,T), \forall x \in \Omega^- \subset \Omega, \ \Omega^- \neq \{\emptyset\}.$$

Using the notations

$$u_2^+(x,t) := \begin{cases} u_2(x,t), & (x,t) \in D^+, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$u_2^-(x,t) := \begin{cases} u_2(x,t), & (x,t) \in D^-, \\ 0, & \text{otherwise}. \end{cases}$$

We have

$$\lambda_2(D)u_2(x,t) =$$
$$\int_0^{T-t}\int_{\Omega^+} K(|x-\xi|, T-t-\tau)u_2(\xi,\tau)d\xi d\tau$$
$$+\int_0^{T-t}\int_{\Omega^-} K(|x-\xi|, T-t-\tau)u_2(\xi,\tau)d\xi d\tau.$$

Multiplying by $u_2^+(x,t)$ and integrating over $\Omega^+ \times (0,T)$, we get

$$\lambda_2(D)\|u_2^+\|^2_{L^2(D^+)} = \int_0^T\int_0^{T-t}\int_{\Omega^+}\int_{\Omega^+} K(|x-\xi|, T-t-\tau)u_2^+(x,t)u_2^+(\xi,\tau)dz$$
$$+\int_0^T\int_0^{T-t}\int_{\Omega^+}\int_{\Omega^-} K(|x-\xi|, T-t-\tau)u_2^-(x,t)u_2^+(\xi,\tau)dz, \tag{9}$$

where $dz = d\xi dx d\tau dt$.

We have $\int_0^{T-t}\int_{\Omega^-} K(|x-\xi|, T-t-\tau)u_2^-(\xi,\tau)d\xi d\tau < 0$. Thus,

$$\lambda_2(D)\|u_2^+\|^2_{L^2(D^+)} =$$
$$\int_0^T\int_0^{T-t}\int_{\Omega^+}\int_{\Omega^+} K(|x-\xi|, T-t-\tau)u_2^+(x,t)u_2^+(\xi,\tau)dz$$
$$+\int_0^T\int_0^{T-t}\int_{\Omega^+}\int_{\Omega^-} K(|x-\xi|, T-t-\tau)u_2^-(x,t)u_2^+(\xi,\tau)dz$$
$$\leqslant \int_0^T\int_0^{T-t}\int_{\Omega^+}\int_{\Omega^+} K(|x-\xi|, T-t-\tau)u_2^+(x,t)u_2^+(\xi,\tau)dz.$$

From the latter, we get

$$\lambda_2(D) \leqslant \frac{\int_0^T\int_0^{T-t}\int_{\Omega^+}\int_{\Omega^+} K(|x-\xi|, T-t-\tau)u_2^+(x,t)u_2^+(\xi,\tau)dz}{\|u_2^+\|^2_{L^2(D^+)}}.$$

Applying Theorem 3.1, we get

$$\lambda_2(D) \leqslant \frac{\int_0^T\int_0^{T-t}\int_{\Omega^+}\int_{\Omega^+} K(|x-\xi|, T-t-\tau)u_2^+(x,t)u_2^+(\xi,\tau)dz}{\|u_2^+\|^2_{L^2(D^+)}}$$
$$\leqslant \sup_{v\in L^2(D^+)} \frac{\int_0^T\int_0^{T-t}\int_{\Omega^+}\int_{\Omega^+} K(|x-\xi|, T-t-\tau)v^+(x,t)v^+(\xi,\tau)dz}{\|v\|^2_{L^2(D^+)}} = \lambda_1(D^+).$$

Similarly, we get

$$\lambda_2(D) \leqslant \lambda_1(D^-).$$

Finally, we have

$$\lambda_2(D) \leqslant \lambda_1(D^-), \quad \lambda_2(D) \leqslant \lambda_1(D^+). \tag{10}$$

Due to Theorem 3.1, we establish that

$$\lambda_1(D^+) < \lambda_1(C^+), \quad \lambda_1(D^-) < \lambda_1(C^-). \tag{11}$$

Applying (10) and (11), we obtain

$$\lambda_2(D) \leqslant \min\{\lambda_1(C^+), \lambda_1(C^-)\}.$$

We now introduce C^+ and C^-, the circular cylinders of the same measure as D^+ and D^-, respectively. Let l be distance between C^+ and C^-, i.e. $l = dist(C^+, C^-)$.

Define the first (normalized) eigenfunction $u_1^{\circledast}(x,t)$ of the operator $P\mathcal{H}_{C^+\cup C^-}$ and take u_+ u_- being the first (normalized) eigenfunctions of each circular cylinder, i.e., of operators $P\mathcal{H}_{C^+\cup C^-}$. The first eigenfunction of the corresponding operators $P\mathcal{H}_{\pm}$. Let $f^{\circledast} \in L^2(C^+ \cup C^-)$ be a function such that

$$f^{\circledast} = \begin{cases} u_+(x,t), & (x,t) \in C^+, \\ \gamma u_-(x,t), & (x,t) \in C^-, \end{cases}$$

where γ is real number, so that $f^{\circledast}$ orthogonal to $u_1^{\circledast}$. $u_+, u_-, u^{\circledast}$ are positive functions.

Noticing that

$$\int_0^T \int_0^{T-t} \int_{B^+\cup B^-} \int_{B^+\cup B^-} K(|x-\xi|, T-t-\tau) f^{\circledast}(\xi,\tau) f^{\circledast}(x,t) d\xi dx d\tau dt = \sum_{i=1}^4 U_i,$$

where

$$U_1 = \int_0^T \int_0^{T-t} \int_{B^+} \int_{B^+} K(|x-\xi|, T-t-\tau) u_+(\xi,\tau) u_+(x,t) d\xi dx d\tau dt,$$

$$U_2 = \int_0^T \int_0^{T-t} \int_{B^+} \int_{B^-} K(|x-\xi|, T-t-\tau) u_-(\xi,\tau) u_+(x,t) d\xi dx d\tau dt,$$

$$U_3 = \gamma \int_0^T \int_0^{T-t} \int_{B^-} \int_{B^+} K(|x-\xi|, T-t-\tau) u_+(\xi,\tau) u_-(x,t) d\xi dx d\tau dt,$$

$$U_4 = \gamma^2 \int_0^T \int_0^{T-t} \int_{B^-} \int_{B^-} K(|x-\xi|, T-t-\tau) u_-(\xi,\tau) u_-(x,t) d\xi dx d\tau dt.$$

By the variational principle, we take

$$\lambda_2(C^+ \cup C^-) =$$

$$\sup_{v \in L^2(C^+\cup C^-), v \perp u_1} \int_0^T \int_0^{T-t} \int_{B^+\cup B^-} \int_{B^+\cup B^-} K(|x-\xi|, T-t-\tau) v(\xi,\tau) v(x,t) d\xi dx d\tau dt$$

$$\geq \int_0^T \int_0^{T-t} \int_{B^+\cup B^-} \int_{B^+\cup B^-} K(|x-\xi|, T-t-\tau) f^{\circledast}(\xi,\tau) f^{\circledast}(x,t) d\xi dx d\tau dt = \sum_{i=1}^4 U_i.$$

On other hand, u_+ and u_- are the first (normalized) eigenfunctions of each circular cylinder C^+ and C^-, we have

$$\lambda_1(C^{\pm}) = \int_0^T \int_0^{T-t} \int_{B^{\pm}} \int_{B^{\pm}} u_{\pm}(x,t) u_{\pm}(\xi,\tau) K(|x-\xi|, T-t-\tau) d\xi dx d\tau dt.$$

Summarizing the above facts, we obtain

$$\lambda_2(C^+ \cup C^-) \geq \frac{U_1 + U_2 + U_3 + U_4}{(\lambda_1(D^+))^{-1} U_1 + (\lambda_1(D^-))^{-1} U_2}.$$

Since the kernel $K(|x-\xi|, T-t-\tau)$ tends to zero as $x \in B^{\pm}$, $\xi \in B^{\mp}$ and $l \to \infty$, we observe that

$$\lim_{l\to 0} U_2 = \lim_{l\to 0} U_3 = 0.$$

Therefore,

$$\lambda_2(C^+ \bigcup C^-) \geqslant \max\{\lambda_1(C^+), \lambda_1(C^-)\}, \tag{12}$$

where $l = dist(C^+, C^-)$. From inequalities (10) and (12), we get

$$\lim_{l\to\infty} \lambda_2 \left(C^+ \bigcup C^-\right) \geq \min\{\lambda_1(C^+), \lambda_1(C^-)\} = \lambda_1(C^+) = \lambda_1(C^-)$$
$$\geq \lim_{l\to\infty} \lambda_2 \left(C^+ \cup C^-\right),$$

and this implies that the maximising sequence for λ_2 is given by a disjoint union of two identical circular cylinders with mutual distance going to ∞. $\square$

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References

[1] H.J. Brascamp, E.H. Lieb, J.M. Luttinger, A general rearrangement inequality for multiple integrals, J. Funct. Anal. 17 (1974) 227–237.
[2] I. Gohberg, M. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, AMS, Providence, RI, 1988.
[3] A. Henrot, Extremum problems for eigenvalues of elliptic operators, Birkhauser Verlag, Basel, 2006.
[4] E.H. Lieb, M. Loss, Analysis, Graduate Studies in Mathematics, vol. 14, AMS, Providence, RI, Second edition, 2001.
[5] G. Rozenblum, M. Ruzhansky, D. Suragan, Isoperimetric inequalities for Schatten norms of Riesz potentials, J. Funct. Anal. 271 (2016) 224–239.
[6] M. Ruzhansky, D. Suragan, Isoperimetric inequalities for the logarithmic potential operator, J. Math. Anal. Appl. 434 (2016) 1676–1689.
[7] M. Ruzhansky, D. Suragan, Schatten's norm for convolution type integral operator, Russ. Math. Surv. 71 (2016) 157–158.
[8] M. Ruzhansky, D. Suragan, On first and second eigenvalues of Riesz transforms in spherical and hyperbolic geometries, Bull. Math. Sci. 6 (2016) 325–334.
[9] D. Suragan, N. Tokmagambetov, On transparent boundary conditions for the high–order heat equation, Siberian Electronic Mathematical Reports 10 (2013) 141-149.