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# Mathematical Model of Heat Transfer in Opening Electrical Contacts with Tunnel Effect

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**Abstract.** The mathematical model describing the dynamics of heating in opening electrical contacts is presented. It takes into account the imperfect thermal contact between anode and cathode due to tunnel effect. The model is based on the system of spherical heat equations in a domain with moving boundary. The analytical solution is found in the form of series containing the integral error functions.

## 1. Introduction

The tunnel effect plays an important role in many fields of engineering applications and modern technologies such as electric vacuum devices, semi-conducting materials, super-conducting contacts, technology of thin films etc. Various problems of tunnel conductivity, magnitude of tunnel current, tunnel voltage across a contact film as well as properties of varied films have been investigated and much progress made in the understanding of these characteristics.

The problem of contact heating owing to tunnel effect is one of them, and it was M. Kohler who discovered theoretically that an analogue to the well-known Wiedemann-Franz' law holds for the electric and thermal currents through the film [8]. It enables to estimate a possible temperature difference between cathode and anode. Now this phenomenon of contact overheating owing to tunnel effect is known in the literature as Kohler effect. Further investigations carried out by R.Holm concerning the influence of thin films on contact superconductivity [1] as well as results of other investigators (I. Dietrich, P. Kisliuk, W. Meissner et al.) led to the conclusion that the contribution of tunnel effect to contact heating is visible only at low current and small contact loading. It has to be noted that such estimations are based on the integral balance of heat transfer between anode and cathode. So they are rather rough and may be used successfully at the limited range of current.

The modern tendency to use extra-low range of the current (about and less) in many fields of engineering leads to the situation when investigators are faced with new serious problems and phenomena that remain

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obscured at ordinary current range but come to the forefront if the range is extra-low. The Kohler effect is one of them. The overheating of anode in comparison with cathode due Kohler effect (not very important at ordinary conditions) is essential at extra-low current, as it will be shown below. It leads to the thermal asymmetry in the microscopic molten bridge that appears between two electrodes during their opening just before arc ignition. If cathode and anode are of the same material the point of the bridge with maximum temperature is displaced toward anode, and when the temperature rises to the boiling point the bridge breaks at this point, and molten metal transfers from anode to cathode. After certain cycles of opening and closing operations one can see the formation of very thin pips (or spires) on the cathode and craters on the anode. This phenomenon, called bridge erosion, is very dangerous for micro-relays. Reducing of the current diminishes the volumes of the spires but not their height. Even the very thin spires are able to make sensitive micro-relays unreliable.

The hypothesis that Kohler effect may be responsible for the bridge erosion of electrical contacts was proposed by E. Justi and H. Schultz [2]. New aspects of this theory and corresponding mathematical models are developed in the papers [3, 6] and some methods for solution of the problem (mainly numerical) are elaborated.

Mathematical modeling is very important for the understanding and calculation of relative contributions of Joule and Kohler components of contact heating and bridge erosion because experimental study of this phenomena is very difficult, sometimes on account of microscopic size of a bridge $\left(10^{-8} - 10^{-6}\ m\right)$ at extra-low current.

This paper is devoted to the development of previous mathematical models and some new analytical approach to the solution of the problem.

## 2. Problem Statement

Let us consider a circular contact spot between cathode and anode covered with a thin (a few$\overset{0}{A}$,) chemisorbed or adhesive films that are penetrated by the conduction electrons by means of tunnel effect. The electrons don't alter their energy level when tunneling. Since they land in anode with a lower negative potential than at the cathode, they have a surplus kinetic energy there. The increment of kinetic energy is given off as heat creating the source on the interface between anode and film with specific capacity

$$\Pi_0 = j \cdot u_f = j^2 \cdot \sigma_f, \tag{1}$$

where $j$ is current density in contact spot, $u_f$ is the tunnel voltage across the film, $\sigma_f$ is the electric tunnel resistivity per unit area of the film. A portion of liberated heat with specific capacity

$$\Pi_1 = \frac{\Theta_f}{W}, \tag{2}$$

where $\Theta_f$ is the temperature difference and $W$ is the specific thermal resistance across the film, flows back to the cathode through the film whereas the remainder

$$\Pi_2 = \Pi - \frac{\Theta_f}{W}, \tag{3}$$

flows into the anode.

Overheating of anode by tunneling electrons is called Kohler effect. The magnitude of tunnel resistance $\sigma_f$ depends on the contact materials, particularly on work function $\Phi$, radius of contact spot $f$ and thickness of the film $d$.

Specific thermal resistance of the film $W$ is given by

$$W = \frac{d}{\lambda_f}, \tag{4}$$

where $\lambda_f$ is the significant parameter of Kohler effect. However, this formula is of little use for practical calculations because of very poor information concerning thermal conductivity of the film $\lambda_f$. But combining Kohler and Wiedemann-Franz' laws $\lambda_f \rho_f = \lambda\rho = LT$, where $\rho_f$ and $\rho$ are electrical resistivity of the film and the metal respectively, $\lambda$ is the thermal conductivity of the contact metal, and using the relation $\sigma_f = \rho_f d$ we can obtain instead of (4) the expression

$$W = \frac{\sigma_f}{\lambda_\rho},$$

that is much more efficient.

The radius of the contact spot at opening decreases according to the expression

$$b(t) = b_0\sqrt{1 - \frac{t^2}{T^2}} = b_0 \sum_{n=0}^{\infty} \frac{t^{2n}}{2n!T^{2n}}. \tag{5}$$

The temperature field for the cathode $\theta_1(r, t)$ and for the anode $\theta_2(r, t)$ can be described in spherical model by the equations

$$\begin{cases} \frac{\partial\theta_1}{\partial t} = a^2\left(\frac{\partial^2\theta_1}{\partial r^2} + \frac{2}{r}\frac{\partial\theta_1}{\partial r} + \frac{q}{r^4}\right), & t > 0 \\ \frac{\partial\theta_2}{\partial t} = a^2\left(\frac{\partial^2\theta_2}{\partial r^2} + \frac{2}{r}\frac{\partial\theta_2}{\partial r} + \frac{q}{r^4}\right). & b\,(t) < r < \infty \end{cases} \tag{6}$$

The initial conditions should be obtained by the solution of corresponding stationary heat equations at the stage preceding contact opening:

$$\begin{aligned} \theta_1\,(r, 0) &= -\tfrac{q}{2r^2} + \tfrac{A_1}{r}, \\ \theta_2\,(r, 0) &= -\tfrac{q}{2r^2} + \tfrac{A_2}{r}. \end{aligned} \tag{7}$$

where

$$A_1 = \frac{q}{b_0}\left(1 + \frac{\xi}{1+\xi}\right), \quad A_2 = \frac{q}{b_0}\left(1 + \frac{\xi\,(1+2\xi)}{1+\xi}\right), \quad \xi = \frac{\Pi_0 b_0^3}{2\lambda q} = \frac{\lambda W}{2b_0}. \tag{8}$$

The boundary conditions at infinity are

$$\frac{\partial\theta_1(\infty, t)}{\partial r} = \frac{\partial\theta_2(\infty, t)}{\partial r} = 0. \tag{9}$$

At the moving boundary $r = b(t)$ we should write the conditions for the imperfect contact

$$\Pi_1 = -\lambda\frac{\partial\theta_1(b, t)}{\partial r} = \frac{\theta_2(b, t) - \theta_1(b, t)}{W}, \tag{10}$$

$$\Pi_2 = -\lambda\frac{\partial\theta_2(b, t)}{\partial r} = \Pi_0 - \frac{\theta_2(b, t) - \theta_1(b, t)}{W}. \tag{11}$$

## 3. Problem Solution

Let us introduce the new unknown functions $U(r, t)$ and $V(r, t)$ instead of the previous functions $\theta_1\,(r, t)$ and $\theta_2\,(r, t)$ using expressions

$$\begin{aligned} \theta_1(r, t) &= \tfrac{U(r,t) - V(r,t)}{2r} - \tfrac{q}{2r^2}, \\ \theta_2(r, t) &= \tfrac{U(r,t) + V(r,t)}{2r} - \tfrac{q}{2r^2}. \end{aligned} \tag{12}$$

Heat equations, initial and boundary conditions can be written in the form

$$\begin{cases} \frac{\partial U}{\partial t} = a^2 \frac{\partial^2 U}{\partial r^2}, \\ \frac{\partial V}{\partial t} = a^2 \frac{\partial^2 V}{\partial r^2}, \end{cases} \tag{13}$$

$$\begin{cases} U(r,0) = A_1 + A_2, \\ V(r,0) = A_2 - A_1, \end{cases} \tag{14}$$

$$\begin{aligned} &\lim_{r\to\infty} \left( \frac{1}{2r} \left( \frac{\partial U(r,t)}{\partial r} - \frac{\partial V(r,t)}{\partial r} \right) - \frac{1}{r^2} \left( U(r,t) - V(r,t) \right) + \frac{q}{r^3} \right) = 0, \\ &\lim_{r\to\infty} \left( \frac{1}{2r} \left( \frac{\partial U(r,t)}{\partial r} + \frac{\partial V(r,t)}{\partial r} \right) - \frac{1}{r^2} \left( U(r,t) + V(r,t) \right) + \frac{q}{r^3} \right) = 0, \end{aligned} \tag{15}$$

$$\begin{aligned} &b^3 \frac{\partial U(b,t)}{\partial r} - b^2 U(b,t) = -q\lambda W - 2qb, \\ &b^3 \frac{\partial V(b,t)}{\partial r} - b^2 \left( \frac{2b}{\lambda W} + 1 \right) V(b,t) = -q\lambda W. \end{aligned} \tag{16}$$

The analytical solutions of this problem are:

$$U(r,t) = A_1 + A_2 + \sum_{n=0}^{\infty} B_n (\sqrt{t})^n i^n erfc \left( \frac{r}{2a\sqrt{t}} \right), \tag{17}$$

$$V(r,t) = A_2 - A_1 + \sum_{n=0}^{\infty} C_n (\sqrt{t})^n i^n erfc \left( \frac{r}{2a\sqrt{t}} \right). \tag{18}$$

The boundary conditions (16) take the form

$$b^3 \left( -\sum_{n=0}^{\infty} B_n (\sqrt{t})^{n-1} i^{n-1} erfc \left( \frac{b}{2a\sqrt{t}} \right) \right) - b^2 \left( A_1 + A_2 + \sum_{n=0}^{\infty} B_n (\sqrt{t})^n i^n erfc \left( \frac{b}{2a\sqrt{t}} \right) \right) = -qW\lambda - 2qb, \tag{19}$$

$$b^3 \left( -\sum_{n=0}^{\infty} C_n (\sqrt{t})^{n-1} i^{n-1} erfc \left( \frac{b}{2a\sqrt{t}} \right) \right) - b^2 \left( \frac{2b}{\lambda W} + 1 \right) \left( A_2 - A_1 + \sum_{n=0}^{\infty} C_n (\sqrt{t})^n i^n erfc \left( \frac{b}{2a\sqrt{t}} \right) \right) = -qW\lambda. \tag{20}$$

To calculate $B_n, C_n$ coefficients we apply Leibniz formula (twice), Faa Di Bruno's formula, Bell polynomials, make substitutions $\sqrt{t} = \tau$, $r = r - b_0$ and multiply both sides of (19), (20) by $\tau$, which yield

$$r_{b(t)} = b(t) - b_0 = -b_0 + b_0 \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)! T^{2n}} = b_0 \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)! T^{2n}}.$$

Let us denote

$$r_{b(t)} = \beta(t) \text{ or } \beta(\tau) = b_0 \sum_{n=1}^{\infty} \frac{\tau^{4n}}{(2n)! T^{2n}}$$

and let

$$\alpha(\tau) = \frac{\beta(\tau)}{2a\tau} = \frac{b_0 \sum_{n=1}^{\infty} \frac{\tau^{4n}}{(2n)! T^{2n}}}{2a\tau} = \frac{b_0}{2a} \sum_{n=1}^{\infty} \frac{\tau^{4n-1}}{(2n)! T^{2n}}.$$

Equations (19) and (20) are transformed to following equations

$$\mu a + \nu b = qW\lambda\tau + 2q\beta\tau, \tag{21}$$

$$\mu c + \eta d = qW\lambda\tau, \tag{22}$$

where

$$a=\left(\sum_{n=0}^{\infty}B_n\tau^n i^{n-1}erfc\left(\alpha(\tau)\right)\right),b=\left(A_1\tau+A_2\tau+\sum_{n=0}^{\infty}B_n\tau^{n+1}i^n erfc\left(\alpha(\tau)\right)\right),$$

$$c=\sum_{n=0}^{\infty}C_n\tau^n i^{n-1}erfc\left(\alpha(\tau)\right),d=\left(A_2\tau-A_1\tau+\sum_{n=0}^{\infty}C_n\tau^{n+1}i^n erfc\left(\alpha(\tau)\right)\right),$$

$$\mu=\beta^3,\nu=\beta^2,\eta=\beta^2\left(\frac{2\beta}{\lambda W}+1\right).$$

It is not difficult to see that coefficients of series $\mu$ and $\eta$ can be calculated by combining like terms and by formula

$$\left(\sum_{n=0}^{\infty}a_k x^k\right)^n=\sum_{k=0}^{\infty}c_k x^k \text{ where } c_m=\tfrac{1}{ma_0}\textstyle\sum_{k=1}^{m}\left(kn-m+k\right)a_k c_{m-k},\ c_0=a_0^n.$$

After taking $n$-th derivative from both sides of equations (21), (22) and utilizing Leibniz's formula twice for products $\mu a$, $\nu b$, $\mu c$, $\eta d$ and

$$\left.\frac{\partial^k\left[2^{\frac{m}{2}}\tau^m i^m erfc\alpha\right]}{\partial\tau^k}\right|_{\tau=0}=\begin{cases}0,k<m\\ \frac{2^{\frac{m}{2}}k!}{(k-m)!}\left[i^m erfc\alpha\right]^{(k-m)},k\geq m\end{cases}$$

where by Faa Di Bruno's formula and Bell polynomials we have

$$\left.\frac{\partial^{(k-m)}\left[i^m erfc\left(\pm\alpha\right)\right]}{\partial\tau^{k-m}}\right|_{\tau=0}=\sum_{p=1}^{k-m}\left.\left(i^m erfc\left(\pm\alpha\right)\right)^{(p)}\right|_{\alpha=0}B_{k-m,p}\left.\left(\alpha'\left(\tau\right)\alpha''\left(\tau\right),...,\alpha^{(k-m-p+1)}\left(\tau\right)\right)\right|_{\tau=0},$$

where

$$\left.\left[i^m erfc\left(\pm\alpha\right)\right]^{(p)}\right|_{\alpha=0}=(-1)^p\,i^{m-p}erfc0=(\mp1)^p\,\frac{\Gamma\left(\frac{m-p+1}{2}\right)}{(m-p)!\,\sqrt{\pi}},$$

$$B_{k-m,p}=\sum\frac{(k-m)!}{j_1!j_2!...j_{(k-m-p+1)}!}\alpha'\left(0\right)^{j_1}\alpha''\left(0\right)^{j_2}...\alpha^{(k-m-p+1)}\left(0\right)^{j_{k-m-p+1}},$$

$j_1, j_2$... satisfy the following conditions

$$j_1+j_2+...+j_{k-m-p+1}=p,$$

$$j_1+2j_2+...+\left(k-m-p+1\right)j_{k-m-p+1}=k-m,$$

we ultimately get $B_n, C_n$ coefficients from following recurrent formulas

$$\begin{aligned}&\textstyle\sum_{k=0}^{n}\binom{n}{k}\mu^{(n-k)}\sum_{m=0}^{k}B_m m!\binom{k}{k-m}\sum_{p=1}^{k-m}i^{m-p-1}erfc\alpha(0)\sum\frac{(k-m)!}{j_1!j_2!...j_{(k-m-p+1)}!}\\ &\alpha'\left(0\right)^{j_1}\alpha''\left(0\right)^{j_2}...\alpha^{(k-m-p+1)}\left(0\right)^{j_{k-m-p+1}}+\left.\left(\nu\left(A_1\tau+A_2\tau\right)\right)^{(n)}\right|_{\tau=0}\\ &+\textstyle\sum_{k=0}^{n}\binom{n}{k}\nu^{(n-k)}\sum_{m=0}^{k}B_m(m+1)!\binom{k}{k-m-1}\sum_{p=1}^{k-m-1}i^{m-p}erfc\alpha(0)\\ &\textstyle\sum\frac{(k-m)!}{j_1!j_2!...j_{(k-m-p)}!}\alpha'\left(0\right)^{j_1}\alpha''\left(0\right)^{j_2}...\alpha^{(k-m-p)}\left(0\right)^{j_{k-m-p}}=\left.\left(qW\lambda\tau+2q\beta\left(\tau\right)\right)^{(n)}\right|_{\tau=0},\end{aligned}\tag{23}$$

$$\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\mu^{(n-k)}\sum_{m=0}^{k}C_m m!\begin{pmatrix} k \\ k-m \end{pmatrix}\sum_{p=1}^{k-m} i^{m-p-1}erfc\alpha(0)\sum \frac{(k-m)!}{j_1!j_2!...j_{(k-m-p+1)}!}$$
$$\alpha'(0)^{j_1}\,\alpha''(0)^{j_2}...\alpha^{(k-m-p+1)}(0)^{j_{k-m-p+1}} + \left(\eta\left(A_2\tau - A_1\tau\right)\right)^{(n)}\Big|_{\tau=0}$$
$$+\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\eta^{(n-k)}\sum_{m=0}^{k}C_m(m+1)!\begin{pmatrix} k \\ k-m-1 \end{pmatrix}\sum_{p=1}^{k-m-1} i^{m-p}erfc\alpha(0) \tag{24}$$
$$\sum \frac{(k-m)!}{j_1!j_2!...j_{(k-m-p)}!}\alpha'(0)^{j_1}\,\alpha''(0)^{j_2}...\alpha^{(k-m-p)}(0)^{j_{k-m-p}} = \left(qW\lambda\tau\right)^{(n)}\Big|_{\tau=0}.$$

Convergence of series (17), (18) can be proved similarly like in the papers [4, 5].

## 4. Conclusion

The method of integral error functions enable us to find the analytical solution of the spherical heat equations which describe the heat transfer in opening, imperfect electrical contacts including tunnel effect.

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