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About Solvability of Some Boundary Value Problems for Poisson Equation in the Ball

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Abstract. In the paper we study properties of some integro - differential operators of fractional order. As an application of the properties of these operators for Poisson equation we examine questions on solvability of a fractional analogue of the Neumann problem and analogues of periodic boundary value problems for circular domains. The exact conditions for solvability of these problems are found.

1. Introduction

Let Q be a bounded domain from R^n with a smooth boundary S. It is known that classical problems for the Poisson equation

$$\Delta u(x) = f(x), x \in Q, \tag{1}$$

are Dirichlet and Neumann problems. Let ν be a normal vector to S, and $D_\nu = \frac{d}{d\nu}$ be an operator of differentiation along the normal, $D_\nu^0 = I$ be a unit operators. Then Dirichlet and Neumann boundary conditions can be given in the following form:

$$D_\nu^\alpha u(x) = g_\alpha(x), x \in S, \tag{2}$$

where $\alpha = 0$ or $\alpha = 1$. It is known that the Dirichlet problem is unconditionally solvable, and for solvability of the Neumann condition the following condition is necessary:

$$\int\limits_Q f(x)dx = \int\limits_S g_1(x)dx. \tag{3}$$

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In this paper, we introduce fractional analogues of the boundary operators D^{α}_{ν}, and for the equation (1) we study the boundary value problem with the boundary condition (2) for all values of the parameter $\alpha \in (0, \infty)$. Moreover, we investigate solvability of some analogues of periodic boundary value problems and an analog of Samarskii-Ionkin type boundary value problems for circular domains.

Let $\Omega = \{x \in R^n : |x| < 1\}$ be a unit ball, $n \geq 2$, $\partial\Omega = \{x \in R^n : |x| = 1\}$ - unit sphere. Suppose further that, $u(x)$ is a smooth function in the domain Ω, $r = |x|, \theta = x/r$, $\delta = r\frac{d}{dr}$- Dirac operator, where $r\frac{d}{dr} = \sum\limits_{j=1}^{n} x_j \frac{\partial}{\partial x_j}$, $\alpha > 0$.

Let $0 < \alpha < \infty$ be a real number. Consider in Ω integration and differentiation operators of the order α in the Hadamard sense [6]:

$$J^{\alpha}[u](x) = \frac{1}{\Gamma(\alpha)} \int\limits_0^r \left(\ln \frac{r}{s}\right)^{\alpha-1} u(s\theta) \frac{ds}{s},\ D^{\alpha}[u](x) = J^{\ell-\alpha}[\delta^{\ell}[u]](x), \ell - 1 < \alpha \leq \ell,\ \ell = 1, 2, \ldots.$$

2. Properties of J^{α} and D^{α} Operators

In this section we study properties of J^{α} and D^{α}operators. The following assertions are proved in [20].

Lemma 2.1. *Let $\alpha > 0$, $0 < \lambda < 1$ and $u(x) \in C^{\lambda+p}\left(\overline{\Omega}\right)$, $p = 0, 1, ...$ If the condition $u(0) = 0$ holds, then $J^{\alpha}[u](x) \in C^{\lambda+p}\left(\overline{\Omega}\right)$ and $J^{\alpha}[u](0) = 0$.*

Lemma 2.2. *Let $\mu \geq 0$, $\ell - 1 < \alpha \leq \ell$, $\ell = 1, 2, \ldots, 0 < \lambda < 1$ and $u(x) \in C^{\lambda+p}(\bar{\Omega}), p \geq \ell$. Then $D^{\alpha}[u](x) \in C^{\lambda+p-\ell}(\bar{\Omega})$ and the equality $D^{\alpha}[u](0) = 0$ holds.*

Lemma 2.3. *Let $\mu \geq 0$, $\ell - 1 < \alpha \leq \ell$, $\ell = 1, 2, \ldots,$ $0 < \lambda < 1$ and $u(x) \in C^{\lambda+p}(\bar{\Omega})$, $p \geq \ell, p = 1, 2, ...$ Then for any $x \in \bar{\Omega}$:*

$$J^{\alpha}[D^{\alpha}[u]](x) = u(x) - u(0), \tag{4}$$

and if $u(0) = 0$, then we get

$$D^{\alpha}[J^{\alpha}[u]](x) = u(x). \tag{5}$$

Lemma 2.4. *Let $\ell - 1 < \alpha \leq \ell$, $\ell = 1, 2, \ldots,$ $0 < \lambda < 1$, $f(x)$ be a smooth function in the domain $\bar{\Omega}$ and $\Delta u(x) = f(x), x \in \Omega$. Then*

$$\Delta D^{\alpha}[u](x) = |x|^{-2} D^{\alpha}[|x|^2 f](x),\ x \in \Omega. \tag{6}$$

3. The Neumann Type Problem

In this section we consider s fractional analogue of the Neumann problem with the boundary condition D^{α}.

Problem 1. (Neumann type problem). Let $0 < \alpha$. Find a function $u(x) \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $D^{\alpha}[u](x) \in C(\overline{\Omega})$, and satisfying the following equation:

$$\Delta u(x) = f(x), x \in \Omega, \tag{7}$$

and the boundary value condition:

$$D^{\alpha}[u](x) = g(x), x \in \partial\Omega. \tag{8}$$

Note that the local and nonlocal boundary value problems with boundary operators of fractional order for the second order elliptic equations were studied in [5, 7, 8, 16, 17, 21] and for higher-order equations in [1–3, 18, 19].

The following proposition is true.

Theorem 3.1. *Let $\ell-1<\alpha\leq\ell,\ell=1,2,...,\ 0<\lambda<1, f(x)\in C^{\lambda+2\ell-1}(\overline{\Omega}), g(x)\in C^{\lambda+\ell+1}(\partial\Omega)$. Then for solvability of the problem 1 it is necessary and sufficient the following condition:*

$$\int\limits_{\Omega} f_{\ell-\alpha}(y)dy=\int\limits_{\partial\Omega} g(y)dy., \tag{9}$$

where the function $f_{\ell-\alpha}(x)=r^{-2}J^{\ell-\alpha}\left[r^2\delta^{\ell-1}[f]\right](x)$.

If a solution of the problem exists, then it is unique up to a constant term, belongs to the class $C^{\lambda+\ell+1}(\overline{\Omega})$.

Proof. Let $u(x)$ be a solution of the problem 1. Apply the operator D^α to the function $u(x)$, and denote $v(x)=D^\alpha[u](x)$. Find conditions, which the function $v(x)$ satisfies. It is obvious that

$$v(x)|_{\partial\Omega}=D^\alpha[u](x)|_{\partial\Omega}=g(x).$$

Applying the operator Δ to the equality $v(x)=D^\alpha[u](x)$, due to (6), we obtain $\Delta v(x)=r^{-2}D^\alpha[r^2f](x)$. Therefore, if $u(x)$ is a solution of the problem 1, then the function $v(x)=D^\alpha[u](x)$ will be a solution of the Dirichlet problem

$$\Delta v(x)=F(x), x\in\Omega, v(x)=g(x), x\in\partial\Omega,$$

with the function $F(x)=r^{-2}D^\alpha[r^2f](x)$. Moreover, the function $v(x)$ satisfies the condition $v(0)=0$. The function $F(x)$ can be represented in the form $F(x)=\left(r\frac{d}{dr}+2\right)f_{\ell-\alpha}(x)$. Then, it is known (see [16]) that for the equality $v(0)=0$, the following condition is necessary:

$$\int\limits_{\Omega} f_{\ell-\alpha}(y)dy=\int\limits_{\partial\Omega} g(y)dS_y.$$

Therefore, necessity of the condition (9) is proved. Applying the operator J^α to the equality $v(x)=D^\alpha[u](x)$, due to (4), we obtain $u(x)-u(0)=J^\alpha[v](x)$. We show that the condition (9) is sufficient for the existence of any solution of the problem 1. Indeed, let $v(x)$ be a solution of the Dirihlet problem with $F(x)=r^{-2}D^\alpha[r^2f](x)$. If $f(x)\in C^{\lambda+2\ell-1}(\overline{\Omega})$, then $F(x)\in C^{\lambda+\ell-1}(\overline{\Omega})$, and since $g(x)\in C^{\lambda+\ell+1}(\partial\Omega)$, then a solution of the Dirichlet problem exists, is unique and belongs to the class $C^{\lambda+p+1}(\bar{\Omega})$ (see e.g. [4]). We represent the function $F(x)=|x|^{-2}D^\alpha[|x|^2f](x)$ as $F(x)=\left(r\frac{d}{dr}+2\right)f_{\ell-\alpha}(x)$. If for the function $f_{\ell-\alpha}(x)$ the condition (9) holds, then corresponding solution of the Dirichlet problem satisfies the condition $v(0)=0$. Then we should to consider the function $u(x)=C+J^\alpha[v](x)$, which satisfies all conditions of the problem 1. By Lemma 2.1 this function belongs to the class $C^{\lambda+p+1}(\bar{\Omega})$. Further, using (5), we obtain $D^\alpha[u](x)|_{\partial\Omega}=D^\alpha[C]+D^\alpha[J^\alpha[v]](x)|_{\partial\Omega}=v(x)|_{\partial\Omega}=g(x)$. Moreover,

$$\Delta u(x)=\Delta\left[\frac{1}{\Gamma(1-\alpha)}\int\limits_0^r\left(\ln\frac{r}{s}\right)^{1-\alpha}v(s\theta)\frac{ds}{s}\right]=\Delta\left[\frac{1}{\Gamma(1-\alpha)}\int\limits_0^1\left(\ln\frac{1}{\xi}\right)^{1-\alpha}v(\xi x)\frac{d\xi}{\xi}\right]=$$

$$=\frac{1}{\Gamma(1-\alpha)}\int\limits_0^1\left(\ln\frac{1}{\xi}\right)^{1-\alpha}\xi^2F(\xi x)\frac{d\xi}{\xi}=\frac{1}{\Gamma(1-\alpha)}\int\limits_0^1\left(\ln\frac{1}{\xi}\right)^{1-\alpha}\xi^2|\xi x|^{-2}D^\alpha[|\xi x|^2f(\xi x)]\frac{d\xi}{\xi}=$$

$$\frac{|x|^{-2}}{\Gamma(1-\alpha)}\int\limits_0^r\left(\ln\frac{r}{s}\right)^{1-\alpha}D^\alpha[s^2f(s\theta)]\frac{ds}{s}=r^{-2}J^\alpha\left[D^\alpha[r^2f]\right](x)=r^{-2}\cdot r^2f(x)=f(x).$$

Thus, the function $u(x)=C+J^\alpha[v](x)$ satisfies all conditions of the problem 1. □

Remark 3.2. If $\alpha=1$, then $f_1(x)=r^{-2}J^0[r^2f](x)=f(x)$ and (9) coincides with the condition of solvability of the Neumann problem (3).

4. Boundary Value Problems with Periodic Conditions

In this section we study some analogues of periodic problems in Ω.

Let $x = (\tilde{x}, x_n) \in \Omega, \tilde{x} = (x_1, ..., x_{n-1})$. For any $x = (\tilde{x}, x_n) \in \Omega$ we put "opposite" point $x^* = (a\tilde{x}, -x_n) \in \Omega$, where $a = (a_1, a_2, ..., a_{n-1})$ and $a_j, j = 1, ..., n-1$ take one of the values ± 1. Denote

$$\partial\Omega_+ = \{x \in \partial\Omega : x_n \geq 0\}, \partial\Omega_- = \{x \in \partial\Omega : x_n \leq 0\}, I = \{x \in \partial\Omega : x_n = 0\}.$$

Let $0 \leq \beta < \alpha \leq 1$ .Consider in Ω the following problem.

Problem 2. Find a function $u(x) \in C^2(\Omega) \cap C(\bar{\Omega})$, such that $D^\beta[u](x) \in C(\bar{\Omega})$, $D^\alpha[u](x) \in C(\bar{\Omega})$ and

$$-\Delta u(x) = f(x), x \in \Omega, \tag{10}$$

$$D^\beta u(x) - (-1)^k D^\beta u(x^*) = g_0(x), x \in \partial\Omega_+, \tag{11}$$

$$D^\alpha[u](x) + (-1)^k D^\alpha[u](x^*) = g_1(x), x \in \partial\Omega_+ \tag{12}$$

where $k = 1, 2$.

These problems are analogous to the classical periodic boundary value problems. The problem (10) - (12) in the case $\beta = 0, \alpha = 1$ have been studied in [12, 13], and in the case $0 < \alpha < 1$ for the Riemann - Liouville and Caputo operators in [15].

If $x = (\tilde{x}, 0) \in I$, then $x^* = (\alpha\tilde{x}, 0) \in I$, therefore, a necessary condition for existence of a solution from the class $u(x) \in C^2(\Omega) \cap C(\bar{\Omega})$, $D^\alpha[u](x) \in C(\bar{\Omega})$ is fulfillment of the following conditions:

$$\partial^p g_0(\tilde{x}, 0) = -(-1)^k \partial^p g_0(a\tilde{x}, 0), \partial^p g_1(\tilde{x}, 0) = (-1)^k \partial^p g_1(a\tilde{x}, 0), (\tilde{x}, 0) \in I, \tag{13}$$

where $p = (p_1, p_2, ..., p_n)$– multiindex with $|p| \leq 2$, $\partial^p = \frac{\partial^{|p|}}{\partial x_1^{p_1} ... \partial x_n^{p_n}}$.

Theorem 4.1. *Let $0 < \lambda < 1$, $f(x) \in C^{\lambda+1}(\bar{\Omega})$, $g_0(x) \in C^{\lambda+2}(\partial\Omega_+)$, $g_1(x) \in C^{\lambda+2}(\partial\Omega_+)$ and the matching conditions (13) hold. Then*

1) if $k = 1$ and $\beta = 0$, then a solution of the problem 2 exists, is unique and belongs to the class $C^{\lambda+2}(\bar{\Omega})$;

2) if $k = 1$ and $\beta > 0$ for solvability of the problem 2, then the following condition is necessary and sufficient:

$$\int_\Omega f_{1-\beta}(y)dy = \int_{\partial\Omega_+} g_0(y)dS_y. \tag{14}$$

3) if $k = 2$, then for solvability of the problem 2 the following condition is necessary and sufficient:

$$\int_\Omega f_{1-\alpha}(y)dy = \int_{\partial\Omega_+} g_1(y)dS_y. \tag{15}$$

If a solution exists, then it is unique up to a constant term, and belongs to the class $C^{\lambda+2}(\bar{\Omega})$;

Proof. First we prove uniqueness. Let $u(x)$ be a solution of the homogenous problem 2 and $D^\beta u(x) = v(x)$. Putting the function $u(x)$ into the boundary value conditions of the problem 2, we have

$$v(x) - (-1)^k v(x^*) = 0, x \in \partial\Omega_+ \tag{16}$$

$$D^{\alpha-\beta}[v](x) + (-1)^k D^{\alpha-\beta}[v](x^*) = 0, x \in \partial\Omega_+ \tag{17}$$

If $x \in \partial\Omega_-$, then $x^* \in \partial\Omega_+$. Then the condition (16) implies $v(x^*) = (-1)^k v(x), x \in \partial\Omega_-$, and (17) yields

$$D^{\alpha-\beta}[v](x^*) = -(-1)^k D^{\alpha-\beta}[v](x),\ x \in \partial\Omega_-.$$

Consequently, the equalities (16) and (17) hold for all points $x \in \partial\Omega$, i.e.

$$v(x) = (-1)^k v(x^*), \quad D^{\alpha-\beta}[v](x) = -(-1)^k D^{\alpha-\beta}[v](x^*), \quad x \in \partial\Omega.$$

Since $D^{\alpha-\beta}[v](x) \in C(\bar{\Omega})$, then from the equality $v(x) = (-1)^k v(x^*),\ x \in \partial\Omega$ it follows:

$$D^{\alpha-\beta}[v](x) = (-1)^k D^{\alpha-\beta}[v](x^*), \quad x \in \partial\Omega.$$

Consequently, $D^{\alpha-\beta}[v](x) = 0, x \in \partial\Omega$, i.e. $v(x)$ is also solution of the homogeneous Problem 1. Then by Theorem 3.1: $v(x) \equiv C, x \in \bar{\Omega}$. Since $v(0) = 0, \beta > 0$, then $v(x) \equiv 0, x \in \bar{\Omega}$, i.e. $D^\beta u(x) \equiv 0, x \in \bar{\Omega}$. Therefore, $u(x) \equiv C, x \in \bar{\Omega}$. Therefore, solution of the problem 2 when $k = 1, \beta = 0$ is unique, and in other cases it is unique up to a constant term. Uniqueness is proved.

Now let us turn to study existence of a solution. Consider the following auxiliary functions:

$$v(x) = \frac{1}{2}(u(x) + u(x^*)), \quad w(x) = \frac{1}{2}(u(x) - u(x^*))$$

Find problems, which these functions satisfy. Let $k = 1$. Applying the operator Δ to the functions $v(x)$ and $w(x)$, we have

$$\Delta v(x) = f^+(x), \quad \Delta w(x) = f^-(x),\ x \in \Omega, \quad f^\pm(x) = \frac{1}{2}[f(x) \pm f(x^*)],$$

Further, from the boundary value conditions (11) and (12) we obtain

$$D^\beta v(x)|_{\partial\Omega_+} = \frac{g_0(x)}{2}, \quad D^\alpha w(x)|_{\partial\Omega_+} = \frac{g_1(x)}{2}.$$

If $x \in \partial\Omega_-$, then $x^* \in \partial\Omega_+$, so the following equalities are true:

$$D^\beta v(x)|_{\partial\Omega_-} = \frac{g_0(x*)}{2}, \quad D^\alpha w(x)|_{\partial\Omega_-} = -\frac{g_1(x*)}{2}.$$

Introduce functions:

$$2\tilde{g}_0(x) = \begin{cases} g_0(x), x \in \partial\Omega_+ \\ g_0(x^*), x \in \partial\Omega_- \end{cases}, \quad 2\tilde{g}_1(x) = \begin{cases} g_1(x), x \in \partial\Omega_+ \\ -g_1(x^*), x \in \partial\Omega_- \end{cases}.$$

Therefore, functions $v(x)$ and $w(x)$ are solutions of the following problems:

$$\Delta v(x) = f^+(x),\ x \in \Omega;\ D^\beta v(x)\Big|_{\partial\Omega} = \tilde{g}_0(x), \tag{18}$$

$$\Delta w(x) = f^-(x),\ x \in \Omega;\ D^\alpha w(x)|_{\partial\Omega} = \tilde{g}_1(x). \tag{19}$$

If for the functions $f(x)$, $g_0(x)$ and $g_1(x)$ conditions of the theorem hold, then $f^\pm(x) \in C^{\lambda+1}(\bar{\Omega})$, $\tilde{g}_0(x) \in C^{\lambda+2}(\partial\Omega)$, $\tilde{g}_1(x) \in C^{\lambda+2}(\partial\Omega)$.Then a solution of the Dirichlet problem ($\beta = 0$) (18) exists, is unique, belongs to the class $C^{\lambda+2}(\bar{\Omega})$. Due to Theorem 3.1 for solvability of the problem (18)($\beta > 0$), (19) it is necessary and sufficient the following condition:

$$\int_\Omega f^+_{1-\beta}(y)dy = \int_{\partial\Omega} \tilde{g}_0(y)dy, \quad \int_\Omega f^-_{1-\alpha}(y)dy = \int_{\partial\Omega} \tilde{g}_1(y)dy, \tag{20}$$

where $f^+_{1-\beta}(y) = r^{-2}J^{1-\beta}[r^2 f^+](x)$, $f^-_{1-\alpha}(y) = r^{-2}J^{1-\alpha}[r^2 f^-](x)$. Since

$$\int_\Omega f^+_{1-\beta}(y)dy = \int_\Omega f_{1-\beta}(y)dy, \quad \int_\Omega f^-_{1-\alpha}(y)dy = 0,$$

$$\int_{\partial\Omega} \tilde{g}_0(y)dS_y = \int_{\partial\Omega_+} g_0(y)dS_y, \quad \int_{\partial\Omega} \tilde{g}_1(y)dS_y = 0,$$

then the second condition of (20) always holds, and therefore, in this case $f^-(x) \in C^{\lambda+1}(\bar{\Omega})$, $\tilde{g}_1(x) \in C^{\lambda+2}(\partial\Omega)$ a solution of the problem (19) exists and belongs to the class $C^{\lambda+1}(\bar{\Omega})$. The first condition of (20) can be rewritten as (15). Under this condition, a solution of the problem (18) exists, is unique up to a constant term, and belongs to the class $C^{\lambda+1}(\bar{\Omega})$. Note that a solution of the problem (19) is unique up to a constant term C. Since the function $w(x)$ should have the property $w(x) = -w(x^*)$, then we get $C \equiv 0$. Therefore, existence of a solution of the problem 2 for the case $k = 1$ is proved. Case $k = 2$ is investigated in the same way. □

5. On an Samarskii-Ionkin Boundary Value Problem for the Poisson Equation in a Disk

If we consider non-classical problems, one of the most famous problems is Samarskii-Ionkin problem, arisen in physics in the 70s of the last century, in connection with the study of the processes occurring in the plasma. Application of the separation of variables method to this problem leads to the spectral problem for the multiple differentiation operator

$$y''(\varphi) = \lambda y(\varphi), 0 < \varphi < 2\pi, y(0) = 0; y'(0) = y'(2\pi).$$

In this section, we consider a Samarskii-Ionkin type boundary value problem for the Poisson equation in the disk and prove its well-posedness.

Let $\Omega = \left\{x \in R^2 : |x| < 1\right\}$ be a unit disk, $r = |x|, \varphi = \arctan(x_2/x_1)$, $0 < \alpha \leq 1$ .Consider in Ω the following problem.

Problem 3. Find a solution of the Poisson equation

$$\Delta u(r,\varphi) = f(r,\varphi), (r,\varphi) \in \Omega \tag{21}$$

satisfying the boundary conditions

$$u(1,\varphi) = g_1(\varphi), 0 \leq \varphi \leq \pi, \tag{22}$$

$$D^\alpha u(1,\varphi) - D^\alpha u(1, 2\pi - \varphi) = g_2(\varphi), 0 \leq \varphi \leq \pi. \tag{23}$$

It is obvious that necessary condition for the solution existence of problem (21) - (23) from the class $u \in C(\bar{\Omega}), D^\alpha u \in C(\bar{\Omega})$ is fulfillment of matching conditions:

$$g_2(0) = g_2(\pi) \tag{24}$$

The problem (21) - (23) in the case $\alpha = 1$ have been studied in [10, 11].

Theorem 5.1. *If a solution of Problem 3 exists, then it is unique.*

Proof. Suppose that there are two functions $u_1(r,\varphi)$ and $u_2(r,\varphi)$ satisfying the conditions of problem (21)-(23).We show that the function $u(r,\varphi) = u_1(r,\varphi) - u_2(r,\varphi)$ is equal to zero. It is obvious that the function $u(r,\varphi)$ is harmonic and satisfies the homogeneous conditions (22)-(23):

$$u(1,\varphi) = 0, 0 \leq \varphi \leq \pi, \tag{25}$$

$$D^\alpha u(1,\varphi) - D^\alpha u(1, 2\pi - \varphi) = 0, 0 \leq \varphi \leq \pi. \tag{26}$$

Let's denote $v(r,\varphi) = u(r,\varphi) - u(r, 2\pi - \varphi)$. Then, we get

$$v(r,\varphi) = u(r,\varphi) - u(r, 2\pi - \varphi) = -\left[u(r, 2\pi - \varphi) - u(r,\varphi)\right] = -v(r, 2\pi - \varphi), 0 \leq \varphi \leq 2\pi.$$

Therefore,

$$\Delta v(r,\varphi) = 0, (r,\varphi) \in \Omega, D^\alpha v(1,\varphi) = 0, 0 \leq \varphi \leq 2\pi.$$

Hence, $v(r,\varphi) \equiv C$. Since $u(1,0) = u(1,2\pi)$, then $v(1,\varphi) = u(1,\varphi) - u(1,2\pi-\varphi) = 0, 0 \leq \varphi \leq 2\pi$, i.e. $u(1,\varphi) = u(1,2\pi-\varphi), 0 \leq \varphi \leq 2\pi$. From this and (25), we have $u(1,\varphi) = 0, 0 \leq \varphi \leq 2\pi$. By the uniqueness of solutions of the Dirichlet problem for harmonic functions, it follows that $u(r,\varphi) \equiv 0$. □

We introduce the auxiliary functions

$$v(r,\varphi)=\frac{u(r,\varphi)+u(r,2\pi-\varphi)}{2}, w(r,\varphi)=\frac{u(r,\varphi)-u(r,2\pi-\varphi)}{2}.$$

It is obvious that $u(r,\varphi)=v(r,\varphi)+w(r,\varphi)$. By direct calculation we find the problems for these functions: the function $v(r,\varphi)$ is a solution of the Dirichlet problem

$$\Delta v(r,\varphi)=f^{+}(r,\varphi),(r,\varphi)\in\Omega, v(1,\varphi)=\tilde{g}_1(\varphi), \tag{27}$$

and the function $w(r,\varphi)$ is a solution of the Neumann type problem

$$\Delta w(r,\varphi)=f^{-}(r,\varphi),(r,\varphi)\in\Omega, D^{\alpha}v(1,\varphi)=\tilde{g}_2(\varphi), 0\le\varphi\le 2\pi. \tag{28}$$

Here

$$f^{+}(r,\varphi)=\frac{f(r,\varphi)+f(r,2\pi-\varphi)}{2}, f^{-}(r,\varphi)=\frac{f(r,\varphi)-f(r,2\pi-\varphi)}{2},$$

$$\tilde{g}_1(\varphi)=\begin{cases} g_1(\varphi)-w(1,\varphi), 0\le\varphi\le\pi \\ g_1(2\pi-\varphi)-w(1,2\pi-\varphi), \pi\le\varphi\le 2\pi \end{cases} \quad \tilde{g}_2(\varphi)=\begin{cases} \frac{g_2(\varphi)}{2}, 0\le\varphi\le\pi \\ -\frac{g_2(2\pi-\varphi)}{2}, \pi\le\varphi\le 2\pi \end{cases}.$$

The Dirichlet problem (27) has a unique solution. It is easily seen that the function $v(r,\varphi)$ has the symmetric property $v(r,\varphi)=v(r,2\pi-\varphi)$.

Since

$$\int_{\Omega} f_{1-\alpha}^{-}(x)dx=0, \int_{0}^{2\pi}\tilde{g}_2(\varphi)d\varphi=0.$$

then we apply the criterion of existence of solutions of the Neumann type problem (28). Its solution is not unique up to an arbitrary $C=\text{constant}$. It is easily seen that the function $w(r,\varphi)$ has the symmetric property $w(r,\varphi)=-w(r,2\pi-\varphi)$ iff $C=0$. Thus, we further assume that this condition holds. The smoothness of the solution of the problem follows from the smoothness of solutions of the corresponding Dirichlet (27) and Neumann type problems (28).

Thus, the following theorem is proved.

Theorem 5.2. *Suppose* $f(r,\varphi)\in C^{\lambda+1}(\bar{\Omega}), g_1(\varphi)\in C^{\lambda+2}[0,\pi], g_2(\varphi)\in C^{\lambda+2}[0,\pi], 0<\lambda<1$, *and the consistency conditions* (24) *hold. Then a solution of the problem 3 exists, is unique and belongs to the class* $C^{\lambda+2}(\bar{\Omega})$.

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