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On the Paranormed Space $\mathcal{L}(t)$ of Double Sequences

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Abstract. In this paper, we introduce the paranormed sequence space $\mathcal{L}(t)$ which is the generalization of the space $\mathcal{L}_q$ of all absolutely $q-$summable double sequences. We examine some topological properties of the space $\mathcal{L}(t)$ and determine its alpha-, beta- and gamma-duals. Finally, we characterize some classes of four-dimensional matrix transformations from the space $\mathcal{L}(t)$ into some spaces of double sequences.

1. Introduction and Notations

We denote the set of all complex valued double sequences by Ω, i.e.,

$$\Omega := \left\{x = (x_{kl}) : x_{kl} \in \mathbb{C} \text{ for all } k, l \in \mathbb{N}\right\},$$

which forms a vector space with coordinatewise addition and scalar multiplication; where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. Any vector subspace of Ω is called as *a double sequence space*. By $\mathcal{M}_u$, we denote the space of all bounded double sequences, that is,

$$\mathcal{M}_u := \left\{x = (x_{kl}) \in \Omega : \|x\|_\infty = \sup_{k,l\in\mathbb{N}} |x_{kl}| < \infty\right\}.$$

A double sequence $x = (x_{kl}) \in \Omega$ is called *convergent to L in the Pringsheim's sense* (shortly, $p-$convergent to L) if for every $\varepsilon > 0$ there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that $|x_{kl} - L| < \varepsilon$ for all $k, l > N$. It is well-known that a $p-$convergent double sequence need not be bounded. If additionally $x \in \mathcal{M}_u$, then x is called *boundedly convergent to L in the Pringsheim's sense* (shortly, $bp-$convergent to L). The spaces of all $p-$ and $bp-$convergent double sequences are denoted by $\mathcal{C}_p$ and $\mathcal{C}_{bp}$, respectively. A double sequence $x = (x_{kl}) \in \mathcal{C}_p$ is said to be *regularly convergent to L* (shortly, $r-$convergent to L) if the limits

$$x_k := \lim_{l\to\infty} x_{kl}\ (k \in \mathbb{N}) \quad \text{and} \quad x^l := \lim_{k\to\infty} x_{kl}\ (l \in \mathbb{N})$$

exist. Note that, in this case $\lim_{k\to\infty} x_k = \lim_{l\to\infty} x^l = L$, where L is the $p-$limit of x. As seen, in addition to the $p-$convergence, the $r-$convergence requires the convergence of rows and columns of a double sequence, and so it is bounded. By $\mathcal{C}_r$, we denote the space of all $r-$convergent double sequences.

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Also the spaces of all null double sequences contained in C_p, C_{bp} and C_r are denoted by C_{p0}, C_{bp0} and C_{r0}, respectively. Referring to Móricz [14], the sets $\mathcal{M}_u$, C_{bp}, C_r, C_{bp0} and C_{r0} are Banach spaces with the norm $\|\cdot\|_\infty$.

Throughout the text, ϑ denotes any of the symbols p, bp or r, and any summation without limits runs from 0 to ∞, for example $\sum_{k,l}$ means $\sum_{k,l=0}^{\infty}$. The sum of a double series $\sum_{k,l} x_{kl}$ with respect to $\vartheta-$convergence rule is defined by $\vartheta - \sum_{k,l} x_{kl} = \vartheta - \lim_{m,n\to\infty} \sum_{k,l=0}^{m,n} x_{kl}$. If there is no confusion, we shall use $\sum_{k,l} x_{kl}$ instead of $\vartheta - \sum_{k,l} x_{kl}$.

The space $\mathcal{L}_q$ of all absolutely $q-$summable double sequences is introduced by Başar and Sever [5], that is,

$$\mathcal{L}_q := \left\{ x = (x_{kl}) \in \Omega : \sum_{k,l} |x_{kl}|^q < \infty \right\}; \;\; (0 < q < \infty).$$

By taking $q = 1$, we obtain the space $\mathcal{L}_u$ of all absolutely summable double sequences.

For more information on double sequence spaces and related topics one can also see [6–8, 10, 13, 15–18, 20–23].

The elementary double seqeunces $\mathbf{e^{kl}} = \left(\mathbf{e}^{\mathbf{kl}}_{ij}\right)$, $\mathbf{e}$, $\mathbf{e_k}$ and $\mathbf{e^l}$ are defined, as follows;

$$\mathbf{e}^{\mathbf{kl}}_{ij} := \begin{cases} 1 & , \quad (i,j) = (k,l), \\ 0 & , \quad (i,j) \neq (k,l), \end{cases}$$

$\mathbf{e} := \sum_{k,l} \mathbf{e^{kl}}$; the double sequence that all terms are one,

$\mathbf{e_k} := \sum_{l} \mathbf{e^{kl}}$; the double sequence that all terms of k-th row are one and other terms are zero,

$\mathbf{e^l} := \sum_{k} \mathbf{e^{kl}}$; the double sequence that all terms of l-th column are one and other terms are zero.

We denote the set of all finitely non-zero double sequences by Φ, i.e.,

$$\begin{aligned} \Phi &:= \left\{ x = (x_{kl}) \in \Omega : \exists\, N \in \mathbb{N} \ni \forall\, (k,l) \in \mathbb{N}^2 \backslash [0,N]^2,\ x_{kl} = 0 \right\} \\ &:= \operatorname{span}\left\{ \mathbf{e^{kl}} : k,l \in \mathbb{N} \right\}. \end{aligned}$$

Let X be a real or complex linear space and g be a function from X to the set $\mathbb{R}$ of real numbers. Then, the pair (X, g) or shortly X is called a paranormed space and g is a paranorm for X, if the following axioms are satisfied for all elements $x, y \in X$:

(i) $g(x) \geq 0$,

(ii) $g(x) = 0$ if $x = \theta$, where θ is the zero vector in X,

(iii) $g(x) = g(-x)$,

(iv) $g(x + y) \leq g(x) + g(y)$,

(v) Scalar multiplication is continuous, i.e., $|\alpha_i - \alpha| \to 0$ and $g(x^i - x) \to 0$ imply $g(\alpha_i x^i - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and all x's in X.

Throughout the text, $t = (t_{kl})$ denotes any double sequence of strictly positive real numbers. We define the double sequence space $\mathcal{L}(t)$, as follows:

$$\mathcal{L}(t) := \left\{ x = (x_{kl}) \in \Omega : \sum_{k,l} |x_{kl}|^{t_{kl}} < \infty \right\}.$$

Clearly, $\mathcal{L}(\mathbf{e}) = \mathcal{L}_u$ and $\mathcal{L}(q\mathbf{e}) = \mathcal{L}_q$; where $0 < q < \infty$.

Let $H = \sup_{k,l\in\mathbb{N}} t_{kl} < \infty$ and $M = \max\{1, H\}$. Now, one can easily check by similar approach used for single sequences that the set $\mathcal{L}(t)$ is complete paranormed space with the paranorm

$$g(x) = \left(\sum_{k,l} |x_{kl}|^{t_{kl}} \right)^{1/M}.$$

2. Dual Spaces of $\mathcal{L}(t)$

In the present section, we determine the dual spaces of the space $\mathcal{L}(t)$. It is important to notice that although the alpha- and gamma-duals of a double sequence space are unique, its beta-dual may be more than one with respect to $\vartheta-$convergence rule. In the rest of the study, ζ denotes any of the symbols α, $\beta(\vartheta)$ or γ and also $\lambda^{n\zeta}$ means that $\left\{\lambda^{(n-1)\zeta}\right\}^{\zeta}$ for a double sequence space λ and $n \in \mathbb{N}_1$, the set of positive integers.

The *α-dual* λ^{α}, the *$\beta(\vartheta)$-dual* $\lambda^{\beta(\vartheta)}$ and *γ-dual* λ^{γ} of a double sequence space λ are defined by

$$\begin{aligned}
\lambda^{\alpha} &:= \left\{a = (a_{kl}) \in \Omega : \sum_{k,l} |a_{kl}x_{kl}| < \infty \text{ for all } x = (x_{kl}) \in \lambda\right\}, \\
\lambda^{\beta(\vartheta)} &:= \left\{a = (a_{kl}) \in \Omega : \vartheta - \sum_{k,l} a_{kl}x_{kl} \text{ exists for all } x = (x_{kl}) \in \lambda\right\}, \\
\lambda^{\gamma} &:= \left\{a = (a_{kl}) \in \Omega : \sup_{m,n\in\mathbb{N}} \left|\sum_{k,l=0}^{m,n} a_{kl}x_{kl}\right| < \infty \text{ for all } x = (x_{kl}) \in \lambda\right\}.
\end{aligned}$$

Definition 2.1. ([21, p. 225]) A double sequence space λ containing Φ is said to be *monotone* if $xu = (x_{kl}u_{kl}) \in \lambda$ for every $x = (x_{kl}) \in \lambda$ and $u = (u_{kl}) \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$, where $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ denotes the set of all sequences consisting of 0's and 1's. If λ is monotone, then $\lambda^{\alpha} = \lambda^{\beta(p)} = \lambda^{\beta(bp)} = \lambda^{\beta(r)} \subset \lambda^{\gamma}$.

Definition 2.2. ([5, p. 154]) A double sequence space λ is called *solid* if

$$\left\{u = (u_{kl}) \in \Omega : \exists\, x = (x_{kl}) \in \lambda \ \ni\ |u_{kl}| \leq |x_{kl}| \text{ for all } k,l \in \mathbb{N}\right\} \subset \lambda.$$

If λ is solid, then it is monotone and $\lambda^{\alpha} = \lambda^{\beta(p)} = \lambda^{\beta(bp)} = \lambda^{\beta(r)} = \lambda^{\gamma}$.

Now, one can easily observe that the set $\mathcal{L}(t)$ is solid. Therefore, to obtain $\zeta-$dual of $\mathcal{L}(t)$, it is sufficient to calculate its $\alpha-$, $\beta(\vartheta)-$ or $\gamma-$dual.

Definition 2.3. ([9, p. 342]) A sequence space λ is called *$\zeta-$space* if $\lambda = \lambda^{2\zeta}$. Further, an $\alpha-$space is also called *Köthe space* or *perfect sequence space*.

Since there are various convergence rules for double sequences (see [10] for other types of convergence), we give a new definition for $\beta-$space.

Definition 2.4. Let λ be a double sequence space and the symbols ν, ϑ denote any kind of convergence rule. Then, we call that λ is a *$\beta(\nu,\vartheta)-$space* if $\lambda = \left\{\lambda^{\beta(\nu)}\right\}^{\beta(\vartheta)}$ for fixed ν, ϑ's and is a *$\beta-$space* if $\lambda = \left\{\lambda^{\beta(\nu)}\right\}^{\beta(\vartheta)}$ for all ν, ϑ's. In this study, we only use this definition for $\nu, \vartheta \in \{p, bp, r\}$.

Theorem 2.5. *Let $0 < t_{kl} \leq 1$. Then, the $\zeta-$dual of the space $\mathcal{L}(t)$ is the set $\mathcal{M}_u(t)$, where*

$$\mathcal{M}_u(t) := \left\{x = (x_{kl}) \in \Omega : \sup_{k,l\in\mathbb{N}} |x_{kl}|^{t_{kl}} < \infty\right\}.$$

Proof. Let $0 < t_{kl} \leq 1$.

$\mathcal{M}_u(t) \subset \{\mathcal{L}(t)\}^{\alpha}$: Take $a = (a_{kl}) \in \mathcal{M}_u(t)$ and $x = (x_{kl}) \in \mathcal{L}(t)$. Then,

$$\sum_{k,l} |a_{kl}x_{kl}|^{t_{kl}} \leq \sup_{k,l\in\mathbb{N}} |a_{kl}|^{t_{kl}} \sum_{k,l} |x_{kl}|^{t_{kl}} < \infty,$$

i.e., $ax \in \mathcal{L}(t)$. Therefore, for a given $\varepsilon > 0$, there exists a positive integer $n_0 = n_0(\varepsilon)$ such that

$$\sum_{k,l=n_0+1,0}^{\infty,n_0} |a_{kl}x_{kl}|^{t_{kl}} + \sum_{k,l=0,n_0+1}^{n_0,\infty} |a_{kl}x_{kl}|^{t_{kl}} + \sum_{k,l=n_0+1}^{\infty} |a_{kl}x_{kl}|^{t_{kl}} < \varepsilon.$$

Then, for every $k > n_0$ or $l > n_0$ or both we have $|a_{kl}x_{kl}|^{t_{kl}} < \varepsilon < 1$ and so $|a_{kl}x_{kl}| < 1$. Thus, we obtain $|a_{kl}x_{kl}| \leq |a_{kl}x_{kl}|^{t_{kl}} < \varepsilon$ for such k and l's. This implies that $ax \in \mathcal{L}_u$, i.e., $a \in \{\mathcal{L}(t)\}^{\alpha}$.

$\{\mathcal{L}(t)\}^{\alpha} \subset \mathcal{M}_u(t)$: Suppose that $a = (a_{kl}) \in \{\mathcal{L}(t)\}^{\alpha} \setminus \mathcal{M}_u(t)$. Then, for every $x = (x_{kl}) \in \mathcal{L}(t)$ we have

$$\sum_{k,l} |a_{kl}x_{kl}| < \infty \quad \text{but} \quad \sup_{k,l\in\mathbb{N}} |a_{kl}|^{t_{kl}} = \infty.$$

In this case, there exist index sequences (k_i) and (l_i) of natural numbers, at least one of them is strictly increasing, such that

$$|a_{k_i,l_i}|^{t_{k_i,l_i}} \geq (i+1)^2$$

for all $i \in \mathbb{N}$. We define $x = (x_{kl}) \in \mathcal{L}(t)$ by

$$x_{kl} := \begin{cases} (i+1)^{-2/t_{kl}} & , \quad k = k_i \text{ and } l = l_i, \\ 0 & , \quad k \neq k_i \text{ or } l \neq l_i \end{cases}$$

for all $k, l \in \mathbb{N}$ which gives

$$\sum_{k,l} |a_{kl}x_{kl}| = \sum_i |a_{k_i,l_i}|(i+1)^{-2/t_{k_i,l_i}} \geq \sum_i 1 = \infty,$$

i.e., $a \notin \{\mathcal{L}(t)\}^{\alpha}$, a contradiction. Hence, a must be in $\mathcal{M}_u(t)$. □

It is known that $\mathcal{M}_u^{\zeta} = \mathcal{L}_u$ and $\mathcal{L}_u^{\zeta} = \mathcal{M}_u$. Now, we have the following corollary:

Corollary 2.6. *Let* $0 < q \leq 1$. *Then, the following statement holds for all* $k \in \mathbb{N}_1$*:*

$$\mathcal{L}_q^{n\zeta} := \begin{cases} \mathcal{M}_u & , \quad n = 2k-1, \\ \mathcal{L}_u & , \quad n = 2k. \end{cases}$$

Let $t = (t_{kl})$ and $s = (s_{kl})$ are connected with the relation

$$\frac{1}{t_{kl}} + \frac{1}{s_{kl}} = 1 \quad \text{for } t_{kl}, s_{kl} > 1.$$

In this case,

$$\frac{s_{kl}}{t_{kl}} = s_{kl} - 1, \quad \frac{t_{kl}}{s_{kl}} = t_{kl} - 1, \quad s_{kl}t_{kl} = s_{kl} + t_{kl}$$

and the inequality

$$|x_{kl}y_{kl}| \leq |x_{kl}|^{t_{kl}} + |y_{kl}|^{s_{kl}}$$

is satisfied for any $x = (x_{kl})$ and $y = (y_{kl})$ in Ω.

Unless stated otherwise, we take $t_{kl}, s_{kl} > 1$ for all $k, l \in \mathbb{N}$ in the rest of the section.

Now, we define the following solid set with an integer $N > 1$:

$$\mathcal{M}_0^{(t)}(s) := \bigcup_{N=2}^{\infty} \left\{ a = (a_{kl}) \in \Omega : \sum_{k,l} |a_{kl}|^{s_{kl}} N^{-s_{kl}/t_{kl}} < \infty \right\}.$$

Theorem 2.7. *The inclusion* $\mathcal{L}(s) \subset \mathcal{M}_0^{(t)}(s)$ *holds.*

Proof. The proof is easy. So, we omit the detail. □

Theorem 2.8. $\mathcal{L}(s) = \mathcal{M}_0^{(t)}(s)$ *if and only if* $s \in \mathcal{M}_u$.

Proof. Let $s = (s_{kl}) \in \mathcal{M}_u$. Then, we have $0 < \inf_{k,l\in\mathbb{N}} N^{-s_{kl}/t_{kl}} \leq \sup_{k,l\in\mathbb{N}} N^{-s_{kl}/t_{kl}} \leq 1$ for some integer $N > 1$. Now, it is easy to see that $\mathcal{L}(s) = \mathcal{M}_0^{(t)}(s)$.

Conversely, suppose that $\mathcal{L}(s) = \mathcal{M}_0^{(t)}(s)$ but $s = (s_{kl}) \notin \mathcal{M}_u$. Then, there exist index sequences (k_i) and (l_i) of natural numbers, at least one of them is strictly increasing such that $s_{k_i,l_i} \geq i+1$ for all $i \in \mathbb{N}$. Define $x = (x_{kl}) \notin \mathcal{L}(s)$ by

$$x_{kl} := \begin{cases} 1 & , \quad k = k_i \text{ and } l = l_i, \\ 0 & , \quad k \neq k_i \text{ or } l \neq l_i \end{cases}$$

for all $k, l \in \mathbb{N}$ which gives the fact

$$\sum_{k,l} |x_{kl}|^{s_{kl}} N^{-s_{kl}/t_{kl}} = \sum_i N^{-(s_{k_i,l_i}-1)} \leq \sum_i N^{-i} < \infty$$

for some integer $N > 1$, i.e., $x \in \mathcal{M}_0^{(t)}(s)$. This contradicts the hypothesis. Hence, $s \in \mathcal{M}_u$. $\square$

Theorem 2.9. *The ζ–dual of the space $\mathcal{L}(t)$ is the set $\mathcal{M}_0^{(t)}(s)$.*

Proof. $\mathcal{M}_0^{(t)}(s) \subset \{\mathcal{L}(t)\}^\alpha$: Let $a \in \mathcal{M}_0^{(t)}(s)$ and $x \in \mathcal{L}(t)$. Then, we get

$$\begin{aligned} \sum_{k,l} |a_{kl}x_{kl}| &= \sum_{k,l} |a_{kl}N^{-1/t_{kl}} x_{kl} N^{1/t_{kl}}| \\ &\leq \sum_{k,l} |a_{kl}|^{s_{kl}} N^{-s_{kl}/t_{kl}} + N \sum_{k,l} |x_{kl}|^{t_{kl}} < \infty \end{aligned}$$

for some integer $N > 1$, i.e., $a \in \{\mathcal{L}(t)\}^\alpha$.

$\{\mathcal{L}(t)\}^\alpha \subset \mathcal{M}_0^{(t)}(s)$: Suppose that $a \in \{\mathcal{L}(t)\}^\alpha \setminus \mathcal{M}_0^{(t)}(s)$. Then, for every $x \in \mathcal{L}(t)$ and all integers $N > 1$ we have

$$\sum_{k,l} |a_{kl}x_{kl}| < \infty \quad \text{but} \quad \sum_{k,l} |a_{kl}|^{s_{kl}} N^{-s_{kl}/t_{kl}} = \infty.$$

Then, there are following three possibilities:

(i) For fixed $l_0 \in \mathbb{N}$ there exists strictly increasing sequence (k_i) of natural numbers such that

$$\sum_{k=k_i+1}^{k_{i+1}} |a_{k,l_0}|^{s_{k,l_0}} (i+2)^{-s_{k,l_0}/t_{k,l_0}} > 1 \text{ or}$$

(ii) For fixed $k_0 \in \mathbb{N}$ there exists strictly increasing sequence (l_i) of natural numbers such that

$$\sum_{l=l_i+1}^{l_{i+1}} |a_{k_0,l}|^{s_{k_0,l}} (i+2)^{-s_{k_0,l}/t_{k_0,l}} > 1 \text{ or}$$

(iii) There exist strictly increasing sequences (k_i) and (l_i) of natural numbers such that

$$\mathbf{M}_i = \sum_{k,l=k_i+1,l_i+1}^{k_{i+1},l_{i+1}} |a_{kl}|^{s_{kl}} (i+2)^{-s_{kl}/t_{kl}} > 1.$$

We continue to the proof of the theorem with Case (iii). One can obtain the similar result for the other cases.

Now, we define $x = (x_{kl})$ by

$$x_{kl} := \begin{cases} |a_{kl}|^{s_{kl}-1} (i+2)^{-s_{kl}} \mathbf{M}_i^{-1} & , \quad k_i < k \leq k_{i+1} \text{ and } l_i < l \leq l_{i+1}, \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $k, l \in \mathbb{N}$. Then, one can see that

$$\begin{aligned}\sum_{k,l}|a_{kl}x_{kl}| &= \sum_i \mathbf{M}_i^{-1}\sum_{k,l=k_i+1,l_i+1}^{k_{i+1},l_{i+1}}|a_{kl}|^{s_{kl}}(i+2)^{-s_{kl}}\\ &= \sum_i \mathbf{M}_i^{-1}(i+2)^{-1}\sum_{k,l=k_i+1,l_i+1}^{k_{i+1},l_{i+1}}|a_{kl}|^{s_{kl}}(i+2)^{-s_{kl}/t_{kl}}\\ &= \sum_i (i+2)^{-1} = \infty,\end{aligned}$$

i.e., $a \notin \{\mathcal{L}(t)\}^{\alpha}$. But, by the inequalities

$$\begin{aligned}|a_{kl}|^{(s_{kl}-1)t_{kl}}(i+2)^{-s_{kl}t_{kl}}\mathbf{M}_i^{-t_{kl}} &\leq |a_{kl}|^{(s_{kl}-1)t_{kl}}(i+2)^{-s_{kl}t_{kl}}\mathbf{M}_i^{-1}\\ &= |a_{kl}|^{s_{kl}}(i+2)^{-s_{kl}-t_{kl}+2}(i+2)^{-2}\mathbf{M}_i^{-1}\\ &\leq |a_{kl}|^{s_{kl}}(i+2)^{-s_{kl}/t_{kl}}(i+2)^{-2}\mathbf{M}_i^{-1}\end{aligned}$$

we conclude that

$$\sum_{k,l}|x_{kl}|^{t_{kl}} \leq \sum_i (i+2)^{-2} < \infty,$$

i.e., $x \in \mathcal{L}(t)$, a contradiction. Hence, $a \in \mathcal{M}_0^{(t)}(s)$. $\square$

Theorem 2.10. *Let $t \in \mathcal{M}_u$. Then, the ζ−dual of the space $\mathcal{M}_0^{(t)}(s)$ is the set $\mathcal{L}(t)$.*

Proof. $\mathcal{L}(t) \subset \left\{\mathcal{M}_0^{(t)}(s)\right\}^{\alpha}$: This is similar to the proof of the inclusion $\mathcal{M}_0^{(t)}(s) \subset \{\mathcal{L}(t)\}^{\alpha}$ in Theorem 2.9.

$\left\{\mathcal{M}_0^{(t)}(s)\right\}^{\alpha} \subset \mathcal{L}(t)$: Suppose that $a \in \left\{\mathcal{M}_0^{(t)}(s)\right\}^{\alpha} \setminus \mathcal{L}(t)$. Then, we have $\sum_{k,l}|a_{kl}|^{t_{kl}} = \infty$. As in the proof of Theorem 2.9, there are three cases. We give the proof only for one case.

There exist strictly increasing sequences (k_i) and (l_i) of natural numbers such that

$$\mathbf{M}_i = \sum_{k,l=k_i+1,l_i+1}^{k_{i+1},l_{i+1}}|a_{kl}|^{t_{kl}} > 2^i.$$

We define $x = (x_{kl})$ by

$$x_{kl} := \begin{cases} |a_{kl}|^{t_{kl}-1}\mathbf{M}_i^{-1} & , \quad k_i < k \leq k_{i+1} \text{ and } l_i < l \leq l_{i+1},\\ 0 & , \quad \text{otherwise}\end{cases}$$

for all $k, l \in \mathbb{N}$ which gives

$$\sum_{k,l}|a_{kl}x_{kl}| = \sum_i \mathbf{M}_i^{-1}\sum_{k,l=k_i+1,l_i+1}^{k_{i+1},l_{i+1}}|a_{kl}|^{t_{kl}} = \sum_i 1 = \infty,$$

i.e., $a \notin \left\{\mathcal{M}_0^{(t)}(s)\right\}^{\alpha}$. Take $\inf_{k,l\in\mathbb{N}} s_{kl} = h$. Then, for some integer $N > 1$, we get that

$$\begin{aligned}
\sum_{k,l} |x_{kl}|^{s_{kl}} N^{-s_{kl}/t_{kl}} &= \sum_i \sum_{k,l=k_i+1,l_i+1}^{k_{i+1},l_{i+1}} |a_{kl}|^{(t_{kl}-1)s_{kl}} \mathbf{M}_i^{-s_{kl}} N^{-s_{kl}/t_{kl}} \\
&\leq \sum_i \sum_{k,l=k_i+1,l_i+1}^{k_{i+1},l_{i+1}} |a_{kl}|^{t_{kl}} \mathbf{M}_i^{-s_{kl}} \\
&= \sum_i \mathbf{M}_i^{-1} \sum_{k,l=k_i+1,l_i+1}^{k_{i+1},l_{i+1}} |a_{kl}|^{t_{kl}} \mathbf{M}_i^{-s_{kl}+1} \\
&\leq \sum_i \mathbf{M}_i^{-1} \sum_{k,l=k_i+1,l_i+1}^{k_{i+1},l_{i+1}} |a_{kl}|^{t_{kl}} 2^{i(-s_{kl}+1)} \\
&\leq \sum_i 2^{i(-h+1)} < \infty,
\end{aligned}$$

i.e., $x \in \mathcal{M}_0^{(t)}(s)$. This contradicts the hypothesis. Hence, $a \in \mathcal{L}(t)$. □

Corollary 2.11. *Let $t, s \in \mathcal{M}_u$. Then, the following statement holds for all $k \in \mathbb{N}_1$:*

$$\{\mathcal{L}(t)\}^{n\zeta} := \begin{cases} \mathcal{L}(s) & , \quad n = 2k-1, \\ \mathcal{L}(t) & , \quad n = 2k. \end{cases}$$

Now, we can give the following corollary:

Corollary 2.12. *The following statement holds for all $k \in \mathbb{N}_1$:*

$$\mathcal{L}_q^{n\zeta} := \begin{cases} \mathcal{L}_s & , \quad n = 2k-1, \\ \mathcal{L}_q & , \quad n = 2k. \end{cases}$$

Also, one can easily derive the following two corollaries:

Corollary 2.13. *Let $0 < t_{kl} < 1$. Then, the set $\mathcal{L}(t)$ is not a ζ–space.*

Corollary 2.14. *Let $1 \leq t_{kl} \leq \sup_{k,l\in\mathbb{N}} t_{kl} < \infty$. Then, the set $\mathcal{L}(t)$ is a ζ–space.*

3. Matrix Transformations

Let λ and μ be two double sequence spaces, and $A = (a_{mnkl})_{m,n,k,l\in\mathbb{N}}$ be any four-dimensional complex infinite matrix. Then, we say that A defines a *matrix transformation* from λ into μ and we write $A : \lambda \to \mu$, if for every sequence $x = (x_{kl}) \in \lambda$ the A-transform $Ax = \{(Ax)_{mn}\}_{m,n\in\mathbb{N}}$ of x exists and belongs to μ; where

$$(Ax)_{mn} = \vartheta - \sum_{k,l} a_{mnnk} x_{kl} \quad \text{for each } m, n \in \mathbb{N}. \tag{1}$$

We define the *ϑ-summability domain* $\lambda_A^{(\vartheta)}$ of A in a space λ of double sequences by

$$\lambda_A^{(\vartheta)} := \left\{ x = (x_{kl}) \in \Omega : Ax = \left(\vartheta - \sum_{k,l} a_{mnkl} x_{kl} \right)_{m,n\in\mathbb{N}} \text{ exists and is in } \lambda \right\}.$$

We say with the notation of (1) that A *maps* the space λ into the space μ if $\lambda \subset \mu_A^{(\vartheta)}$ and we denote the set of all four-dimensional matrices, transforming the space λ into the space μ, by $(\lambda : \mu)$. Thus, $A = (a_{mnkl}) \in (\lambda : \mu)$ if and only if the double series on the right side of (1) converges in the sense of ϑ for each $m, n \in \mathbb{N}$, i.e,

$A_{mn} \in \lambda^{\beta(\vartheta)}$ for each $m, n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax \in \mu$ for all $x \in \lambda$; where $A_{mn} = (a_{mnkl})_{k,l\in\mathbb{N}}$ for each $m, n \in \mathbb{N}$. Here, we note concerning with four-dimensional matrix transformations that ϑ must be fixed, otherwise the results may be incorrect. In this paper, we do not fix ϑ since the $\beta(\vartheta)-$duals of corresponding spaces are identical.

For all $m, n, k, l \in \mathbb{N}$, we say that $A = (a_{mnkl})$ is *a triangular matrix* if $a_{mnkl} = 0$ for $k > m$ or $l > n$ or both, [1]. Following Adams [1], we also say that a triangular matrix $A = (a_{mnkl})$ is called a *triangle* if $a_{mnmn} \neq 0$ for all $m, n \in \mathbb{N}$. Referring to Cooke [11, Remark (a), p. 22], one can conclude that every triangle matrix has an unique inverse which is also a triangle.

Theorem 3.1. *Let $t = (t_{kl})$ be any double sequence of strictly positive real numbers. Then, the necessary and sufficient conditions for $A \in (X : Y)$ can be read from the following table:*

$X \downarrow$ $\quad Y \rightarrow$	$\mathcal{M}_u$	$\mathcal{C}_\vartheta$	$\mathcal{C}_{\vartheta 0}$
$\mathcal{L}(t)$ $(0 < t_{kl} \leq 1)$	**1.**	**2.**	**3.**
$\mathcal{L}(t)$ $(1 < t_{kl} < \infty)$	**4.**	**5.**	**6.**

where

1.

$$\sup_{m,n,k,l\in\mathbb{N}} |a_{mnkl}| < \infty. \tag{2}$$

2. *(2) and*

$$\exists\, a_{kl} \in \mathbb{C} \ni \quad \vartheta - \lim_{m,n\to\infty} a_{mnkl} = a_{kl} \quad \textit{for each } k, l \in \mathbb{N}. \tag{3}$$

3. *(2) and*

$$A^{kl} = (a_{mnkl})_{m,n\in\mathbb{N}} \in \mathcal{C}_{\vartheta 0} \quad \textit{for each } k, l \in \mathbb{N}. \tag{4}$$

4.

$$D = \sup_{m,n\in\mathbb{N}} \sum_{k,l} |a_{mnkl}|^{s_{kl}} N^{-s_{kl}/t_{kl}} < \infty \quad \textit{for some integer } N > 1. \tag{5}$$

5. *(3) and (5).*

6. *(4) and (5).*

Proof. We only give the proof of the class $(\mathcal{L}(t) : \mathcal{C}_\vartheta)$ for all $t_{kl} > 1$ for all $k, l \in \mathbb{N}$.

Necessity. The necessity of (5) is immediate from $\beta(\vartheta)-$dual of $\mathcal{L}(t)$. Besides, since the set $\left\{\mathbf{e^{kl}}; k, l \in \mathbb{N}\right\} \subset \mathcal{L}(t)$, $A\mathbf{e^{kl}} = A^{kl} = (a_{mnkl})_{m,n\in\mathbb{N}} \in \mathcal{C}_\vartheta$ for each $k, l \in \mathbb{N}$ by the hypothesis. Hence, the condition (3) is also necessary.

Sufficiency. Let the conditions (3) and (5) hold, and take $x \in \mathcal{L}(t)$. Then, there exists a positive constant K such that $\sum_{k,l} |x_{kl}|^{t_{kl}} \leq K$. So, we get that

$$\begin{aligned} \sum_{k,l} |a_{mnkl} x_{kl}| &\leq \sum_{k,l} |a_{mnkl}|^{s_{kl}} N^{-s_{kl}/t_{kl}} + N \sum_{k,l} |x_{kl}|^{t_{kl}} \\ &\leq D + NK < \infty \end{aligned}$$

for some integer $N > 1$ and all $m, n \in \mathbb{N}$. Thus, we see that the series $\sum_{k,l} a_{mnkl}x_{kl}$ converges absolutely, and similary $\sum_{k,l} a_{kl}x_{kl}$, too. Since $x \in \mathcal{L}(t)$, we can write that

$$\sum_{k,l=n_0+1,0}^{\infty,n_0} |x_{kl}|^{t_{kl}}, \quad \sum_{k,l=0,n_0+1}^{n_0,\infty} |x_{kl}|^{t_{kl}} \quad \text{and} \quad \sum_{k,l=n_0+1}^{\infty} |x_{kl}|^{t_{kl}} \tag{6}$$

are less than $\varepsilon/12(D + NK) < 1$ for some $n_0 \in \mathbb{N}$. Also, one can write by the condition (3) that

$$\sum_{k,l=0}^{n_0} |a_{mnkl} - a_{kl}||x_{kl}| < \frac{\varepsilon}{2} \tag{7}$$

for all sufficiently large m, n's. Thus, we obtain by (6) and (7) that

$$\begin{aligned} \left|\sum_{k,l} a_{mnkl}x_{kl} - \sum_{k,l} a_{kl}x_{kl}\right| &\leq \sum_{k,l=0}^{n_0} |a_{mnkl} - a_{kl}||x_{kl}| + \sum_{k,l=n_0+1,0}^{\infty,n_0} |a_{mnkl} - a_{kl}||x_{kl}| \\ &\quad + \sum_{k,l=0,n_0+1}^{n_0,\infty} |a_{mnkl} - a_{kl}||x_{kl}| + \sum_{k,l=n_0+1}^{\infty} |a_{mnkl} - a_{kl}||x_{kl}| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \varepsilon \end{aligned}$$

for all sufficiently large m, n's. This implies that $Ax \in \mathcal{C}_\vartheta$ for $\vartheta \in \{p, bp\}$. To obtain $Ax \in \mathcal{C}_r$, we also have to show that the rows and columns of Ax converges. Since $A^{kl} = (a_{mnkl})_{m,n\in\mathbb{N}} \in \mathcal{C}_r$ from (3), there exist some scalars a^n_{kl} and a^m_{kl} such that $\lim_{m\to\infty} a_{mnkl} = a^n_{kl}$ $(n \in \mathbb{N})$ and $\lim_{n\to\infty} a_{mnkl} = a^m_{kl}$ $(m \in \mathbb{N})$. Note that, $\lim_{n\to\infty} a^n_{kl} = \lim_{m\to\infty} a^m_{kl} = a_{kl}$. Now, replacing a_{kl} with a^m_{kl} and a^n_{kl} in the inequalities, above, we derive the desired result. This completes the proof. □

Now, we can give the following corollary:

Corollary 3.2. *The necessary and sufficient conditions for $A \in (X : Y)$ can be read from the following table:*

$X \downarrow$ $\quad Y \rightarrow$	$\mathcal{M}_u$	$\mathcal{C}_\vartheta$	$\mathcal{C}_{\vartheta 0}$
$\mathcal{L}_q$ $(0 < q \leq 1)$	**1.**	**2.**	**3.**
$\mathcal{L}_q$ $(1 < q < \infty)$	**7.**	**8.**	**9.**

where

7.

$$\sup_{m,n\in\mathbb{N}} \sum_{k,l} |a_{mnkl}|^s < \infty. \tag{8}$$

8. *(3) and (8).*

9. *(4) and (8).*

Conclusion

In this paper, we have worked on some algebraic and topological properties of the paranormed space $\mathcal{L}(t)$. This space were also studied by Gökhan and Çolak in [12]. However, there are some missing points in [12]. For instance, in the proof of Part (i) of Theorem 8 in [12], they wrote for a double sequence (p_{mn}) of strictly positive real numbers that if $\inf_{m,n\in\mathbb{N}_1} p_{mn} = 0$ then there are two cases:

(a) There exist strictly increasing sequence $(m(i))$ of positive integers and $n(1) < n(2) < \cdots < n(k_0)$ for some fixed $k_0 \in \mathbb{N}_1$ such that $p_{m(i),n(j)} < 1/i$ for all positive integers i and for $1 \leq j \leq k_0$ (or there exist strictly increasing sequence $(n(j))$ of positive integers and $m(1) < m(2) < \cdots < m(k_0)$ for some fixed $k_0 \in \mathbb{N}_1$ such that $p_{m(i),n(j)} < 1/j$ for all positive integers j and for $1 \leq i \leq k_0$) or

(b) There exist strictly increasing sequences $(m(i))$ and $(n(j))$ of positive integers such that $p_{m(i),n(j)} < (i+j)^{-1}$ for all positive integers i, j.

Nevertheless, these cases do not include all possibilities whenever $\inf_{m,n\in\mathbb{N}_1} p_{mn} = 0$. One can easily observe this by means of the sequence $p = (p_{mn})$ defined by

$$p_{mn} := \begin{cases} 1/m & , \quad m = n, \\ 1 & , \quad m \neq n \end{cases}$$

for all $m, n \in \mathbb{N}_1$. Clearly, $\inf_{m,n\in\mathbb{N}_1} p_{mn} = 0$ and Part (a) is invalid. To obtain $p_{m(i),n(j)} < (i+j)^{-1}$, we must take $m(i) = n(j)$ for all $i, j \in \mathbb{N}_1$. This implies that $m(i) = n(j) = k$ for all $i, j \in \mathbb{N}_1$ and a fixed integer $k \in \mathbb{N}_1$. Therefore, they are not increasing sequences. Also, even if they are strictly increasing sequences, we get for infinitely many $m(i)$ and $n(j)$'s that $p_{m(i),n(j)} = 1$. Thus, Part (b) is invalid too. In this study, we obviate missing points and also characterize some classes of matrix transformations from the space $\mathcal{L}(t)$ to the spaces $\mathcal{M}_u$, $\mathcal{C}_\vartheta$ and $\mathcal{C}_{\vartheta 0}$ for all t's. So, the present study may consider as a complement of Gökhan and Çolak [12].

Let ℓ_p denotes the space of all absolutely $p-$summable single sequences, and $\ell(p)$ be paranormed counterpart of ℓ_p, where $0 < p < \infty$. Altay and Başar [3] and Başar and Altay [4] investigated the space bv_p as the domains of two-dimensional backward difference matrix Δ in the space ℓ_p. Also, in [2], they studied the domain of Riesz mean R^q in the paranormed space $\ell(p)$. Here, we note that as a continuation of the present paper, to obtain more general spaces of double sequences with some algebraic and topological properties, one can investigate the domain of certain four dimensional triangles, for example four dimensional backward difference matrix Δ or Riezs mean R^{qs} with respect to the sequences $q = (q_k)$ and $s = (s_l)$ of non-negative numbers which are not all zero, in the space $\mathcal{L}(t)$.

As a natural continuation of Altay and Başar [2], Yeşilkayagil and Başar [19] have recently investigated the domain $R^{qt}(\mathcal{L}_s)$ of four dimensional Riesz mean R^{qt} in the space $\mathcal{L}_s$ of absolutely s-summable double sequences. Of course, following the present paper one can extend the normed space $R^{qt}(\mathcal{L}_s)$ to the paranormed case.

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