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# Necessary Optimality Condition for the Singular Controls in an Optimal Control Problem with Nonlocal Conditions

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**Abstract.** In this paper, we continue investigation of the problem considered in our earlier works. The paper deals with an optimal control problem for an ordinary differential equation with integral boundary conditions that generalizes the Cauchy problem. The problem is investigated the case when Pontryagin's maximum principle is degenerate. Moreover, the second order optimality conditions are derived for the considered problem.

## 1. Introduction

L.S. Pontryagin's maximum principle is of great importance in the optimal control theory. However, in some cases of the maximum principle degenerates, i.e. is fulfilled trivially. Note that such controls usually are called as the singular controls. They are divided into three types considering the specification. To the first type occurs in the case when Pontryagin's maximum principle degenerates. The second type occurs in the case when linearized maximum principle degenerates. In the third type the necessary condition degenerates in the classical sense. Actually, the last case of singularity is investigated in [9].

Since in the case of singular controls the first order necessary condition does not give any information, it is necessary to derive higher order conditions.

In spite of the theory of necessary optimality conditions of higher order, while the theory of singular controls has been developed completely for the control problems described by the systems with local conditions [3, 5, 6, 12, 13], the study of optimal problems with nonlocal conditions has recently started [1, 2, 4, 7–11, 14].

Now we shall note some earlier works in this direction. O. Vasilieva and K. Mizukami [14] obtained optimality conditions in the optimal control problems, with two point boundary conditions. Lately they developed the algorithms for the numerical solution of such problems. Sharifov Y.A. also considered the same problem with conditions that provide enough complete information on the solvability of the problem

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and obtained necessary optimality condition of the first and second orders [10]. Further this idea was developed for the impulsive two point optimal control problems and the first order necessary conditions were derived [11]. Then using the developed technique optimal control problems with three, multipoint, integral, integral impulsive and other boundary conditions have been investigated and corresponding necessary optimality conditions in the classical sense have been derived [2, 7].

The present paper is a continuation of the paper [9]. In this paper, we consider an optimal control problem for an ordinary differential equation with integral boundary conditions that generalizes the Cauchy problem. The problem is investigated in the case when Pontryagin's maximum principle degenerates in other words it is fulfilled trivially. Then the second order optimality conditions are derived for the considered problem.

Note that investigation of such problems is stimulated by the fact that they have strong relations to many applied problems as well as chemical engineering, thermo-elasticity, heat conduction, plasma physics and etc. For the further reference one can see [4, 8].

The paper is organized as follows. In Section 2 the formulation of the problem is given and some preliminary results are presented. In Section 3 a formula is derived for the increment of the functional. Section 4 is devoted to obtaining the necessary optimality condition for the singular in the sense of Pontryagin's maximum principle controls.

## 2. Problem Statement

Consider the following system of differential equations with integral boundary conditions

$$\frac{dx}{dt} = f\left(t, x, u\left(t\right)\right), \quad 0 \leq t \leq T, \tag{1}$$

$$x\left(0\right) + \int_0^T m\left(t\right) x\left(t\right) dt = C, \tag{2}$$

$$u\left(t\right) \in U, \; t \in \left[0, T\right]. \tag{3}$$

Here $x(t)$ is $n$-dimensional vector of phase variables, $u(t)$ is $r$-dimensional piecewise-continuous (from the right) vector of the control actions with the values from some nonempty bounded set $U$ of $r$-dimensional Euclidian space $R^r$; $T$ and $C \in R^n$ are fixed; $m(t) \in R^{n\times n}$ is the given matrix function. It is assumed that the function $f(t, x, u)$ is continuous on totality of arguments on $[0, T] \times R^n \times R^r$ together with partial derivatives with respect to $x$ up to second order, inclusively.

On the solutions of system (1)-(3) it is required to minimize the functional

$$J\left(u\right) = \varphi\left(x\left(0\right), x\left(T\right)\right) + \int_0^T F\left(t, x\left(t\right), u\left(t\right)\right) dt. \tag{4}$$

Here it is supposed that, the scalar functions $\varphi(x, y)$ and $F(t, x, u)$ are continuous in totality of arguments, have continuous and bounded partial derivatives with respect to $x$ and $y$up to second order, inclusively. As a solution of boundary value problem (1)-(3) corresponding to the fixed control parameter $u(\cdot) \in U$ we consider the function $x(t) : [0, T] \to R^n$ absolutely continuous on $[0, T]$.
Denote by $AC([0, T], R^n)$ the space of such functions and by $C([0, T], R^n)$the space of the determined on $[0, T]$ continuous functions with the values from $R^n$.

Obviously, this is a Banach space with the norm

$$\|x\|_{C[0,T]} = \max_{t\in[0,T]} |x(t)|,$$

where $|\cdot|$ is a norm in $R^n$.

The admissible control together with appropriate solution of boundary value problem (1)-(3) is called an admissible process. The admissible process $\{u(t), x(t,u)\}$ being the solution of problem (1)-(4), i.e. minimizing the functional (4) at constraints (1)-(3) is said to be an optimal process, while $u(t)$ - an optimal control.

It is assumed that optimal control problem (1)-(4) has a solution.

The existence and uniqueness of the solution of boundary value problem (1)-(3) was studied in the paper [9]. Now let us state some conditions that will be called further:

A1) Let$\|B\| < 1$, where $B = \int_0^T m(t)\,dt$.

A2) Let the function $f : [0,T] \times R^n \times R^r \to R^n$ be continuous, and there exists a constant $K \geq 0$ such that

$$\left| f(t,x,u) - f(t,y,u) \right| \leq K \left| x - y \right|$$

for all $t \in [0,T]$ , $x,\ y \in R^n, u \in U$.

A3) $L = (1 - \|B\|)^{-1} KTN < 1$ , where $N = \max_{0 \leq t,s \leq T} \|N(t,s)\|$ ,

$$N(t,s) = \begin{cases} E + \int_0^s m(t)\,d\tau, & 0 \leq t \leq s \\ -\int_s^T m(t)\,d\tau, & s < t \leq T, \end{cases}$$

$E \in R^{n \times n}$ is a unit matrix

**Theorem 2.1.** ([9]) *Let condition A1) be fulfilled. Then the function $x(\cdot) \in ([0,T],\ R^n)$ is an absolutely continuous solution of boundary value problem* (1)-(3) *if and only if when it satisfies the*

$$x(t) = (E+B)^{-1} C + \int_0^T k(t,\tau) f(\tau, x(\tau), u(\tau))\,d\tau, \tag{5}$$

*where $k(t,\tau) = (E+B)^{-1} N(t,\tau)$.*

**Theorem 2.2.** ([9]) *Let conditions A1)-A3) be satisfied. Then for any $C \in R^n$ and for any fixed admissible control, boundary value problem* (1)-(3) *has a unique solution.*

## 3. Formula for the Increment of the Functional

Let $\{u, x(t,u)\}$ and $\{\bar{u} = u + \Delta u,\ \bar{x} = x\Delta = x(t,\bar{u})\}$ be two admissible processes. Then we can represent boundary value problem (1)-(3) in increments in the form

$$\Delta\dot{x} = \Delta f(t,x,u)\ ,\ \ t \in [0,T]\,, \tag{6}$$

$$\Delta x(0) + \int_0^T m(t)\,\Delta x(t)\,dt = 0, \tag{7}$$

where $\Delta f(t,x,u) = f(t,\bar{x},\bar{u}) - f(t,x,u)$ denotes the total increment of the function, while $\Delta_{\bar{u}} f(t,x,u) = f(t,x,\bar{u}) - f(t,x,u)$ denotes the partial increment of the function $f(t,x,u)$.

Then the increment of functional (4) takes the form:

$$\Delta J(u) = J(\bar{u}) - J(u) = \Delta\varphi(x(0), x(\tau)) + \int_0^T \Delta F(t,x,u)\,dt. \tag{8}$$

Adding to (8) the zero terms

$$\int_0^T \langle \psi(t), \Delta\dot{x} - \Delta f(t,x,u)\rangle \, dt,$$

$$\left\langle \lambda, \ \ \Delta x(0) - \int_0^T m(t)\,\Delta x dt \right\rangle$$

one can check that the increment of functional (8) takes the form

$$\Delta J(u) = \Delta\varphi(x(0), x(T)) + \int_0^T \Delta F(t, x(t), u(t))\, dt$$

$$+ \int_0^T \langle \psi(t), \Delta\dot{x} - \Delta f(t,x,u)\rangle \, dt + \left\langle \lambda, \Delta x(0) + \int_0^T m(t)\,\Delta x(t)\, dt \right\rangle. \tag{9}$$

Here $\psi(t) \in R^n$ is an arbitrary trivial vector-function, $\lambda \in R^n$ is a nonzero arbitrary constant vector.

After some standard operations, usually carried out by deriving necessary optimality conditions of second order, for the increment of the functional (9) we get the formula

$$\Delta J(u) = J(\bar{u}) - J(u) = -\int_0^T \Delta_{\bar{u}} H(t, x(t), \psi(t), u(t))\, dt$$

$$-\int_0^T \left[ \left\langle \Delta_{\bar{u}} \frac{\partial H(t, x(t), \psi(t), u(t))}{\partial x}, \Delta x(t) \right\rangle + \frac{1}{2} \left\langle \Delta x(t)' \, \frac{\partial^2 H(t, x(t), \psi(t), u(t))}{\partial x^2}, \Delta x(t) \right\rangle \right] dt$$

$$+\frac{1}{2} \left\langle \Delta x(0)' \, \frac{\partial^2 \varphi}{\partial x(0)^2} + \Delta x(T)' \, \frac{\partial^2 \varphi}{\partial x(0)\,\partial x(T)}, \Delta x(0) \right\rangle$$

$$+\frac{1}{2} \left\langle \Delta x(0)' \, \frac{\partial^2 \varphi}{\partial x(T)\,\partial x(0)} + \Delta x(T)' \, \frac{\partial^2 \varphi}{\partial x(T)^2}, \Delta x(T) \right\rangle$$

$$+ \int_0^T \left\langle \dot{\psi}(t) + \frac{\partial H(t,x,\psi,u)}{\partial x} + m'(t)\,\lambda, \Delta x(t) \right\rangle dt + \left\langle \frac{\partial \varphi}{\partial x(0)} - \psi(0) + \lambda, \Delta x(0) \right\rangle$$

$$+ \left\langle \frac{\partial \varphi}{\partial x(T)} + \psi(0), \Delta x(T) \right\rangle - \frac{1}{2} \int_0^T \left\langle \Delta x(t)' \, \Delta_{\bar{u}} \frac{\partial^2 H(t,x,\psi,u)}{\partial x^2}, \Delta x(t) \right\rangle dt$$

$$+ o_1\left( \|\Delta x(0)\|^2 + \|\Delta x(T)\|^2 \right) - \int_0^T o_H\left( \|\Delta x(t)\|^2 \right) dt \ . \tag{10}$$

Here $H(t,x,\psi,u) = \langle \psi, f(t,x,u)\rangle - F(t,x,u)$.

Now suppose that the arbitrary vector-function $\psi(t) \in R^n$ and the constant vector $\lambda \in R^n$ is a solution of the following boundary value problem

$$\dot{\psi}(t) = -\frac{\partial H(t, x, \psi, u)}{\partial x} - m'(t)\lambda, \ t \in [0, T] \ , \tag{11}$$

$$\psi(0) = \frac{\partial \varphi}{\partial x(0)} + \lambda, \ \ \psi(T) = -\frac{\partial \varphi}{\partial x(T)} = 0. \tag{12}$$

The problem (11), (12) is said to be a conjugated problem.

Taking into account the equalities (11), (12) in (10), for the increment of the functional we get the auxiliary formula

$$\Delta J(u) = J(\bar{u}) - J(u) = -\int_0^T \Delta_{\bar{u}} H(t, x(t), \psi(t), u(t))\, dt$$

$$-\int_0^T \left[\left\langle \Delta_{\bar{u}} \frac{\partial H(t, x(t), \psi(t), u(t))}{\partial x}, \Delta x(t) \right\rangle + \frac{1}{2}\left\langle \Delta x(t)' \frac{\partial^2 H(t, x(t), \psi(t), u(t))}{\partial x^2}, \Delta x(t) \right\rangle\right] dt$$

$$+\frac{1}{2}\left\langle \Delta x(0)' \frac{\partial^2 \varphi}{\partial x(0)^2} + \Delta x(T)' \frac{\partial^2 \varphi}{\partial x(0)\,\partial x(T)}, \Delta x(0) \right\rangle$$

$$+\frac{1}{2}\left\langle \Delta x(0) \frac{\partial^2 \varphi}{\partial x(T)\,\partial x(0)} + \Delta x(T)' \frac{\partial^2 \varphi}{\partial x(T)^2}, \Delta x(T) \right\rangle + \eta(u, \Delta u), \tag{13}$$

where

$$\eta(u, \Delta u) = -\frac{1}{2}\int_0^T \left\langle \Delta x'(t)\, \Delta_{\bar{u}} \frac{\partial^2 H(t, x, \psi, u)}{\partial x^2}, \Delta x(t) \right\rangle dt$$

$$+o_1\left(\|\Delta x(0)\|^2 + \|\Delta x(T)\|^2\right) - \int_0^T o_H\left(\|\Delta x(t)\|^2\right) dt.$$

On the other hand, from the smoothness conditions imposed on the right hand side of system (1) it follows the solution of (6)-(7) satisfies also the following boundary value problem

$$\Delta \dot{x}(t) = \frac{\partial f(t, x, u)}{\partial x} \Delta x(t) + \Delta_{\bar{u}} f(t, x, u) + \eta_1(t) \ , \tag{14}$$

$$\Delta x(0) + \int_0^T m(t)\, \Delta x(t)\, dt = 0, \tag{15}$$

where by definition

$$\eta_1(t) = \Delta_{\bar{u}} \frac{\partial f(t,x,u)}{\partial x} \Delta x(t) + o_2(\|\Delta x(t)\|).$$

It is known that [1] any solution of the equation (14) may be represented in the form

$$\Delta x(t) = \phi(t)\Delta x(0) + \phi(t) \int_0^t \phi^{-1}(\tau)\Delta_{\bar{u}} f(\tau, x(\tau), u(\tau))\, d\tau + \eta_2(t), \tag{16}$$

where the matrix-function $\phi(t)$ is a solution of the following differential equation

$$\frac{d\phi(t)}{dt} = \frac{\partial f(t, x(t), u(t))}{\partial x} \phi(t), \;\; \phi(0) = E,$$

where $E \in R^{n\times n}$ is a unit matrix.

Now let us require that the function determined by equality (16) satisfies condition (15).

Then

$$\left[E + \int_0^T m(t)\phi(t)\, dt\right] \Delta x(0)$$
$$= -\int_0^T m(t)\phi(t)\, dt \int_0^t \phi^{-1}(\tau)\Delta_{\bar{u}} f(t,x,u)\, d\tau dt \int_0^T m(t)\eta(t)\, dt.$$

Suppose that the matrix $E + B_1$ is invertible, where $B_1 = \int_0^T m(t)\phi(t)\, dt$.

Then

$$\Delta x(0) = -(E + B_1)^{-1} \int_0^T m(t)\phi(t) \int_0^t \phi^{-1}(\tau)\Delta_{\bar{u}} f(\tau,x,u)\, d\tau dt - (E + B_1)^{-1} \times \int_0^T m(t)\eta_2(t)\, dt. \tag{17}$$

Now, considering (17) in (16) for $\Delta x(t)$ we get the formula

$$\Delta x(t) = -\phi(t)(E + B_1)^{-1} \int_0^T m(\tau)\phi(\tau) \int_0^t \phi^{-1}(\tau)\Delta_u f(\tau, x(\tau), u(\tau))\, d\tau dt$$

$$+\phi(t) \int_0^t \phi^{-1}(\tau)\Delta_{\bar{u}} f(\tau, x(\tau), u(\tau))\, d\tau + \eta_2(t) - \phi(t)(E + B_1)^{-1} \int_0^T m(\tau)\eta_2(t)\, dt. \tag{18}$$

After some simplifications, we can rewrite equality (18) in the form

$$\Delta x(t) = \int_0^T G(t,\tau)\Delta_{\bar{u}} f(\tau,x,u)\, d\tau + \eta_3(t), \tag{19}$$

where

$$\eta_3(t) = \eta_2(t) - \phi(t)(E+B_1)^{-1}\int_0^T m(t)\,\eta_2(t)\,dt,$$

$$G(t,\tau) = \begin{cases} \phi(t)(E+B_1)^{-1}\left(E+\int_0^t m(\tau)\phi(\tau)\,d\tau\right)\phi^{-1}(\tau), & 0\le\tau\le t, \\ -\phi(t)(E+B_1)^{-1}\int_t^T m(\tau)\phi(\tau)\,d\tau\phi^{-1}(\tau), & t<\tau\le T. \end{cases}$$

From (19) it follows

$$\Delta x(0) = \int_0^T G(0,\tau)\,\Delta_{\bar{u}} f(\tau, x(\tau), u(\tau))\,d\tau + \eta_3(0), \tag{20}$$

$$\Delta x(T) = \int_0^T G(T,\tau)\,\Delta_u f(\tau, x(\tau), u(\tau))\,d\tau + \eta_3(T). \tag{21}$$

Introduce the matrix-function

$$R(\tau,s) = -G'(0,\tau)\frac{\partial^2\varphi}{\partial x(0)^2}G(0,s) - G'(T,\tau)\frac{\partial^2\varphi}{\partial x(T)\,\partial x(0)}G(0,s)$$

$$-G'(0,\tau)\frac{\partial^2\varphi}{\partial x(0)\,\partial x(T)}G(T,s) - G(T,\tau)\frac{\partial^2\varphi}{\partial x(T)^2}G(T,s) + \int_0^T G'(t,\tau)\frac{\partial^2 H}{\partial x^2}G(t,s)\,dt.$$

Then it is obvious that

$$-\frac{1}{2}\int_0^T \left\langle \Delta x(t)'\frac{\partial^2 H(t,x(t),\psi(t),u(t))}{\partial x^2}, \Delta x(t)\right\rangle dt$$

$$+\frac{1}{2}\left\langle \Delta x(0)'\frac{\partial^2\varphi}{\partial x(0)^2} + \Delta x(T)'\frac{\partial^2\varphi}{\partial x(T)\,\partial x(0)}, \Delta x(0)\right\rangle \tag{22}$$

$$+\frac{1}{2}\left\langle \Delta x(0)'\frac{\partial^2\varphi}{\partial x(0)\,\partial x(T)} + \Delta x(T)'\frac{\partial^2\varphi}{\partial x(T)^2}, \Delta x(T)\right\rangle$$

$$= -\frac{1}{2}\int_0^T\int_0^T \langle \Delta'_{\bar{u}} f(t,x(t),u(t))\,R(t,s), \Delta_u f(s,x(s),u(s))\rangle\,dtds - \frac{1}{2}\eta_4,$$

where

$$\eta_4 = \int_0^T\int_0^T \left\langle \eta_3(t)\frac{\partial^2 H(t,x(t),\psi(t),u(t))}{\partial x^2}, G(t,s)\,\Delta_{\bar{u}} f(s,x(s),\psi(s),u(s))\right\rangle dtds$$

$$+\int_0^T\int_0^T\left\langle \Delta_u^{'}f\left(s,x\left(s\right),u\left(s\right)\right),G'\left(t,s\right)\frac{\partial^2 H\left(t,x\left(t\right),\psi\left(t\right),u\left(t\right)\right)}{\partial x^2},\eta_3\left(t\right)\right\rangle dsdt$$

$$+\int_0^T\left\langle \eta_3^{'}\left(t\right)\frac{\partial^2 H\left(t,x\left(t\right),\psi\left(t\right),u\left(t\right)\right)}{\partial x^2},\eta_3\left(t\right)\right\rangle dt\ .$$

Taking into account (22) in (13) for the increment of the functional we finally get the formula

$$\Delta J\left(u\right)=J\left(\bar{u}\right)-J\left(u\right)=-\int_0^T\Delta_{\bar{u}}H\left(t,x\left(t\right),\psi\left(t\right),u\left(t\right)\right)dt$$

$$-\int_0^T\left\langle \Delta_{\bar{u}}\frac{\partial H\left(t,x\left(t\right),\psi\left(t\right),u\left(t\right)\right)}{\partial x},\Delta x\left(t\right)\right\rangle dt$$

$$-\frac{1}{2}\int_0^T\int_0^T\left\langle \Delta_{\bar{u}}'f\left(t,x\left(t\right),u\left(t\right)\right)R\left(t,s\right),\Delta_u f\left(s,x\left(s\right),u\left(s\right)\right)\right\rangle dtds-\frac{1}{2}\eta_4$$

$$+\eta\left(u,\Delta u\right).$$

## 4. Necessary Optimality Condition for Controls Singular in the Sense of Pontryagin's Maximum Principle

It is known that, [9] for the optimality of the admissible control $u\left(t\right)$ in problem (1)-(4), the fulfillment of the inequality

$$\Delta_v H\left(\tau,x,\psi,u\right)\le 0 \tag{23}$$

is necessary for all $v\in U$ and $\tau\in(0,T)$. It is clear that conditions (23) being indeed the first order necessary optimality condition give limited information on controls suspicious for optimality. Besides, there exists the possibility the degeneration of the Pontryagin's maximum condition (23) or its consequences. In this case one has to derive other necessary optimality conditions that allow one to find out non-optimality of the admissible controls for which the Pontryagin's maximum principle or its consequences degenerate.

**Definition 4.1.** The admissible control $u\left(t\right)$ is called singular in the sense of Pontryagin's maximum principle if for all $v\in U$and $\theta\in[0,T]$

$$\Delta_v H\left(\theta,x\left(\theta\right),\psi\left(\theta\right),u\left(\theta\right)\right)\equiv 0\ . \tag{24}$$

Obviously, subject to condition (24), Pontryagin's maximum principle becomes ineffective.

Now we derive the necessary optimality conditions for the singular in the sense of Pontryagin's maximum principle controls.

Assuming that $u\left(t\right)$ is a singular optimal control, we define its special increment by the formula

$$\Delta u_\varepsilon\left(t\right)=\begin{cases} v,t\in[\theta,\theta+\varepsilon)\ , \\ u\left(t\right),\ t\in[0,T]\setminus[\theta,\theta+\varepsilon)\ . \end{cases} \tag{25}$$

Denote by $\Delta x_\varepsilon(t)$ special increment of $x(t)$ responding to the increment (15) of the control $u(t)$. Following [9], we can prove the validity of the following estimation

$$\|\Delta x_\varepsilon(t)\| \le \bar{L}\varepsilon,\ \ t \in [0,T]\ , \bar{L} = const > 0.$$

Taking into account formulas (22)-(25), the formula for the increment of the functional and the fact that $\eta(u,u_\varepsilon) = o\left(\varepsilon^2\right)$, from (13) by the optimality of the control $u(t)$ singular in the sense of Pontryagin's maximum principle, we get

$$\begin{aligned}\Delta J(u) = -\frac{\varepsilon^2}{2}\left\{\left\langle 2\Delta_v \frac{\partial H(\theta, x(\theta), \psi(\theta), u(\theta))}{\partial x}, \Delta x(\theta)\right\rangle\right. \\ \left. + \left\langle \Delta_v' f(\theta, x(\theta), u(\theta)) R(\theta,\theta), \Delta_v f(\theta, x(\theta), u(\theta))\right\rangle\right\} + o\left(\varepsilon^2\right).\end{aligned} \tag{26}$$

By rather smallness of $\varepsilon > 0$ it follows that

$$\begin{aligned}2\left\langle \Delta_v \tfrac{\partial H\left(\theta, x(\theta), \psi(\theta), u(\theta)\right)}{\partial x},\ G(\theta\,,\theta)\Delta_v f(\theta, x(\theta), u(\theta))\right\rangle \\ + \left\langle \Delta_v' f(\theta, x(\theta), u(\theta)) R(\theta,\theta), \Delta_v f(\theta, x(\theta), u(\theta))\right\rangle \le 0.\end{aligned} \tag{27}$$

Consequently, we present the following theorem.

**Theorem 4.2.** *For the optimality of the singular in the sense of Pontryagin's maximum principle control $u(t)$ in problem* (1)-(4) *the fulfillment of inequality* (27) *is necessary for all $v \in U, \theta \in [0,T)$.*

Inequality (27) is a necessary condition for the singular in the sense of Pontryagin's maximum principle controls and allows one essentially narrow the set of singular controls suspected for optimality.

Note that the scheme used in this paper is applicable for more general classes of problems.

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