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On Discrete Pseudo-Differential Operators and Equations

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Abstract. We introduce discrete pseudo-differential operators in appropriate discrete Sobolev–Slobodetskii spaces. Using discrete Fourier transform and factorization concept we study invertibility of such operators in some discrete spaces. Some examples for discrete Calderon–Zygmund operators and difference operators are considered.

1. Introduction

A classical pseudo-differential operator in Euclidean space $\mathbb{R}^m$ is defined by the formula [2, 8–10]

$$(Au)(x) = \int\limits_{\mathbb{R}^m} \tilde{A}(x,\xi)e^{-ix\cdot\xi}\tilde{u}(\xi)d\xi, \tag{1}$$

where the sign $\sim$ over a function denotes its Fourier transform

$$\tilde{u}(\xi) = \int\limits_{\mathbb{R}^m} u(x)e^{ix\cdot\xi}dx,$$

and the function $\tilde{A}(x,\xi)$ is called a symbol of a pseudo-differential operator A.

Our main goal here is describing a periodic variant of this definition and studying its certain properties related to solvability of corresponding equations in canonical domains of an Euclidean space.

1.1. *Discrete Functions and Operators: Preliminaries and Examples*

Given function u_d of a discrete variable $\tilde{x} \in \mathbb{Z}^m$ we define its discrete Fourier transform by the series

$$(F_d u_d)(\xi) \equiv \widetilde{u_d}(\xi) = \sum_{\tilde{x}\in\mathbb{Z}^m} e^{i\tilde{x}\cdot\xi}u_d(\tilde{x}), \quad \xi \in \mathbb{T}^m,$$

where $\mathbb{T}^m = [-\pi,\pi]^m$ and partial sums are taken over cubes

$$Q_N = \{\tilde{x} \in \mathbb{Z}^m : \tilde{x} = (\tilde{x}_1,\cdots,\tilde{x}_m), \max_{1\le k\le m} |\tilde{x}_k| \le N\}.$$

One can define some discrete operators for such functions u_d.

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Example 1.1. If $K(x), x \in \mathbb{R}^m \setminus \{0\}$, is a Calderon–Zygmund kernel, then the corresponding operator is defined by [12]

$$(K_d u_d)(\tilde{x}) = \sum_{\tilde{y} \in \mathbb{Z}^m, \tilde{y} \neq \tilde{x}} K(\tilde{x} - \tilde{y}) u_d(\tilde{y}), \quad \tilde{x} \in \mathbb{Z}^m.$$

Example 1.2. If a first order finite difference of a discrete variable $\tilde{x}_k$ is defined by

$$\delta_k u_d(\tilde{x}) = u_d(\tilde{x}_k + 1) - u_d(\tilde{x}_k),$$

then the discrete Laplacian is

$$(\Delta_d u_d)(\tilde{x}) = \sum_{k=1}^{m} \left(u_d(\tilde{x}_k + 2) - 2u_d(\tilde{x}_k + 1) + u_d(\tilde{x}_k) \right),$$

and its discrete Fourier transform is the function

$$(F_d \Delta_d u_d)(\xi) = \sum_{k=1}^{m} (e^{i\xi_k} - 1)^2.$$

Let $D \subset \mathbb{R}^m$ be a sharp convex cone, $D_d \equiv D \cap \mathbb{Z}^m$, and $L_2(D_d)$ be a space of functions of discrete variable defined on D_d, and $A_d(\tilde{x})$ be a given function of a discrete variable $\tilde{x} \in \mathbb{Z}^m$. We consider the following types of operators

$$(A_d u_d)(\tilde{x}) = \int_{\mathbb{T}^m} \sum_{\tilde{y} \in D_d} e^{i(\tilde{y} - \tilde{x}) \cdot \xi} \widetilde{A}_d(\xi) u_d(\tilde{y}) d\xi, \quad \tilde{x} \in D_d, \tag{2}$$

where

$$\widetilde{A}_d(\xi) = \sum_{\tilde{x} \in \mathbb{Z}^m} e^{i\tilde{x} \cdot \xi} A_d(\tilde{x}), \quad \xi \in \mathbb{T}^m.$$

Definition 1.3. The function $\widetilde{A}_d(\xi)$ is called a symbol of the operator A_d, and this symbol is called an elliptic symbol if $\tilde{A}_d(\xi) \neq 0, \forall \xi \in \mathbb{T}^m$.

Remark 1.4. If $D = \mathbb{R}^m$ then an ellipticity is necessary and sufficient condition for the operator A_d to be invertible in the space $L_2(\mathbb{Z}^m)$.

Remark 1.5. One can define a general pseudo-differential operator with symbol $\tilde{A}_d(\tilde{x}, \xi)$ depending on a spatial discrete variable $\tilde{x}$ by the similar formula

$$(A_d u_d)(\tilde{x}) = \int_{\mathbb{T}^m} \sum_{\tilde{y} \in D_d} e^{i(\tilde{y} - \tilde{x}) \cdot \xi} \widetilde{A}_d(\tilde{x}, \xi) u_d(\tilde{y}) d\xi, \quad \tilde{x} \in D_d,$$

but taking into account a local principle [6] the main aim in this situation is describing invertibility conditions for model operators like (2) in canonical domains D_d.

Below we will refine the lattice $\mathbb{Z}^m$ and introduce more convenient space scale.

2. Discrete Sobolev-Slobodetskii Spaces

2.1. Discrete Fourier Transform

We consider here refined lattice $h\mathbb{Z}^m, h > 0$, and define corresponding discrete Fourier transform. If a function of a discrete variable is defined on a lattice $h\mathbb{Z}^m$ then its discrete Fourier transform can be introduced by the formula

$$(\tilde{u}_d)(\xi) = \sum_{\tilde{x} \in h\mathbb{Z}^m} u_d(\tilde{x}) e^{i\tilde{x} \cdot \xi} h^m, \quad \xi \in \hbar\mathbb{T}^m,$$

where $\hbar = h^{-1}$.

2.2. Discrete Spaces

Let $H^s(h\mathbb{Z}^m)$ denotes a space of functions of a discrete variable for which

$$||u_d||_s^2 \equiv \int\limits_{\hbar\mathbb{T}^m} |\widetilde{u}_d(\xi)|^2(1+|\sigma_{\Delta_d}(h)(\xi)|)^s d\xi < +\infty,$$

where

$$\sigma_{\Delta_d}(h)(\xi) = h^{-2}\sum_{k=1}^{m}(e^{ih\xi_k}-1)^2, \quad \xi \in \hbar\mathbb{T}^m.$$

Remark 2.1. A lot of variants for definition of discrete H^s-spaces were introduced in the paper [3], there are also significant properties of these spaces like continual case.

2.3. Periodic Integral Transforms

Let us denote by P_{D_d} projection operator on hD_d, $P_{D_d} : L_2(h\mathbb{Z}^m) \to L_2(hD_d)$ so that for arbitrary function $u_d \in L_2(h\mathbb{Z}^m)$

$$(P_{D_d}u_d)(\tilde{x}) = \begin{cases} u_d(\tilde{x}), & \text{if } \tilde{x} \in hD_d; \\ 0, & \text{otherwise.} \end{cases}$$

For a half-space case, the Fourier image of the operator P_{D_d} is evaluated [12, 13, 15] and we will demonstrate it in the following

Example 2.2. If $D = \mathbb{R}^m_+ \equiv \{x \in \mathbb{R}^m : x = (x_1, \cdots, x_m), x_m > 0\}$ then

$$(F_dP_{D_d}u_d)(\xi', \xi_m) = \frac{1}{4\pi i}\lim_{\tau\to 0+}\int\limits_{-\hbar\pi}^{\hbar\pi} u_d(\xi', \eta_m)\cot\frac{h(\xi_m-\eta_m+i\tau)}{2}d\eta_m.$$

If D is a sharp convex cone, then $\overset{*}{D}$ is the conjugate cone i.e.

$$\overset{*}{D} = \{x \in \mathbb{R}^m : (x, y) > 0, \ \forall\, y \in D\},$$

(for example, $C^a_+ = \{x \in \mathbb{R}^m : x = (x_1, \cdots, x_m), x_m > a|x'|, x' = (x_1, \cdots, x_{m-1}), a > 0\}, \overset{*}{C^a_+} = \{x \in \mathbb{R}^m : ax_m > |x'|\}$). Now we introduce the function

$$B_d(z) = \sum_{\tilde{x}\in hD_d} e^{i\tilde{x}\cdot z}, \quad z = \xi + i\tau, \quad \xi \in \hbar\mathbb{T}^m, \quad \tau \in \overset{*}{D},$$

and define the operator

$$(B_d\tilde{u}_d)(\xi) = \lim_{\tau\to 0}\int\limits_{\hbar\mathbb{T}^m} B_d(z-\eta)\tilde{u}_d(\eta)d\eta.$$

Lemma 2.3. *For arbitrary $u_d \in L_2(h\mathbb{Z}^m)$ the following property*

$$F_dP_{D_d}u_d = B_dF_du_d$$

holds.

Proof. Let $\chi_+(\tilde{x})$ be an indicator of the set hD_d. Thus

$$(P_{D_d}u_d)(\tilde{x}) = \chi_+(\tilde{x}) \cdot u_d(\tilde{x}).$$

Further since the function $\chi_+(\tilde{x})$ is not summable we cannot apply directly a convolution property of the Fourier transform. We choose the function $e^{i\tilde{x}\cdot\tau}$ so the product $\chi_+(\tilde{x})e^{i\tilde{x}\cdot\tau}$ will be summable for some admissible τ. Taking into account a forthcoming passing to a limit under $\tau \to 0+$ we have

$$F_d(\chi_+(\tilde{x})e^{i\tilde{x}\cdot\tau}) = B_d(z).$$

Thus we can use the Fourier transform obtaining convolution of functions $B_d(z)$ and $\widetilde{u}_d(\xi)$. It is left passing to a limit. □

3. Discrete Pseudo-Differential Operators

First two subsections of this section is devoted to some special cases of discrete pseudo-differential operators for which the author knows some preliminary results. Results of such kind are very desirable for general pseudo-differential operators and related equations.

3.1. Discrete Calderon–Zygmund Operators

A simplest Calderon–Zygmund operator is defined by the formula [6]

$$(Ku)(x) = \lim_{\substack{\varepsilon \to 0 \\ N \to +\infty}} \int\limits_{\varepsilon<|x-y|<N} K(x-y)u(y)dy, \quad x \in \mathbb{R}^m, \tag{3}$$

where a kernel $K(x)$ is called Calderon–Zygmund kernel and satisfies the following conditions

- $K(x)$ is homogeneous of order $-m$;
- $K(x)$ has vanishing mean value on the unit sphere $S^{m-1} \subset \mathbb{R}^m$

$$\int\limits_{S^{m-1}} K(\theta)d\theta = 0;$$

- $K(x) \in C^1(\mathbb{R}^m \setminus \{0\})$.

Such an integral (3) is called an integral in principal value sense [6].

One can put by definition $K(0) \equiv 0$ and define discrete Calderon–Zygmund operator (see example 1.1) by the formula

$$(K_d u_d)(\tilde{x}) = \sum_{\tilde{y}\in\mathbb{Z}^m} K(\tilde{x}-\tilde{y})u_d(\tilde{y}), \quad \tilde{x} \in \mathbb{Z}^m.$$

Moreover, one can refine the lattice $\mathbb{Z}^m$ and consider a family of discrete Calderon–Zygmund operators $\{K_d(h)\}$ on the lattices $h\mathbb{Z}^m, h > 0$,

$$(K_d(h)u_d)(\tilde{x}) = \sum_{\tilde{y}\in h\mathbb{Z}^m} K(\tilde{x}-\tilde{y})u_d(\tilde{y})h^m, \quad \tilde{x} \in h\mathbb{Z}^m.$$

Symbols for Calderon–Zygmund operators and their discrete analogues are defined by Fourier integral and series [6, 12, 13]

$$\sigma_K(\xi) = v.p. \int\limits_{\mathbb{R}^m} K(x)e^{ix\cdot\xi}dx, \quad \xi \in \mathbb{R}^m \setminus \{0\},$$

$$\sigma_{K_d(h)}(\xi) = \sum_{\tilde{x}\in h\mathbb{Z}^m} K(\tilde{x})e^{i\tilde{x}\xi)}h^m, \quad \xi \in \hbar\mathbb{T}^m.$$

Let us denote $\mathbb{Z}^m_+ = \mathbb{Z}^m \cap \mathbb{R}^m_+$.

Proposition 3.1. *Operators K and K_d simultaneously both are invertible or non-invertible in spaces $L_2(\mathbb{R}^m)$ and $L_2(h\mathbb{Z}^m)$ respectively $\forall\ h > 0$. The same is valid for spaces $L_2(\mathbb{R}^m_+)$ and $L_2(h\mathbb{Z}^m_+)$ under additional condition $\sigma_K(0, \cdots, 0, -1) = \sigma_K(0, \cdots, 0, +1)$.*

One can find a proof for this proposition in [12, 13]. These achievements in studying discrete operators have moved the author to generalize these constructions on more large classes of discrete operators.

3.2. Discrete Difference Operators

Another class of discrete operators very similar to pseudo-differential operators is a class of difference operators [16–18]. Such difference operators can be defined by the formula

$$\sum_{|k|=0}^{+\infty} a_k(x)u(x+\beta_k) = v(x), \quad x \in D, \tag{4}$$

where D is $\mathbb{R}^m$ or $\mathbb{R}^m_+$, k is multi-index $k = (k_1, \cdots, k_m)$, $\beta_k = (\beta_{k_1}, \cdots, \beta_{k_m}) \in D$.

If we consider the equation (4) with constant coefficients in the whole space $\mathbb{R}^m$

$$\sum_{|k|=0}^{+\infty} a_k u(x+\beta_k) = v(x), \quad x \in \mathbb{R}^m, \tag{5}$$

then we can use the Fourier transform

$$\tilde{u}(\xi) = \int_{\mathbb{R}^m} e^{ix\cdot\xi}u(x)dx$$

and obtain an equivalent equation in the space $L_2(\mathbb{R}^m)$

$$\sigma(\xi)\tilde{u}(\xi) = \tilde{v}(\xi),$$

where

$$\sigma(\xi) = \sum_{k=0}^{+\infty} a_k e^{i\beta_k\cdot\xi}, \quad \xi \in \mathbb{R}^m. \tag{6}$$

It implies necessary and sufficient condition for a unique solvability of the equation (2): if $\sigma \in L_\infty(\mathbb{R}^m)$ then

$$ess \inf_{\xi\in\mathbb{R}^m} |\sigma(\xi)| > 0.$$

Unfortunately, we cannot use this approach if we are in the space $\mathbb{R}^m_+$ and consider the equation (1) with constant coefficients

$$\sum_{|k|=0}^{+\infty} a_k u(x+\beta_k) = v(x), \quad x \in \mathbb{R}^m_+, \tag{7}$$

since we have no description for Fourier image of the space $L_2(\mathbb{R}^m_+)$. Hence the first step is to obtain such a description, it was done in [2].

Remark 3.2. To obtain such a description we will use a factorization technique for all cases (see below). For the case $D = \mathbb{R}^m_+$ it was used in [4] where one has considered special homogeneous symbols.

3.3. General Pseudo-Differential Operators

We introduce a general discrete pseudo-differential operator by the formula (2) taking into account refinement of the lattice

$$(A_d(h)u_d)(\tilde{x}) = \int_{\hbar\mathbb{T}^m} \sum_{\tilde{y}\in hD_d} e^{i(\tilde{y}-\tilde{x})\cdot\xi} \widetilde{A}_d(\xi) u_d(\tilde{y}) d\xi, \quad \tilde{x} \in hD_d,$$

Definition 3.3. We say a discrete operator A_d has an order α if its symbol $\widetilde{A}_d(\xi)$ satisfies the condition

$$c_1(1 + |\sigma_{\Delta_d}(h)(\xi)|)^{\frac{\alpha}{2}} \le |\widetilde{A}_d(\xi)| \le c_2(1 + |\sigma_{\Delta_d}(h)(\xi)|)^{\frac{\alpha}{2}}$$

with constants c_1, c_2 non-depending on h.

The class of such symbols will be denoted by $S_\alpha(\hbar\mathbb{T}^m)$.

Lemma 3.4. *Pseudo-differential operator A_d of order α is a linear bounded operator $H^s(h\mathbb{Z}^m) \to H^{s-\alpha}(h\mathbb{Z}^m)$.*

Proof. Indeed,

$$||A_d u_d||_s^2 = \int_{\mathbb{T}^m} |\widetilde{A}_d(\xi)\widetilde{u}_d(\xi)|^2 (1 + |\sigma_{\Delta_d}(h)(\xi)|)^s d\xi \le c \int_{\mathbb{T}^m} |\widetilde{u}_d(\xi)|^2 (1 + |\sigma_{\Delta_d}(h)(\xi)|)^{s+\alpha} d\xi,$$

Q.E.D. □

Remark 3.5. It is very important for forthcoming considerations that according to definition 3.3 a norm of the discrete pseudo-differential operator A_d does not depend on h (see also [14]).

3.4. Discrete Pseudo-Differential Equations

Consider corresponding discrete equation

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x} \in hD_d. \tag{8}$$

Let $D \subset \mathbb{R}^m$ be a sharp convex cone and $\overset{*}{D}$ be its conjugate cone.

Definition 3.6. Periodic tube domain $T(D)$ over the cone D is called a subset of multidimensional complex space $\mathbb{C}^m$ of the following type

$$T_\hbar(D) = \hbar\mathbb{T}^m + iD.$$

Let us define the subspace $A(\hbar\mathbb{T}^m) \subset L_2(\hbar\mathbb{T}^m)$ consisting of functions which admit a holomorphic continuation into $T(\overset{*}{D})$ and satisfy the condition

$$\sup_{\tau\in\overset{*}{D}} \int_{\hbar\mathbb{T}^m} |\tilde{u}_d(\xi + i\tau)|^2 d\xi < +\infty.$$

In other words, the space $A(\hbar\mathbb{T}^m) \subset L_2(\hbar\mathbb{T}^m)$ consists of boundary values of holomorphic in $T_h(\overset{*}{D})$ functions.

Let

$$B(\hbar\mathbb{T}^m) = L_2(\hbar\mathbb{T}^m) \ominus A(\hbar\mathbb{T}^m),$$

$B(\hbar\mathbb{T}^m)$ be a direct complement of $A(\hbar\mathbb{T}^m)$ in $L_2(\hbar\mathbb{T}^m)$.

3.4.1. A jump problem

We formulate the problem in the following way: finding a pair of functions $\Phi^{\pm}, \Phi^{+} \in A(\hbar\mathbb{T}^m), \Phi^{-} \in B(\hbar\mathbb{T}^m)$, such that

$$\Phi^{+}(\xi) - \Phi^{-}(\xi) = g(\xi), \quad \xi \in \hbar\mathbb{T}^m, \tag{9}$$

where $g(\xi) \in L_2(\hbar\mathbb{T}^m)$ is given.

Lemma 3.7. *The operator $B_d : L_2(\hbar\mathbb{T}^m) \to A(\hbar\mathbb{T}^m)$ is a bounded projector. A function $u_d \in L_2(hD_d)$ iff its Fourier transform $\tilde{u}_d \in A(\hbar\mathbb{T}^m)$.*

Proof. According to standard properties of the discrete Fourier transform F_d we have

$$F_d(\chi_{+}(\tilde{x})u_d(\tilde{x})) = \lim_{\tau\to 0} \int\limits_{\hbar\mathbb{T}^m} B_d(z-\eta)\widetilde{u_d}(\eta)d\eta,$$

where $\chi_{+}(\tilde{x})$ is an indicator of the set hD_d. It implies a boundedness of the operator B_d. The second assertion follows from holomorphic properties of the kernel $B_d(z)$. In other words for arbitrary function $v \in A(\hbar\mathbb{T}^m)$ we have

$$v(z) = \int\limits_{\hbar\mathbb{T}^m} B_d(z-\eta)v(\eta)d\eta, \quad z \in T_h(\overset{*}{D}).$$

It is an analogue of the Cauchy integral formula. □

Theorem 3.8. *The jump problem has unique solution for arbitrary right-hand side from $L_2(\hbar\mathbb{T}^m)$.*

Proof. Indeed it is equivalent to one-to-one representation of the space $L_2(hD_d)$ as a direct sum of two subspaces. If we will denote by $\chi_{+}(x), \chi_{-}(x)$ indicators of discrete sets $hD_d, h(\mathbb{Z}^m \setminus D_d)$ respectively then the following representation

$$u_d(\tilde{x}) = \chi_{+}(\tilde{x})u_d(\tilde{x}) + \chi_{-}(\tilde{x})u_d(\tilde{x})$$

is unique and holds for arbitrary function $u_d \in L_2(h\mathbb{Z}^m)$. After applying the discrete Fourier transform we have

$$F_d u_d = F_d(\chi_{+}u_d) + F_d(\chi_{-}u_d),$$

where $F_d(\chi_{+}u_d) \in A(\hbar\mathbb{T}^m)$ according to lemma 2, and thus $F_d(\chi_{-}u_d) = F_d u_d - F_d(\chi_{+}u_d) \in B(\hbar\mathbb{T}^m)$ since $F_d u_d \in L_2(\hbar\mathbb{T}^m)$. □

Example 3.9. If $m = 2$ and D is the first quadrant in a plane then a solution of a jump problem is given by formulas

$$\Phi^{+}(\xi) = \frac{1}{(4\pi i)^2} \lim_{\tau\to 0} \int\limits_{-\hbar\pi}^{\hbar\pi} \int\limits_{-\hbar\pi}^{\hbar\pi} \cot\frac{h(\xi_1 + i\tau_1 - t_1)}{2} \cot\frac{h(\xi_2 + i\tau_2 - t_2)}{2} g(t_1, t_2)dt_1 dt_2$$

$$\Phi^{-}(\xi) = \Phi^{+}(\xi) - g(\xi), \quad \tau = (\tau_1, \tau_2) \in D.$$

3.4.2. A general statement

It looks as follows. Finding a pair of functions $\Phi^{\pm}, \Phi^{+} \in A(\hbar\mathbb{T}^m), \Phi^{-} \in B(\hbar\mathbb{T}^m)$, such that

$$\Phi^{+}(\xi) = G(\xi)\Phi^{-}(\xi) + g(\xi), \quad \xi \in \hbar\mathbb{T}^m, \tag{10}$$

where $G(\xi), g(\xi)$ are given periodic functions. If $G(\xi) \equiv 1$, we have the jump problem (9).

Like classical studies [5, 7], we want to use a special representation for an elliptic symbol to solve the problem (10).

3.4.3. Periodic wave factorization

Let us denote by $H^s(hD_d)$ a subspace of $H^s(h\mathbb{Z}^m)$ consisting of functions of discrete variable $\tilde{x}$ for which their supports belong to hD_d, and $\widetilde{H}^s(hD_d), \widetilde{H}^s(h\mathbb{Z}^m)$ their Fourier images.

Lemma 3.10. *For $|s| < 1/2$, the operator B_d is a bounded projector $\widetilde{H}^s(h\mathbb{Z}^m) \to \widetilde{H}^s(hD_d)$, and a jump problem has unique solution $\Phi^+ \in \widetilde{H}^s(hD_d), \Phi^- \in \widetilde{H}^s(h\mathbb{Z}^m \setminus hD_d)$ for arbitrary $g \in \widetilde{H}^s(h\mathbb{Z}^m)$.*

A proof for this assertion can be obtained by methods of functions theory of many complex variables [1, 20] by reasoning like Paley–Wiener theorem [2, 20].

Definition 3.11. Periodic wave factorization for elliptic symbol $\tilde{A}(\xi)$ is called its representation in the form

$$\tilde{A}_d(\xi) = \tilde{A}_{\neq}(\xi)\tilde{A}_{=}(\xi)$$

where the factors $A_{\neq}(\xi), A_{=}(\xi)$ admit holomorphic continuation into domains $T_h(\pm \overset{*}{D})$ respectively and $A_{\neq}(\xi) \in S_{æ}(\hbar\mathbb{T}^m), A_{=}(\xi) \in S_{\alpha-æ}(\hbar\mathbb{T}^m)$. The number æ is called index of periodic wave factorization.

Theorem 3.12. *If the elliptic symbol $\tilde{A}_d(\xi) \in S_\alpha(h\mathbb{T}^m)$ admits periodic wave factorization with index æ so that $|æ - s| < 1/2$ then the the equation* (8) *has unique solution in the space $H^s(hD_d)$ for arbitrary right-hand side $v_d \in H^{s-\alpha}(hD_d)$.*

Proof. Let ℓv_d be an arbitrary continuation of v_d on a whole $h\mathbb{Z}^m$ so that $\ell v_d \in H^{s-\alpha}(h\mathbb{Z}^m)$. Let

$$w_d(\tilde{x}) = (\ell v_d)(\tilde{x}) - (A_d u_d)(\tilde{x})$$

and rewrite

$$(A_d u_d)(\tilde{x}) + w_d(\tilde{x}) = (\ell v_d)(\tilde{x}).$$

Further applying the discrete Fourier transform and using the periodic wave factorization we write

$$\tilde{A}_{\neq}(\xi)\tilde{u}_d(\xi) + \tilde{A}_{=}^{-1}(\xi)\tilde{w}_d(\xi) = \tilde{A}_{=}^{-1}(\xi)\widetilde{\ell v_d}(\xi).$$

According to lemma 3.4 we have $\tilde{A}_{\neq}(\xi)\tilde{u}_d(\xi) \in \widetilde{H}^{s-æ}(h\mathbb{Z}^m), \tilde{A}_{=}^{-1}(\xi)\tilde{w}_d(\xi) \in \widetilde{H}^{s-\alpha+\alpha-æ}(h\mathbb{Z}^m)$ and analogously $\tilde{A}_{=}^{-1}(\xi)\widetilde{\ell v_d}(\xi) \in \widetilde{H}^{s-æ}(h\mathbb{Z}^m)$. Moreover, really $\tilde{A}_{\neq}(\xi)\tilde{u}_d(\xi) \in \widetilde{H}^{s-æ}(hD_d)$ in view of a holomorphy property, and accurate considerations with supports of $A_{=}(\xi)$ and $\tilde{w}_d(\xi)$ show that in fact $\tilde{A}_{=}^{-1}(\xi)\tilde{w}_d(\xi) \in \widetilde{H}^{s-æ}(h\mathbb{Z}^m \setminus hD_d)$.

Thus we obtain a variant of a jump problem for the space $\widetilde{H}^{s-æ}(\mathbb{Z}^m)$ which can be solved by the lemma 3.10. According to this lemma we have

$$\tilde{A}_{\neq}(\xi)\tilde{u}_d(\xi) = B_d(\tilde{A}_{=}^{-1}(\xi)\widetilde{\ell v_d}(\xi))$$

or finally

$$\tilde{u}_d(\xi) = \tilde{A}_{\neq}^{-1}(\xi)B_d(\tilde{A}_{=}^{-1}(\xi)\widetilde{\ell v_d}(\xi))$$

Q.E.D. □

Remark 3.13. It is easy to see that the solution does not depend on choice of continuation ℓv_d.

4. Future Extensions: Discrete (Periodic) Boundary Value Problems

4.1. Ellipticity and Solvability

As before in a continual case pseudo-differential operators considered in a whole space $\mathbb{Z}^m$ are invertible in corresponding functional spaces iff these are elliptic. For example, this assertion is valid for discrete Calderon–Zygmund operator $K_d : L_2(h\mathbb{Z}^m) \to L_2(h\mathbb{Z}^m)$ and for general pseudo-differential operator $A_d : H^s(h\mathbb{Z}^m) \to H^{s-\alpha}(h\mathbb{Z}^m)$ of order α. Thus the equation (8) for $D = \mathbb{R}^m$ is uniquely solvable in appropriate functional spaces for arbitrary right-hand side under an ellipticity condition only. If $D \neq \mathbb{R}^m$ there are a lot of difficulties defined by a type of canonical domain D. We will discuss these difficulties and possible generalizations in next sections.

4.2. Discrete Half-Space Case

If we consider the equation (8) in a discrete half-space $h\mathbb{Z}_+^m$ we have an explicit form for a Fourier image of the projector P_{D_d} (see example 2.2) but it is applicable to solve an auxiliary jump problem if the following condition

$$Ind\ A_d = 0$$

holds. Here $Ind\ A_d$ denotes a topological index of the symbol $\tilde{A}_d(\xi)$ in other words it is a variation of an argument of the symbol $\tilde{A}_d(\xi)$ on the last variable ξ_m under fixed others $\xi' = (\xi_1, \cdots, \xi_{m-1})$. This definition can be given by the following integral

$$æ = \frac{1}{2\pi} \int_{-\hbar\pi}^{\hbar\pi} d \arg \tilde{A}_d(\cdot, \xi_m).$$

The æ is an integer and it does not depend on ξ' according to a homotopy property. It was shown earlier [16–18] at least for the space $L_2(h\mathbb{Z}_+^m)$ if $æ \neq 0$ the equation (8) is underdetermined ($æ > 0$) or overdetermined ($æ < 0$). For these cases to obtain unique solution one needs to add some additional conditions (as usual these are boundary conditions) or to introduce additional unknowns (as usual these are represented by potential like operators).

Taking into account all mentioned above this leads to the following

Problem 4.1. *Adapting a factorization concept to discrete H^s-spaces and general discrete pseudo-differential operators to describe all possible statements of well posed boundary value problems for discrete elliptic pseudo-differential equations in $H^s(h\mathbb{Z}_+^m)$ like continual case* [2].

4.3. General Conical Case

It is more complicated case in a comparison with a half-space case because for corresponding multidimensional analogue [11] of classical Riemann boundary value problem [5, 7] we have no explicit description for the space $B(\hbar\mathbb{T}^m)$. But a concept of the wave factorization introduced by the author has permitted to describe solvability cases for continual elliptic pseudo-differential equation (8) [11]. Moreover, according to sec.3 it is possible to apply this idea for a discrete case also. Collecting these facts one can formulate the following

Problem 4.2. *Applying the periodic wave factorization for a periodic elliptic symbol to include into consideration an arbitrary index of periodic wave factorization, order of an operator and parameter s of the Sobolev–Slobodetskii space $H^s(hD_d)$.*

5. Conclusion

These considerations have to be useful for statements of boundary value problems for discrete elliptic pseudo-differential equations in canonical non-smooth domains. Such boundary value problems will appear when an index of the wave factorization is not zero. Moreover, we hope to establish a correspondence between discrete and continual [11] cases and to describe a limit transfer from discrete case to continual one.

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