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# Non-Commutative $H_E(\mathcal{A};\ell_\infty)$ and $H_E(\mathcal{A};\ell_1)$ Spaces

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**Abstract.** In this paper, we introduce non-commutative $H_E(\mathcal{A};\ell_\infty)$ and $H_E(\mathcal{A};\ell_1)$ spaces. Then it is shown that these spaces possess many of the properties of non-commutative $H_p(\mathcal{A})$ spaces, such as various factorization results including a Riesz type factorization theorem and contractibility of conditional expectation.

## 1. Introduction and Preliminaries

### 1.1. *Quasi-Banach symmetric function spaces*

Let $L_0[0,1]$ be the space of all measurable real-valued functions on $[0,1]$ equipped with the Lebesgue measure $m$ (functions which coincide almost everywhere are considered identical). Define $S[0,1]$ to be the subset of $L_0[0,1]$ which consists of all functions $x$ such that $m(\{t : |x(t)| > s\})$ is finite for some $s > 0$.

For $x \in S[0,1]$ we denote by $\mu(x)$ the decreasing rearrangement of the function $|f|$. That is,

$$\mu(t,x) = \inf\{s \geq 0 : \ m(\{|x| > s\}) \leq t\}, \quad t > 0.$$

**Definition 1.1.** We say that $(E, \|\cdot\|_E)$ is a symmetric quasi-Banach function space if the following holds.

(a) $E$ is a subset of $S[0,1]$.
(b) $(E, \|\cdot\|_E)$ is a quasi-Banach space.
(c) If $x \in E$ and if $y \in S[0,1]$ are such that $|y| \leq |x|$, then $y \in E$ and $\|y\|_E \leq \|x\|_E$.

Furthermore we recall that the quasi-norm in $E$ is said to be order continuous if, for every sequence $\{x_n\}_{n\geq 0} \subset E$ such that $x_n \downarrow 0$ in $S[0,1]$, we have that $\|x_n\|_E \to 0$. Order continuity of the quasi-norm is equivalent to separability of the space $E$ (see [10, 16]).

Special examples of such quasi-Banach function spaces are the spaces $L_p[0,1]$, $0 < p \leq \infty$, equipped with their usual quasi-norm $\|\cdot\|_p$.

We recall that that every symmetric Banach function space satisfies

$$L_\infty[0,1] \subset E \subset L_1[0,1]$$

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with continuous embeddings. For more details see [15].

We say that $y$ is submajorized by $x$ in the sense of Hardy-Littlewood (written $y << x$) if

$$\int_0^t \mu(s,y)ds \leq \int_0^t \mu(s,x)ds, \quad t > 0.$$

Now let $E$ be a quasi-Banach lattice. Let $0 < r < \infty$. Then $E$ is said to be $r$-convex and $r$-concave, if there exists a constant $C > 0$ such that for all finite sequence $(x_n)$ in $E$

$$\left\|\left(\sum_{k=1}^n |x_k|^r\right)^{1/r}\right\|_E \leq C\left(\sum_{k=1}^n \|x_k\|_E^r\right)^{1/r},$$

and

$$\left(\sum_{k=1}^n \|x_k\|_E^r\right)^{1/r} \leq C\left\|\left(\sum_{k=1}^n |x_k|^r\right)^{1/r}\right\|_E,$$

and as usual the best constant $C > 0$ is denoted by $M^{(r)}(E)$ and $M_{(r)}(E)$, respectively. We recall that for $r_1 \leq r_2$ we have

$$M^{r_1}(E) \leq M^{r_2}(E)$$

and

$$M_{r_2}(E) \leq M_{r_1}(E).$$

To see example: each $L_p(m)$ with Lebesgue measure $m$ is $p-$convex and $p-$concave with constant 1, and as a sequence $M^{(2)}(L_p(m)) = 1$ for $2 \leq p$ and $M_{(2)}(L_p(m)) = 1$ for $p \leq 2$. For all needed information on convexity and concavity we once again refer to [16]. If $M^{max(1,r)}(E) = 1$, then the r'th power

$$E^r := \{x \in L_0(\Omega) : |x|^{1/r} \in E\}$$

endowed with the norm

$$\|x\|_{E^r} = \||x|^{1/r}\|_E^r$$

is again a Banach function space which is $1/min(1,r)$-convex.

### 1.2. Quasi-Banach symmetric operator spaces

Let $\mathbb{H}$ be a Hilbert space. The closed densely defined linear operator $x$ in $\mathbb{H}$ with domain $D(x)$ is said to be affiliated with $\mathcal{M}$ if and only if $uxu = x$ for all unitary operators $u$ which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If $x$ is affiliated with $\mathcal{M}$; then $x$ is said to be $\tau$-measurable if for every $\varepsilon > 0$ there exists a projection $e$ in $\mathcal{M}$ such that $e(\mathbb{H}) \subseteq D(x)$ and $\tau(1-e) < \varepsilon$. The set of all $\tau$-measurable operators will be denoted by $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a $*$-algebra with sum and product being the respective closure of the algebraic sum and product [19]. For each $x$ on $\mathbb{H}$ affiliated with $\mathcal{M}$, all spectral projection $e_s^{\perp}(|x|) = \chi_{(s;\infty)}(|x|)$ corresponding to the interval $(s;\infty)$ belong to $\mathcal{M}$, and $x \in L_0(\mathcal{M})$ if and only if $\chi_{(s;\infty)}(|x|) < \infty$ for some $s \in \mathbb{R}$. Recall that the decreasing rearrangement (or generalized singular numbers) of an operator $x \in L_0(\mathcal{M})$ is defined as follows

$$\mu(s,x) = \inf\{t > 0 : \lambda_t(x) \leq s\},\ s > 0$$

where

$$\lambda_t(x) = \tau(e_t^{\perp}(|x|));\ t > 0.$$

The function $s \mapsto \lambda_s(x)$ is called the distribution function of $x$. For more details on generalized singular value function of measurable operators we refer to [12]. Recall the construction of a quasi-Banach symmetric operator space $L_E(\mathcal{M},\tau)$ (for convenience $L_E(\mathcal{M})$). Let $E$ be a quasi-Banach symmetric function space. Set

$$L_E(\mathcal{M},\tau) = \Big\{x \in L_0(\mathcal{M},\tau) :\ \mu(x) \in E\Big\}.$$

We equip $L_E(\mathcal{M},\tau)$ with a natural quasi-norm

$$\|x\|_{L_E(\mathcal{M},\tau)} = \|\mu(x)\|_E, \quad x \in E(\mathcal{M},\tau).$$

It was further established in [20] (see also [23]) that $E(\mathcal{M},\tau)$ is quasi-Banach.

Since for each operator $x \in L_0(\mathcal{M})$

$$\mu(|x|^r) = \mu(x)^r,$$

we conclude for every symmetric Banach function space $E$ on the interval $[0,1]$ which satisfies $M^{max(1,r)}(E) = 1$ that

$$L_{E^r}(\mathcal{M}) := \{x \in L_0(\mathcal{M}) : |x|^{1/r} \in L_E(\mathcal{M})\}$$

and

$$\|x\|_{L_{E^r}(\mathcal{M})} = \|\mu(|x|)\|_{E^r} = \|\mu(|x|^{1/r})\|_E^r = \||x|^{1/r}\|_{L_E(\mathcal{M})}^r.$$

See [8, 10].

### 1.3. *Non-commutative Hardy spaces*

Let $\mathcal{M}$ be a finite von Neumann algebra on the Hilbert space $\mathbb{H}$ equipped with a normal faithful tracial state $\tau$. Let $\mathcal{D}$ be a von Neumann subalgebra of $\mathcal{M}$, and let $\Phi : \mathcal{M} \to \mathcal{D}$ be the unique normal faithful conditional expectation such that $\tau \circ \Phi = \tau$. A finite subdiagonal algebra of $\mathcal{M}$ with respect to $\Phi$ is a $w^*$-closed subalgebra $\mathcal{A}$ of $\mathcal{M}$ satisfying the following conditions:

(i) $\mathcal{A} + J(\mathcal{A})$ is $w^*$-dense in $\mathcal{M}$;

(ii) $\Phi$ is multiplicative on $\mathcal{A}$, i.e., $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a, b \in \mathcal{A}$;

(iii) $\mathcal{A} \cap J(\mathcal{A}) = \mathcal{D}$, where $J(\mathcal{A})$ is the family of all adjoint elements of the element of $\mathcal{A}$, i.e., $J(\mathcal{A}) = \{a^* : a \in \mathcal{A}\}$.

The algebra $\mathcal{D}$ is called the diagonal of $\mathcal{A}$. It's proved by Exel [11] that a finite subdiagonal algebra $\mathcal{A}$ is automatically maximal in the sense that if $\mathcal{B}$ is another subdiagonal algebra with respect to $\Phi$ containing $\mathcal{A}$, then $\mathcal{B} = \mathcal{A}$. Given $0 < p \leq \infty$ we denote by $L_p(\mathcal{M})$ the usual non-commutatve $L_p$-spaces associated with $(\mathcal{M},\tau)$. Recall that $L_\infty(\mathcal{M}) = \mathcal{M}$, equipped with the operator norm $\|\cdot\|_\infty := \|\cdot\|$ (see [19]). The norm of $L_p(\mathcal{M})$ will be denoted by $\|\cdot\|_p$. For $p < \infty$ we define $H_p(\mathcal{A})$ to be closure of $\mathcal{A}$ in $L_p(\mathcal{M})$, and for $p = \infty$ we simply set $H_\infty(\mathcal{A}) = \mathcal{A}$ for convenience. These are so called Hardy spaces associated with $\mathcal{A}$. They are non-commutative extensions of the classical Hardy space on the torus $\mathbb{T}$. These non-commutative Hardy spaces have received a lot of attention since Arveson's pioneer work. For more details on non-cammutative Hardy space we refer to [1, 7, 17] and [19].

### 1.4. *Non-commutative $\ell_\infty-$ and $\ell_1-$valued symmetric Hardy spaces*

For brevity, we introduce the following definition which was defined in [4].

**Definition 1.2.** Let $E$ be a symmetric quasi Banach space on [0;1] and $\mathcal{A}$ be a finite subdiagonal subalgebra of $\mathcal{M}$. Then $H_E(\mathcal{A}) = [\mathcal{A}]_{L_E(\mathcal{M})}$ called symmetric Hardy space associated with $\mathcal{A}$, where $[\cdot]_{L_E(\mathcal{M})}$ means closure in the norm of $L_E(\mathcal{M})$. We denote $[\mathcal{A}_0]_{L_E(\mathcal{M})}$ by $H_E^0(\mathcal{A})$.

The theory of vector-valued non-commutative $L_p$-spaces were introduced by Pisier in [18] for the case, when $\mathcal{M}$ is hyperfinite and Junge introduced these spaces for general setting in [13] (see also [9, 14]). The theory for the spaces $L_E(\mathcal{M};\ell_\infty)$ and $L_E(\mathcal{M};\ell_1)$ was developed by Defant in [8] and Dirksen in [10] and in full analogy with the special case $L_E = L_p$ considered in [9, 13, 14].

Denote by $L_E(\mathcal{M};\ell_\infty)$ the space of all families $x = (x_n)_{n\geq 1}$ in $L_E(\mathcal{M},\tau)$ for which there are operators $a, b \in L_{E^{1/2}}(\mathcal{M})$ and a uniformly bounded sequence $(y_n)_{n\geq 1}$ in $\mathcal{M}$ such that $x_n = ay_nb$ for all $n \in \mathbb{N}$. We set

$$\|x\|_{L_E(\mathcal{M};\ell_\infty)} := \inf\{\|a\|_{L_{E^{1/2}}(\mathcal{M})} \sup_n \|y_n\|_\infty \|b\|_{L_{E^{1/2}}(\mathcal{M})}\},$$

where the infimum is taken over all such possible factorizations. Moreover, we denote by $L_E(\mathcal{M};\ell_\infty^{col})$ (here "col" should remind on the word "column") the space of all $x=(x_n)_{n\geq 1}$ in $L_E(\mathcal{M})$ for which there are $b\in L_E(\mathcal{M})$ and a bounded sequence $(y_n)_{n\geq 1}$ in $\mathcal{M}$ such that $x_n=y_nb$ for all $n$. We then put

$$\|x\|_{L_E(\mathcal{M};\ell_\infty^{col})} := \inf\{\sup\|y_n\|_\infty\|b\|_{L_E(\mathcal{M})}\}.$$

Similarly, the row version consisting of all families $x=(x_n)_{n\geq 1}$ admitting a factorization $x_n=ay_n$ with $a\in L_E(\mathcal{M})$ and $(y_n)_{n\geq 1}$ bounded in $\mathcal{M}$ is denoted by $L_E(\mathcal{M};\ell_\infty^{row})$ and we define

$$\|x\|_{L_E(\mathcal{M};\ell_\infty^{row})} := \inf\{\|a\|_{L_E(\mathcal{M})}\sup\|y_n\|_\infty\}.$$

In both cases the infimum is again taken over all possible factorizations. The space $L_E(\mathcal{M};\ell_1)$ is defined as the space of all sequences $x=(x_n)_{n\geq 1}$ in $L_E(\mathcal{M})$ which can be decomposed as

$$x_n=\sum_{k=1}^{\infty}u_{kn}v_{nk},\forall n\geq 1$$

for two families $(u_{kn})_{k,n\geq 1}$ and $(v_{nk})_{n,k\geq 1}$ in $L_{E^{1/2}}(\mathcal{M})$ such that

$$\sum_{k,n=1}^{\infty}u_{kn}u_{kn}^*\in L_E(\mathcal{M}) \quad\text{and}\quad \sum_{n,k=1}^{\infty}v_{nk}^*v_{nk}\in L_E(\mathcal{M}),$$

where the series converge in norm. For $x\in L_E(\mathcal{M};\ell_1)$ we define

$$\|x\|_{L_E(\mathcal{M};\ell_1)} := \inf\{\|\sum_{k,n=1}^{\infty}u_{kn}u_{kn}^*\|_{L_E(\mathcal{M})}^{1/2}\|\sum_{n,k=1}^{\infty}v_{nk}^*v_{nk}\|_{L_E(\mathcal{M})}^{1/2}\},$$

where the infimum runs over all decompositions of $x$ as above.

Now we define the Hardy space analogue of these spaces by a similar way.

**Definition 1.3.** (i) We define $H_E(\mathcal{A};\ell_\infty)$ as the space of all sequences $x=(x_n)_{n\geq 1}$ in $H_E(\mathcal{A})$ which admit a factorization of the following form: there are $a,b\in H_{E^{1/2}}(\mathcal{A})$, and a bounded sequence $y=(y_n)\subset\mathcal{A}$ such that

$$x_n=ay_nb,\forall n\geq 1.$$

Given $x\in H_E(\mathcal{A},\ell_\infty)$ define

$$\|x\|_{H_E(\mathcal{A};\ell_\infty)} := \inf\{\|a\|_{H_{E^{1/2}}(\mathcal{A})}\sup_n\|y_n\|_\infty\|b\|_{H_{E^{1/2}}(\mathcal{A})}\},$$

where the infimum runs over all factorizations of $(x_n)$ as above. Moreover, let us define $H_E(\mathcal{A};\ell_\infty^{col})$ as the space of all $(x_n)_{n\geq 1}$ in $H_E(\mathcal{A})$ for which there are $b\in H_E(\mathcal{A})$ and bounded sequence $(y_n)_{n\geq 1}$ in $\mathcal{M}$ such that $x_n=y_nb$ and

$$\|x\|_{H_E(\mathcal{A};\ell_\infty)} := \inf\{\sup_n\|y_n\|_\infty\|b\|_{H_E(\mathcal{A})}\}.$$

Similarly, we define the row version $H_E(\mathcal{A};\ell_\infty^{row})$ all sequences which allow a uniform factorization $x_n=ay_n$, again with $a\in H_E(\mathcal{A})$ and $(y_n)_{n\geq 1}$ uniformly bounded in $\mathcal{M}$.

(ii) We define $H_E(\mathcal{A};\ell_1)$ as the space of all sequences $x=(x_n)_{n\geq 1}$ in $H_E(\mathcal{A})$ which can be decomposed as

$$x_n=\sum_{k=1}^{\infty}u_{kn}v_{nk},\forall n\geq 1$$

for two families $(u_{kn})_{k,n\geq 1}$ and $(v_{nk})_{n,k\geq 1}$ in $H_{E^{1/2}}(\mathcal{A})$ such that

$$\sum_{k,n=1}^{\infty}u_{kn}u_{kn}^*\in L_E(\mathcal{M}) \quad\text{and}\quad \sum_{n,k=1}^{\infty}v_{nk}^*v_{nk}\in L_E(\mathcal{M}).$$

In this space we define norm in the following form:

$$\|x\|_{H_E(\mathcal{A};\ell_1)} := \inf\left\{\left\|\sum_{k,n=1}^{\infty} u_{kn}u_{kn}^*\right\|_{H_E(\mathcal{A})}^{1/2}\left\|\sum_{n,k=1}^{\infty} v_{nk}^*v_{nk}\right\|_{H_E(\mathcal{A})}^{1/2}\right\},$$

where the infimum runs over all decompositions of $x$ as above.

**Example 1.4.** For $E = L_p$, we obtain with $H_E(\mathcal{A}) = H_p(\mathcal{A})$ and $H_{E^{1/2}}(\mathcal{A}) = H_{2p}(\mathcal{A})$ the symmetric case of the spaces $H_p^{(r,s)}(\mathcal{A};\ell_\infty)$, i.e.

$$H_E(\mathcal{A};\ell_\infty) = H_p^{(2p,2p)}(\mathcal{A};\ell_\infty).$$

Moreover, we then have

$$H_E(\mathcal{A};\ell_\infty^{col}) = H_p^{right}(\mathcal{A};\ell_\infty)$$

and

$$H_E(\mathcal{A};\ell_\infty^{row}) = H_p^{left}(\mathcal{A};\ell_\infty).$$

Particular cases which are shown in Example 1.4 with $H_p(\mathcal{A};\ell_1)$ were introduced in [5, 21, 22] with some basic properties. Section 1 contains some preliminary definitions. In section 2, we prove that $H_E(\mathcal{A},\ell_\infty)$ and $H_E(\mathcal{A};\ell_1)$ are quasi-Banach spaces and an analogue Saito's theorem (see [20, Proposition 2 ]). In section 3, we extend that the conditional expectation $\Phi$ to a contractive projection from $H_E(\mathcal{A};\ell_\infty)$ onto $L_E(\mathcal{D};\ell_\infty)$ and from $H_E(\mathcal{A};\ell_1)$ onto $L_E(\mathcal{D};\ell_1)$, respectively.

## 2. Some Properties of $H_E(\mathcal{A};\ell_\infty)$ and $H_E(\mathcal{A};\ell_1)$ Spaces

The following is our key lemma.

**Lemma 2.1.** (i) *Let $E$ be a $r-$convex symmetric quasi Banach function space on* $[0;1]$ *for some $0 < r < \infty$ and $E$ do not contain $c_0$, where $c_0 = \{(a_n) : \lim_{n\to\infty} a_n = 0\}$. If $(x_n) \in L_E(\mathcal{M};\ell_\infty)$, then there exist $h, g \in H_{E^{1/2}}(\mathcal{A})$ and $(z_n) \subset \mathcal{M}$ such that $h^{-1}, g^{-1} \in \mathcal{A}$, and for all $n$, $x_n = hz_ng$, and $\sup_n \|z_n\|_\infty \leq 1$. Moreover,*

$$\|(x_n)\|_{L_E(\mathcal{M};\ell_\infty)} = \inf\{\|h\|_{H_{E^{1/2}}(\mathcal{A})}\sup_n \|z_n\|_\infty\|g\|_{H_{E^{1/2}}(\mathcal{A})}\},$$

*where the infimum runs over all factorizations of $(x_n)$ as above.*

(ii) *Let $E$ be a symmetric quasi Banach function space on* $[0;1]$, *then*

$$L_E(\mathcal{M};\ell_\infty) = L_{E^{1/2}}(\mathcal{M};\ell_\infty^{row})\cdot L_{E^{1/2}}(\mathcal{M};\ell_\infty^{col}).$$

*Proof.* (i) If $x \in L_E(\mathcal{M};\ell_\infty)$, then for any $\varepsilon > 0$ there is a bounded sequence $y = (y_n)$ in $\mathcal{M}$ and operators $a, b \in L_{E^{1/2}}(\mathcal{M})$ such that for all $n$

$$x_n = ay_nb, \quad \|y_n\| \leq 1,$$

and $\|a\|_{L_{E^{1/2}}(\mathcal{M})}\|b\|_{L_{E^{1/2}}(\mathcal{M})} < \|x\|_{L_E(\mathcal{M};\ell_\infty)} + \varepsilon$. Let $a^* = u|a^*|$ and $b = v|b|$ the polar decompositions of $a^*$ and $b$, respectively. Put $c = (|a^*|^2 + \varepsilon)^{\frac{1}{2}}$ and $d = (|b|^2 + \varepsilon)^{\frac{1}{2}}$. Clearly, $|a^*|^2 \leq c^2$ and $|b|^2 \leq d^2$. Then by Remark 2.3. in [9] there exist contractions $\omega, \theta \in \mathcal{M}$ such that $|a^*| = \omega c$, $|b| = \theta d$. Since $c, d \in L_{E^{1/2}}(\mathcal{M})$ and $c^{-1}, d^{-1} \in \mathcal{M}$, by Proposition 3.3 (i) in [4] there exist $h, g \in H_{E^{1/2}}(\mathcal{A})$ and unitary operators $\nu, w \in \mathcal{M}$ such that $c = h\nu$, $d = wg$, and $h^{-1}, g^{-1} \in \mathcal{A}$. Obviously,

$$x_n = h[\nu\omega^*u^*y_nv\theta w]g.$$

Put

$$z_n = \nu\omega^*u^*y_nv\theta w,$$

then it is clear that $\sup_n \|z_n\|_\infty \leq 1$. The norm estimate is clear.

(ii) Let $x \in L_E(\mathcal{M};\ell_\infty)$, then $x_n = ay_nb$. Choosing $x_n^{(1)} = a$ and $x_n^{(2)} = y_nb$ for all $n$, we see that

$$x_n = x_n^{(1)}x_n^{(2)}, \quad \forall\, n.$$

Since $\mathcal{M}$ is finite and $a, b \in L_{E^{1/2}}(\mathcal{M})$, we have $(x_n^{(1)}) \in L_{E^{1/2}}(\mathcal{M};\ell_\infty^{row})$, $(x_n^{(2)}) \in L_{E^{1/2}}(\mathcal{M};\ell_\infty^{col})$.

This completes the proof. □

**Proposition 2.2.** *Let $E$ be a $r-$convex symmetric quasi Banach function space on $[0;1]$ for some $0 < r < \infty$ and $E$ do not contain $c_0$. Then*

$$H_E(\mathcal{A};\ell_\infty) = \{(x_n) \in L_E(\mathcal{M};\ell_\infty) : \quad (x_n) \subset H_E(\mathcal{A})\}. \tag{1}$$

*Moreover,*

$$\|(x_n)\|_{L_E(\mathcal{M};\ell_\infty)} = \|(x_n)\|_{H_E(\mathcal{A};\ell_\infty)}, \quad \forall\, (x_n) \in H_E(\mathcal{A};\ell_\infty). \tag{2}$$

*Proof.* The inclusion $H_E(\mathcal{A};\ell_\infty) \subset \{(x_n) \in L_E(\mathcal{M};\ell_\infty) : \quad (x_n) \in H_E(\mathcal{A})\}$ is clearly. Let $(y_n) \in \{(x_n) \in L_E(\mathcal{M};\ell_\infty) : (x_n) \in H_E(\mathcal{A})\}$. Then by (i) of Lemma 2.1 there exist $a, b \in H_{E^{1/2}}(\mathcal{A})$, and $z_n \in \mathcal{M}$ such that

$$y_n = az_nb, \quad \forall\, n,$$

and $a^{-1}, b^{-1} \in \mathcal{A}$, and $\sup_n \|z_n\|_\infty \le 1$. By Proposition 3.3. (ii) in [4], we have that

$$z_n = a^{-1}y_nb^{-1} \in H_r(\mathcal{A}) \cap \mathcal{M} = \mathcal{A}, \quad \forall\, n.$$

Hence $(y_n) \in H_E(\mathcal{A};\ell_\infty)$. So (1) holds. Using (i) of Lemma 2.1 we get (2). □

**Theorem 2.3.** *Let $E$ be a $r-$convex symmetric quasi Banach function space on $[0;1]$ for some $0 < r < \infty$ and $E$ do not contain $c_0$. Then $H_E(\mathcal{A},\ell_\infty)$ is a quasi-Banach space.*

*Proof.* By (2), it is suffices to show $H_E(\mathcal{A},\ell_\infty)$ is a closed linear subspace of $L_E(\mathcal{M},\ell_\infty)$. Let $(x_n^{(1)})$, $(x_n^{(2)}) \in H_E(\mathcal{A},\ell_\infty)$ and $\alpha,\ \beta \in \mathbb{C}$. Then $(\alpha x_n^{(1)} + \beta x_n^{(2)}) \in L_E(\mathcal{M},\ell_\infty)$ and for all $n$, $\alpha x_n^{(1)} + \beta x_n^{(2)} \in H_E(\mathcal{A})$. By Proposition 2.2, we have that $(\alpha x_n^{(1)} + \beta x_n^{(2)}) \in H_E(\mathcal{A},\ell_\infty)$, i.e., $H_E(\mathcal{A},\ell_\infty)$ is a linear subspace of $L_E(\mathcal{M},\ell_\infty)$. Next to prove $H_E(\mathcal{A},\ell_\infty)$ is closed. Let $(x_n^{(j)}) \in H_E(\mathcal{A},\ell_\infty)$ $(j = 1, 2, ...)$ and $(x_n) \in L_E(\mathcal{M},\ell_\infty)$ such that

$$\lim_{j\to\infty} \|(x_n^{(j)}) - (x_n)\|_{H_E(\mathcal{A},\ell_\infty)} = 0.$$

Since

$$\|x_n^{(j)} - x_n\|_{H_E(\mathcal{A})} \le \|(x_n^{(j)}) - (x_n)\|_{H_E(\mathcal{A},\ell_\infty)}, \quad \forall n \in \mathbb{N},$$

it follows that $\lim_{j\to\infty} \|x_n^{(j)} - x_n\|_{H_E(\mathcal{A})} = 0$, so $x_n \in H_E(\mathcal{A})$. Using Proposition 2.2 we obtain $(x_n) \in H_E(\mathcal{A},\ell_\infty)$, i.e., $H_E(\mathcal{A},\ell_\infty)$ is closed. □

**Corollary 2.4.** *Let $E$ be a $r-$convex symmetric quasi Banach function space on $[0;1]$ for some $0 < r < \infty$ and $E$ do not contain $c_0$. Then*

$$H_E(\mathcal{A};\ell_\infty) = H_{E^{1/2}}(\mathcal{A};\ell_\infty^{row}) \cdot H_{E^{1/2}}(\mathcal{A};\ell_\infty^{col}).$$

**Lemma 2.5.** *Let $E$ be a $r-$convex symmetric quasi Banach function space on $[0;1]$ for some $0 < r < \infty$ and $E$ do not contain $c_0$. If $x \in L_E(\mathcal{M};\ell_1)$, then for each $n$ there exist $(a_{kn})_{k\ge1} \subset H_{E^{1/2}}(\mathcal{A})$, $(b_{nk})_{k\ge1} \subset H_{E^{1/2}}(\mathcal{A})$ and $(y_{nk})_{k\ge1} \subset \mathcal{M}$ such that*

$$x_n = \sum_{k=1}^{\infty} a_{kn}y_{nk}b_{nk},$$

*where $(a_{kn}^{-1})_{k\ge1}$, $(b_{nk}^{-1})_{k\ge1} \subset \mathcal{A}$, and $\sup_n \|y_{nk}\|_\infty \le 1$ for all $n$ and $k$. Moreover,*

$$\|x\|_{L_E(\mathcal{M};\ell_1)} = \inf\left\{ \left\| \sum_{k,n=1}^{\infty} a_{kn}a_{kn}^* \right\|_{H_E(\mathcal{A})}^{1/2} \sup_n \|y_{nk}\|_\infty \left\| \sum_{n,k=1}^{\infty} b_{nk}^* b_{nk} \right\|_{H_E(\mathcal{A})}^{1/2} \right\},$$

*where the infimum runs over all decompositions of $x$ as above.*

*Proof.* Let $(x_n) \in L_E(\mathcal{M};\ell_1)$. Then for $\varepsilon > 0$ there are two families $(u_{kn}),(v_{nk}) \in L_{E^{1/2}}(\mathcal{M})$ such that $x_n = \sum_{k=1}^{\infty} u_{kn}v_{nk} \in L_E(\mathcal{M})$, $\sum v^*_{nk}v_{nk}$, $\sum u_{kn}u^*_{kn} \in L_E(\mathcal{M})$ and

$$\left\|\sum_{k,n=1}^{\infty} u_{kn}u^*_{kn}\right\|^{1/2}_{L_E(\mathcal{M})} \left\|\sum_{n,k=1}^{\infty} v^*_{nk}v_{nk}\right\|^{1/2}_{L_E(\mathcal{M})} < \|x\|_{L_E(\mathcal{M};\ell_1)} + \varepsilon.$$

Let $u^*_{kn} = \vartheta_{kn}|u^*_{kn}|$ and $v_{nk} = \nu_{nk}|v_{nk}|$ be the polar decompositions of $u^*_{kn}$ and $v_{nk}$, for all $n$ and $k$, respectively. Put $c_{kn} := (|u^*_{kn}|^2 + \frac{\varepsilon}{2^{k+n}})^{\frac{1}{2}}$ and $d_{nk} := (|v_{nk}|^2 + \frac{\varepsilon}{2^{k+n}})^{\frac{1}{2}}$. It is clear that $|u^*_{kn}|^2 \le c^2_{kn}$ and $|v_{nk}|^2 \le d^2_{nk}$. By Remark 2.3 in [9], there exist contractions $\omega_{kn}, \theta_{nk} \in \mathcal{M}$ such that $|u^*_{kn}| = \omega_{kn}c_{kn}$, $|v_{nk}| = \theta_{nk}d_{nk}$. Notice that $c_{kn} \in L_{E^{1/2}}(\mathcal{M})$, $d_{nk} \in L_{E^{1/2}}(\mathcal{M})$ and $c^{-1}_{kn}, d^{-1}_{nk} \in \mathcal{M}$. Hence, by Proposition 3.3 (i) in [4], there exist unitary operators $v_{kn}$, $w_{nk} \in \mathcal{M}$ and $h_{kn} \in H_{E^{1/2}}(\mathcal{A})$, $g_{nk} \in H_{E^{1/2}}(\mathcal{A})$ such that $c_{kn} = h_{kn}v_{kn}$ and $d_{nk} = w_{nk}g_{nk}$, and $h^{-1}_{kn}, g^{-1}_{nk} \in \mathcal{A}$. Clearly,

$$x_n = \sum_{k=1}^{\infty} h_{kn}[v_{kn}\omega^*_{kn}u^*_{kn}v_{nk}\theta_{nk}w_{nk}]g_{nk}.$$

Set

$$y_{nk} = v_{kn}\omega^*_{kn}u^*_{kn}v_{nk}\theta_{nk}w_{nk}.$$

Then

$$x_n = \sum_{k=1}^{\infty} h_{kn}y_{nk}g_{nk} \quad \text{and} \quad \sup_n \|y_{nk}\|_\infty \le 1.$$

The norm estimate is clear. □

Similar to Proposition 2.2, we have the following result.

**Proposition 2.6.** *Let E be an r−convex symmetric quasi Banach function space on* $[0;1]$ *for some* $0 < r < \infty$ *and E do not contain* $c_0$. *Then*

$$H_E(\mathcal{A};\ell_1) = \{(x_n) \in L_E(\mathcal{M};\ell_1) : \quad (x_n) \subset H_E(\mathcal{A})\}.$$

*Moreover,*

$$\|(x_n)\|_{L_E(\mathcal{M};\ell_1)} = \|(x_n)\|_{H_E(\mathcal{A};\ell_1)}, \quad \forall\ (x_n) \in H_E(\mathcal{A};\ell_1).$$

Using Lemma 2.5 and Proposition 2.6 we obtain the following result.

**Theorem 2.7.** *Let E be a r−convex symmetric quasi Banach function space on* $[0;1]$ *for some* $0 < r < \infty$ *and E do not contain* $c_0$. *Then* $H_E(\mathcal{A};\ell_1)$ *is a quasi-Banach space.*

**Proposition 2.8.** *Let E be an r−convex symmetric quasi Banach function space on* $[0;1]$ *for some* $0 < r < \infty$ *and E do not contain* $c_0$. *Then*

$$H_E(\mathcal{A};\ell_\infty) = H_r(\mathcal{A};\ell_\infty) \cap L_E(\mathcal{M};\ell_\infty) \ \text{ and } \ H^0_E(\mathcal{A};\ell_\infty) = H^0_r(\mathcal{A};\ell_\infty) \cap L_E(\mathcal{M};\ell_\infty).$$

*Proof.* We prove only the first equivalence. The proof of the second equivalence is similar. It is obvious that $H_E(\mathcal{A};\ell_\infty) \subset H_r(\mathcal{A};\ell_\infty) \cap L_E(\mathcal{M};\ell_\infty)$. To prove the converse inclusion let $(y_n)_{n\ge1} \in H_r(\mathcal{A};\ell_\infty) \cap L_E(\mathcal{M};\ell_\infty)$, then $(y_n)_{n\ge1} \in L_E(\mathcal{M};\ell_\infty)$. By Proposition 3.3. in [4], $(y_n) \subset H_r(\mathcal{A}) \cap L_E(\mathcal{M}) = H_E(\mathcal{A})$. Applying Lemma 2.2 we find $(y_n) \in H_E(\mathcal{A};\ell_\infty)$. □

**Proposition 2.9.** *Let E be a r−convex symmetric quasi Banach function space on* $[0;1]$ *for some* $0 < r < \infty$ *and E do not contain* $c_0$. *Then*

$$H_E(\mathcal{A};\ell_1) = H_r(\mathcal{A};\ell_1) \cap L_E(\mathcal{M};\ell_1) \ \text{ and } \ H^0_E(\mathcal{A};\ell_1) = H^0_r(\mathcal{A};\ell_1) \cap L_E(\mathcal{M};\ell_1).$$

The following proposition is analogue of Proposition 2 in [20] on the $H_E(\mathcal{A};\ell_\infty)$ space.

**Proposition 2.10.** *Let E be a symmetric Banach function space on* $[0;1]$ *and E do not contain* $c_0$. *Then we have the following, where* $H_E^0(\mathcal{A};\ell_\infty) = \{x \in H_E(\mathcal{A};\ell_\infty) : \Phi(x_n) = 0, \ \forall n\}$ :

$$H_E(\mathcal{A};\ell_\infty) = \{x \in L_E(\mathcal{M};\ell_\infty) : \tau(x_n c) = 0, \ for \ all \ c \in \mathcal{A}_0 \ and \ n\}.$$

*Moreover,*

$$H_E^0(\mathcal{A};\ell_\infty) = \{x \in L_E(\mathcal{M};\ell_\infty) : \tau(x_n c) = 0, \ for \ all \ c \in \mathcal{A} \ and \ n\}. \tag{3}$$

*Proof.* The inclusion $H_E(\mathcal{A};\ell_\infty) \subset \{x \in L_E(\mathcal{M};\ell_\infty) : \tau(x_n c) = 0, \ for \ all \ c \in \mathcal{A}_0 \ and \ n\}$ is clearly. Let $y \in \{x \in L_E(\mathcal{M};\ell_\infty) : \tau(x_n c) = 0, \ for \ all \ c \in \mathcal{A}_0 \ and \ n\}$. Then by Lemma 2.1 (i) there exist $a \in H_{E^{1/2}}(\mathcal{A})$, $b \in H_{E^{1/2}}(\mathcal{A})$ and $z_n \in \mathcal{M}$ such that

$$y_n = a z_n b, \ \forall n$$

where $a^{-1}, b^{-1} \in \mathcal{A}$ and $\sup_n \|y_n\|_\infty \leq 1$. On the other hand, we have $\tau(y_n c) = 0, \ \forall c \in \mathcal{A}_0$. Since $a^{-1}sb^{-1} \in \mathcal{A}_0, \ \forall s \in \mathcal{A}_0$, substituting $c$ by $a^{-1}sb^{-1}$ we obtain $z_n \in \mathcal{A}$ (see [4, Lemma 3.1.]), so $(y_n) \in H_E(\mathcal{A};\ell_\infty)$. Now we prove the (3). It is obvious that $H_E^0(\mathcal{A};\ell_\infty) \subset \{x \in L_E(\mathcal{M};\ell_\infty) : \tau(x_n c) = 0, \ for \ all \ c \in \mathcal{A} \ and \ n\}$. Let $x \in L_E(\mathcal{M};\ell_\infty)$, then as above by using Lemma 2.1 (i) and since $\tau(x_n d) = 0, \ \forall \ d \in \mathcal{A}_0$ we get that $x \in H_E(\mathcal{A};\ell_\infty)$. On the other hand we have $\tau(x_n c) = 0, \ \forall \ c \in \mathcal{A}$. Then since $x_n \in L_E(\mathcal{M})$, we deduce $x_n \in H_E^0(\mathcal{A})$ (see [4, Lemma 3.1.]), which is the conclusion. □

## 3. Contractibility of Φ on $H_E(\mathcal{A};\ell_\infty)$ and $H_E(\mathcal{A};\ell_1)$ Spaces

It is well-known that conditional expectation $\Phi$ extends to a contractive projection from $L_p(\mathcal{M})$ onto $L_p(\mathcal{D})$ for every $1 \leq p \leq \infty$. In general, $\Phi$ cannot be, of course, continuously extended to $L_p(\mathcal{M})$ for $p < 1$. However, in [3] proved that $\Phi$ contractive projection from $H_p(\mathcal{A})$ onto $L_p(\mathcal{D})$ for $p < 1$. In this section we prove that $\Phi$ extends to a contractive projection on $H_E(\mathcal{A};\ell_\infty)$ and $H_E(\mathcal{A};\ell_1)$ spaces.

**Theorem 3.1.** *Let E be a symmetric quasi-Banach function space on [0;1] with* $M^{(r)}(E) = 1$ *for some* $0 < r < \infty$ *and let* $h = (h_n)_{n\geq1} \subset H_E(\mathcal{A})$. *Define* $(h_n)_{n\geq1} \mapsto (\Phi(h_n))_{n\geq1}$, *then* $\Phi$ *extends to a contractive projection from* $H_E(\mathcal{A};\ell_\infty)$ *onto* $L_E(\mathcal{D};\ell_\infty)$, *i.e.*

$$\|\Phi(h)\|_{L_E(\mathcal{D};\ell_\infty)} \leq \|h\|_{H_E(\mathcal{A};\ell_\infty)}$$

*for all* $h \in H_E(\mathcal{A};\ell_\infty)$. *The extension will be denoted still by* $\Phi$.

*Proof.* Let $h = (h_n)_{n\geq1} \in H_E(\mathcal{A};\ell_\infty)$, then for all $\varepsilon > 0$ there exist $a, b \in H_{E^{1/2}}(\mathcal{A})$ and a bounded sequence $(x_n) \subset \mathcal{A}$ such that for all $n$, $h_n = a x_n b$, and

$$\|(h_n)_{n\geq1}\|_{H_E(\mathcal{A};\ell_\infty)} + \varepsilon \geq \|a\|_{H_{E^{1/2}}(\mathcal{A})} \sup_n \|x_n\|_\infty \|b\|_{H_{E^{1/2}}(\mathcal{A})} .$$

Hence, by Corollary 2.3. and Theorem 2.2. in [4],

$$\Phi(h_n) = \Phi(a x_n b) = \Phi(a)\Phi(x_n)\Phi(b),$$

where

$$\Phi(a) \in L_E(\mathcal{D}), \ \ \Phi(x_n) \in \mathcal{D}, \ \ \Phi(b) \in L_E(D)$$

and

$$\|\Phi(a)\|_{L_{E^{1/2}}(\mathcal{D})} \leq \|a\|_{H_{E^{1/2}}(\mathcal{A})}, \|\Phi(x_n)\|_\infty \leq \|x_n\|_\infty, \|\Phi(b)\|_{L_{E^{1/2}}(\mathcal{D})} \leq \|b\|_{H_{E^{1/2}}(\mathcal{A})} .$$

Therefore,

$$\|(\Phi(h_n))_{n\geq1}\|_{L_E(\mathcal{D};\ell_\infty)} \leq \|\Phi(a)\|_{L_{E^{1/2}}(\mathcal{D})} \sup_n \|\Phi(x_n)\|_{L_\infty(\mathcal{D})} \|\Phi(b)\|_{L_{E^{1/2}}(\mathcal{D})}$$

$$\leq \|a\|_{H_{E^{1/2}}(\mathcal{A})} \sup_n \|x_n\|_{H_\infty(\mathcal{A})} \|b\|_{H_{E^{1/2}}(\mathcal{A})} \leq \|(h_n)\|_{H_E(\mathcal{A};\ell_\infty)} + \varepsilon.$$

Then letting $\varepsilon \to 0$ we obtain the desired inequality . □

**Theorem 3.2.** *Let E be a symmetric quasi-Banach function space on [0;1] with $M^{(r)}(E) = 1$ for some $0 < r < \infty$ and let $y = (y_n)_{n\geq 1} \subset H_E(\mathcal{A})$. Define $(y_n)_{n\geq 1} \mapsto (\Phi(y_n))_{n\geq 1}$, then $\Phi$ extends to a contractive projection from $H_E(\mathcal{A};\ell_1)$ onto $L_E(\mathcal{D};\ell_1)$, i.e.*

$$\left\|\Phi(y)\right\|_{L_E(\mathcal{D};\ell_1)} \leq \left\|y\right\|_{H_E(\mathcal{A};\ell_1)}$$

*for all $y \in H_E(\mathcal{A};\ell_1)$. The extension will be denoted still by $\Phi$.*

*Proof.* Let $y = (y_n)_{n\geq 1} \in H_E(\mathcal{A};\ell_1)$, then for all $\varepsilon > 0$ there are $(u_{kn})_{k,n\geq 1}$ and $(v_{nk})_{n,k\geq 1}$ in $H_{E^{1/2}}(\mathcal{A})$ such that

$$x_n = \sum_{k=1}^{\infty} u_{kn}v_{nk}, \quad \forall n,$$

and $\sum_{k,n=1}^{\infty} u_{kn}u_{kn}^* \in L_E(\mathcal{M})$ and $\sum_{n,k=1}^{\infty} v_{nk}^* v_{nk} \in L_E(\mathcal{M})$, and

$$\left\|(y_n)_{n\geq 1}\right\|_{H_E(\mathcal{A};\ell_1)} + \varepsilon \geq \left\|\sum_{k,n=1}^{\infty} u_{kn}u_{kn}^*\right\|_{H_E(\mathcal{A})}^{\frac{1}{2}} \left\|\sum_{k,n=1}^{\infty} v_{nk}^* v_{nk}\right\|_{H_E(\mathcal{A})}^{\frac{1}{2}}.$$

Hence, by Corollary 2.3. and Theorem 2.2. in [4]

$$\Phi(y_n) = \Phi\left(\sum_{k=1}^{\infty} u_{kn}v_{nk}\right) = \sum_{k=1}^{\infty} \Phi(u_{kn}v_{nk}) = \sum_{k=1}^{\infty} \Phi(u_{kn})\Phi(v_{nk}), \quad \forall\ n.$$

Then by using the inequality in the proof of Lemma 5.1 in [4] we obtain

$$\begin{aligned} \left\|(\Phi(y_n))_{n\geq 1}\right\|_{L_E(\mathcal{D};\ell_1)} &= \left\|\left(\ \sum_{k=1}^{\infty} \Phi(u_{kn})\Phi(v_{nk})\ \right)_{n\geq 1}\right\|_{L_E(\mathcal{D};\ell_1)} \\ &\leq \left\|\left(\sum_{k,n=1}^{\infty} |\Phi(u_{kn}^*)|^2\right)^{1/2}\right\|_{L_{E^{1/2}}(\mathcal{D})} \left\|\left(\sum_{k,n=1}^{\infty} |\Phi(v_{nk})|^2\right)^{1/2}\right\|_{L_{E^{1/2}}(\mathcal{D})} \\ &\leq \left\|\left(\sum_{k,n=1}^{\infty} |u_{kn}^*|^2\right)^{1/2}\right\|_{H_{E^{1/2}}(\mathcal{A})} \left\|\left(\sum_{k,n=1}^{\infty} |v_{nk}|^2\right)^{1/2}\right\|_{H_{E^{1/2}}(\mathcal{A})} \\ &\leq \left\|(y_n)_{n\geq 1}\right\|_{H_E(\mathcal{A};\ell_1)} + \varepsilon. \end{aligned} \quad \textit{equation}$$

So letting $\varepsilon \to 0$ we obtain verifies inequality. □

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