Filomat 32:3 (2018), 1079–1089
https://doi.org/10.2298/FIL1803079M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

G-Connectedness in Topological Groups with Operations

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Abstract. It is a well known fact that for a Hausdorff topological group X, the limits of convergent sequences in X define a function denoted by $\lim$ from the set of all convergent sequences in X to X. This notion has been modified by Connor and Grosse-Erdmann for real functions by replacing $\lim$ with an arbitrary linear functional G defined on a linear subspace of the vector space of all real sequences. Recently some authors have extended the concept to the topological group setting and introduced the concepts of G-continuity, G-compactness and G-connectedness. In this paper we present some results about G-hulls, G-connectedness and G-fundamental systems of G-open neighbourhoods for a wide class of topological algebraic structures called groups with operations, which include topological groups, topological rings without identity, R-modules, Lie algebras, Jordan algebras, and many others.

1. Introduction

Connor and Grosse-Erdmann in [19] investigated the impact of changing the definition of the convergence of sequences on the structure of sequential continuity of real functions. Çakallı extended this concept to topological group setting in [16] introducing the concept of G-compactness and he obtained further results on G-continuity and G-compactness in [12]. One is often relieved to find that the standard closed set definition of connectedness for metric spaces can be replaced by a sequential definition of connectedness and that many of the properties of connectedness of sets can be easily derived using sequential arguments. Connectedness is much more useful, for example for the covering spaces of topological groups. For the non-connected case, see, for example [8]. The notion of G-connectedness for topological groups was introduced in [11] and some further properties of this continuity were developed in [10].

In [37] Orzech introduced a certain algebraic category C called category of groups with operations including groups, rings without identity, R-modules, Lie algebras, Jordan algebras, and many others. The internal category and crossed module in C was studied in [38] and the studies have resumed by the works of Datuashvili [21–24]. Recently some works for topological groups with operations and their internal categories have been carried out in [1, 2, 32, 33, 35, 36].

In this paper some results about G-continuity, G-connectedness and G-fundamental system of G-open neighbourhoods for a wide class of topological algebraic structures called topological groups with operations, which include topological groups, topological rings without identity, R-modules, Lie algebras, Jordan algebras, and many others are presented.

2010 *Mathematics Subject Classification*. Primary 40J05; Secondary 54A05, 22A05
Keywords. Sequences, summability, G-hull, G-continuity, G-connectedness, topological group with operations
Received: 26 October 2016; Revised: 22 February 2017; Accepted: 25 February 2017
Communicated by Ljubiša D.R. Kočinac
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We acknowledge that an extended abstract including the statements without proofs of some results of this paper appears in [31] as AIP Conference Proceedings.

2. Preliminaries

Following the idea given in a 1946 American Mathematical Monthly problem [9], a number of authors Posner [39], Iwinski [27], Srinivasan [41], Antoni [3], Antoni and Salat [4], Spigel and Krupnik [42] have studied A-continuity defined by a regular summability matrix A. Some authors Öztürk [43], Savaş and Das [44], Savaş [45], Borsik and Salat [7] have studied A-continuity for methods of almost convergence and for related methods. See also [5] for an introduction to summability matrices and [18] for summability in topological groups. Di Maio and Kočinac [30] defined statistical convergence in topological spaces, introduced statistically sequential spaces and statistically Frchet spaces, and considered their applications in selection principles theory, function spaces and hyperspaces.

Throughout the paper, X denotes a Hausdorff topological group with operations as defined in Definition 3.1, the boldface letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ... represent the sequences of terms in X; and $s(X)$ and $c(X)$ respectively denote the set of all sequences in X and the set of all convergent sequences in X.

By a G-method of sequential convergence for X, we mean a morphism defined on a subgroup with operations $c_G(X)$ of $s(X)$ into X. A sequence $\mathbf{x} = (x_n)$ is said to be G-convergent to ℓ if $\mathbf{x} \in c_G(X)$ and $G(\mathbf{x}) = \ell$. In particular, $\lim$ denotes the limit function $\lim\mathbf{x} = \lim_n x_n$ on $c(X)$. A method G is called *regular* if every convergent sequence $\mathbf{x} = (x_n)$ is G-convergent with $G(\mathbf{x}) = \lim\mathbf{x}$. A map $f\colon X \to X$ is called *G- continuous* if $G(f(\mathbf{x})) = f(G(\mathbf{x}))$ for $\mathbf{x} \in c_G(X)$ [12].

We define the operations on methods of sequential convergence G_1 and G_2 as $(G_1 \star G_2)(\mathbf{x}) = G_1(\mathbf{x}) \star G_2(\mathbf{x})$ where $c_{G_1 \star G_2}(X) = c_{G_1}(X) \cap c_{G_2}(X)$ for $\star \in \Omega_2$.

The notion of regularity introduced above coincides with the classical notion of regularity for summability matrices. See [5] for an introduction to regular summability matrices and see [47] for a general view of sequences of reals or complex.

Let $A \subseteq X$ and $\ell \in X$. Then ℓ is said in the *G-hull* of A if there is a sequence $\mathbf{x} = (x_n)$ of points in A such that $G(\mathbf{x}) = \ell$ and the G-hull of A is denoted by $\overline{A}^G$ in [19]. Following the notations in [29], we denote *G-hull* of a set A by $[A]_G$ and say that A is *G-closed* if $[A]_G \subseteq A$. If G is a regular method, then $A \subseteq [A]_G$, and hence A is G-closed if and only if $[A]_G = A$. Even for regular methods $[[A]_G]_G = [A]_G$ is not always true and the union of any two G-closed subsets of X need not also be a G-closed subset of X [12, Counterexample 1]. If $B \subseteq A \subseteq X$ and $a \in A$, then we say a is in the *G-hull of B* in A if there is a sequence $\mathbf{x} = (x_n)$ of points in B such that $G(\mathbf{x}) = a$. A subset F of A is called *G-closed* in A if there exists a G-closed subset K of X such that $F = K \cap A$. We say that a subset U of A is *G-open* in A if $A \backslash U$ is G-closed in A. Here note that a subset U of A is G-open in A if and only if there exists a G-open subset V of X such that $U = A \cap V$. The union of any G-open subsets of X is G-open. A subset V is a *G-neighborhood* of a if there exists a U-open subset of X with $a \in U$ such that $U \subseteq V$. The union of G-open subsets of A is called *G-interior* of A and denoted by $A^{\circ G}$ is also G-open.

Çakallı [16] has introduced the concept of G-compactness and has proved that the G-continuous image of any G-compact subset of X is also G-compact [16, Theorem 7]. He investigated G-continuity, and obtained further results in [12] (see also [17], [20], [14] and [15] for some other types of continuities which can not be given by any sequential method).

We recall that as it is stated in [29, Remark 2.2] since the definition of G-method already involves sequences the term *'sequentially'* in G-sequentially closed sets seems redundant, so they choose the terminology of G-closed sets. By the same idea we use the similar terminology G-open sets, G-continuity, G-connectedness, G-compactness and etc.

3. *G*-Hulls in Topological Groups with Operations

The idea of the definition of category of groups with operations comes from Higgins [26] and Orzech [37]; and the definition below is from Porter [38] and Datuashvili [25, p.21], which is adapted from Orzech [37].

Definition 3.1. Let C be a category of groups with a set of operations Ω and with a set E of identities such that E includes the group laws, and the following conditions hold: If Ω_i is the set of i-ary operations in Ω, then

(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

(b) The group operations written additively $0, -$ and $+$ are the elements of Ω_0, Ω_1 and Ω_2 respectively. Let $\Omega_2' = \Omega_2 \backslash \{+\}$, $\Omega_1' = \Omega_1 \backslash \{-\}$ and assume that if $\star \in \Omega_2'$, then $\star^\circ$ defined by $x \star^\circ y = y \star x$ is also in Ω_2'. Also assume that $\Omega_0 = \{0\}$;

(c) For each $\star \in \Omega_2'$, E includes the identity $x \star (y + z) = x \star y + x \star z$;

(d) For each $\omega \in \Omega_1'$ and $\star \in \Omega_2'$, E includes the identities $\omega(x + y) = \omega(x) + \omega(y)$ and $\omega(x) \star y = \omega(x \star y)$.

Then the category C satisfying the conditions (a)-(d) is called a *category of groups with operations*.

From now on C will be a category of groups with operations.

A *morphism* between any two objects of C is a group homomorphism, which preserves the operations of Ω_1' and Ω_2'.

Remark 3.2. The set Ω_0 contains exactly one element, the group identity; hence for instance the category of associative rings with unit is not a category of groups with operations.

Example 3.3. The categories of groups, rings generally without identity, R-modules, associative, associative commutative, Lie, Leibniz, alternative algebras are examples of categories of groups with operations.

The subobject in the category C can be defined as follows.

Definition 3.4. Let X be a group with operations, i.e., an object of C. A subset $A \subseteq X$ is called a *subgroup with operations* subject to the the following conditions:

1. $a \star b \in A$ for $a, b \in A$ and $\star \in \Omega_2$;
2. $\omega(a) \in A$ for $a \in A$ and $\omega \in \Omega_1$.

The normal subobject in the category C is defined as follows.

Definition 3.5. ([37, Definition 1.7]) Let X be an object in C and A a subgroup with operations of X. A is called a *normal subgroup with operations or ideal* if

1. $(A, +)$ is a normal subgroup of $(X, +)$;
2. $x \star a \in A$ for $x \in X$, $a \in A$ and $\star \in \Omega_2'$.

The category of topological groups with operations is defined in [1, pp. 228] (see also [35, Definition 3.4]) as follows:

Definition 3.6. A category $\mathsf{Top}^{\mathsf{C}}$ of topological groups with a set Ω of continuous operations and with a set E of identities such that E includes the group laws such that the conditions (a)-(d) of Definition 3.1 are satisfied, is called a *category of topological groups with operations*.

A *morphism* between any two objects of $\mathsf{Top}^{\mathsf{C}}$ is a continuous group homomorphism, which preserves the operations in Ω_1' and Ω_2'.

The categories of topological groups, topological rings and topological R-modules are examples of categories of topological groups with operations.

In the rest of the paper $\mathsf{Top}^{\mathsf{C}}$ will denote the category of topological groups with operations and X will denote and object of $\mathsf{Top}^{\mathsf{C}}$ and call a topological group with operations; and G will be a regular method unless otherwise is stated.

Theorem 3.7. *For any $a \in X$ and $\star \in \Omega_2$, the function $f_a \colon X \to X, x \mapsto a \star x$ is G-continuous.*

Proof. Let $\mathbf{x}$ be a G-convergent sequence with $G(\mathbf{x}) = u \in X$ and let $\mathbf{a}$ be the constant sequence $(a, a, \dots)$. Since the constant sequence $\mathbf{a} = (a, a, \dots,)$ converges to a and G is regular, we have that $G(\mathbf{a}) = a$. Since the sequence $\mathbf{a} \star \mathbf{x} = (a_n \star x_n)$ converges to $a \star u$, the regularity of G implies that $G(\mathbf{a} \star \mathbf{x}) = a \star u$. Therefore

$$G(f_a(\mathbf{x})) = G(\mathbf{a} \star \mathbf{x}) = a \star u = f_a(u) = f_a(G(\mathbf{x}))$$

and hence f_a is G-continuous. □

The following lemma which is adopted from [11, Lemma 6] is very useful in the proofs of Theorems 3.10 and 3.11.

Lemma 3.8. *For subsets $A, B \subseteq X$ the following are satisfied for $\star \in \Omega_2$:*

1. *If $A \subset B$ then $[A]_G \subset [B]_G$.*
2. *$[A]_G \star [B]_G \subset [A \star B]_G$.*

Proof. 1. The proof follows from the related definitions.

2. Let $u \in [A]_G$ and $v \in [B]_G$. Let $\mathbf{x}$ and $\mathbf{y}$ be the sequences respectively in A and B with $G(\mathbf{x}) = u \in A$ and $G(\mathbf{y}) = v \in B$. Hence $\mathbf{x} \star \mathbf{y}$ is a sequence in $A \star B$ and since G preserves the operations we have that $G(\mathbf{x} \star \mathbf{y}) = u \star v$. Hence $u \star v \in [A \star B]_G$. □

The following theorem in topological group case was given in [11, Theorem 5].

Theorem 3.9. *Let A be a subgroup with operations of X. If A is G-open, then it is G-closed.*

Theorem 3.10. *The G-hull $[A]_G \subseteq X$ of any subgroup with operations A of X is still a subgroup with operations.*

Proof. Let A be a subgroup with operations of X. Hence $A \star A \subseteq A$ for $\star \in \Omega_2$ and by Lemma 3.8, $[A]_G \star [A]_G \subseteq [A \star A]_G \subseteq [A]_G$. Therefore $[A]_G \star [A]_G \subseteq [A]_G$. Moreover if $u \in [A]_G$, there exists a sequence $\mathbf{x}$ of points in A such that $G(\mathbf{x}) = u$. Since $G(\omega(\mathbf{x})) = \omega G(\mathbf{x}) = \omega(x)$, we have that $\omega(x) \in [A]_G$. Consequently $[A]_G$ becomes a subgroup with operations of X. □

Theorem 3.11. *The G-hull $[A]_G \subseteq X$ of any normal subgroup with operations A of X is still a normal subgroup with operations.*

Proof. If A is a normal subgroup with operations we have that $x + A - x \subseteq A$, for each $x \in X$. Hence by Lemma 3.8 we have that,

$$[\{x\}]_G + [A]_G - [\{x\}]_G \subseteq [\{x\} + A - \{x\}]_G \subseteq [A]_G. \tag{1}$$

Since G is regular, $\{x\} \subseteq [\{x\}]_G$ and hence

$$\{x\} + [A]_G - \{x\} \subseteq [\{x\}]_G + [A]_G - [\{x\}]_G. \tag{2}$$

Therefore by (1) and (2) we have $\{x\} + [A]_G - \{x\} \subseteq [A]_G$. Hence $([A]_G, +)$ is a normal subgroup.

Moreover if $x \in X$ and $u \in [A]_G$, there exists a sequence $\mathbf{a} = (a_n)$ of points in A with $G(\mathbf{a}) = u$. Since A is a subgroup with operations for $\star \in \Omega_2{}'$, $\mathbf{x} \star \mathbf{a} = (x \star a_n)$ is a sequence of the points in A, where $\mathbf{x}$ is the constant sequence $(x, x, \dots)$. Since G is regular, it follows that $G(\mathbf{x}) = x$ and hence

$$G(\mathbf{x} \star \mathbf{a}) = G(\mathbf{x}) \star G(\mathbf{a}) = x \star u \in [A]_G.$$

□

Recall that in [19], a method is called *subsequential* if, whenever $\mathbf{x}$ is G-convergent with $G(\mathbf{x}) = u$, then there is a subsequence $\mathbf{y}$ of $\mathbf{x}$ with $\lim \mathbf{y} = u$. We say a method G *preserves the G-convergence of subsequences* if, whenever a sequence $\mathbf{x}$ is G-convergent with $G(\mathbf{x}) = u$, then any subsequence of $\mathbf{x}$ is G-convergent to the same point u.

Let G be a method on X and $A \subseteq X$. In [29, Definition 3.3] the *G-kernel* of A denoted by $ker_G(A)$ or $(A)_G$ is defined as the set of ℓ's such that there is no any sequence $\mathbf{x}$ in $s(X \setminus A) \cap c_G(X)$ with $G(\mathbf{x}) = \ell$, and it was proved in [29, Theorem 3.5] that $(A)_G = X \setminus [X \setminus A]_G$ and $A^{\circ G} = X \setminus \overline{[X \setminus A]}^G$. It is easy to see that $A^{\circ G} \subseteq (A)_G$ and by [29, Corollary 3.6] A is G-open in X if and only if $A \subseteq (A)_G$.

Then we can give the following theorem.

Theorem 3.12. *Let G be a regular method preserving the G-convergence of subsequences; and A a subset of X. Then the following are equivalent:*

1. $a \in (A)_G$.
2. *Any sequence $\mathbf{x} = (x_n)$ which is G-convergent to a is almost in A.*

Proof. (1) ⇒ (2): Let $a \in (A)_G$ and $\mathbf{x}$ a sequence of the points in X such that $G(\mathbf{x}) = a$. Then the sequence $\mathbf{x} = (x_n)$ is almost in A. Otherwise $\mathbf{x}$ has a subsequence $\mathbf{y} = (x_{n_1}, x_{n_2}, \dots)$ of the points in $X \backslash A$ and since G preserves the G-convergence of subsequences $G(\mathbf{y}) = a$. Hence $a \in [X \backslash A]_G$ which is a contradiction with $a \in (A)_G$ since $(A)_G = X \setminus [X \setminus A]$.

(2) ⇒ (1): If the point $a \notin (A)_G$, then $a \in [X \setminus A]_G$, thus there exists a sequence $\mathbf{x}$ in $X \setminus A$ such that $G(\mathbf{x}) = a$, which is a contradiction with the condition (2). □

Theorem 3.13. *Let G be a method preserving the G-convergence of subsequences and A a subset of X. Then the following are equivalent.*

1. *A is G-open, i.e., $X \setminus A$ is G-closed.*
2. *If $a \in A$, then any sequence $\mathbf{x} = (x_n)$ of the points in X such that G-convergent with $G(\mathbf{x}) = a$ is almost in A.*

Proof. (1) ⇒ (2): Let $X \setminus A$ be a G-closed subset of X and $\mathbf{x} = (x_n)$ a sequence of the points in X which G-converges with $G(\mathbf{x}) = a \in A$. Then the sequence $\mathbf{x}$ is almost in A, i.,e., there is an $n_0 \in \mathbb{N}$ such that $x_n \in A$ for $n \geq n_0$. Otherwise $\mathbf{x}$ has a subsequence $\mathbf{y} = (x_{n_1}, x_{n_2}, \dots)$ of the points in $X \backslash A$ and since G preserves the G-convergence of subsequences $G(\mathbf{y}) = a$. Since $X \backslash A$ is G-closed a becomes in $X \backslash A$ which is a contradiction.

(2) ⇒ (1): Assuming (2) if $\mathbf{x}$ is a sequence of the points in $X \backslash A$ with $G(\mathbf{x}) = u$, then $u \in X \backslash A$. Otherwise if $u \in A$, then by (2) the sequence $\mathbf{x}$ becomes almost in A which is a contradiction. □

Remark 3.14. If G is a method defined on X, then we can also obtain a similar method on $X \times X$ defined by $G(\mathbf{x}, \mathbf{y}) = (G(\mathbf{x}), G(\mathbf{y}))$ when $\mathbf{x}$ and $\mathbf{y}$ are G-convergent sequences in X. Then we have the following theorem.

Proposition 3.15. *Let A and B be subsets of X. Then we have the following.*

1. $[A \times B]_G = [A]_G \times [B]_G$.
2. *If A and B are G-closed, then $A \times B$ is G-closed.*

Proof. 1. If $(u, v) \in [A \times B]_G$, then there is a sequence $(\mathbf{x}, \mathbf{y}) = (x_n, y_n)$ of the points in $A \times B$ such that $G(\mathbf{x}, \mathbf{y}) = (u, v)$. Hence $G(\mathbf{x}) = u$ and $G(\mathbf{y}) = v$ and therefore $u \in [A]_G$ and $v \in [B]_G$. This implies that $[A \times B]_G \subseteq [A]_G \times [B]_G$. On the other hand, if $(a, b) \in [A]_G \times [B]_G$, then there are sequences $(\mathbf{x}) = (x_n)$ and $(\mathbf{y}) = (y_n)$ of the points in A and B respectively such that $G(\mathbf{x}) = a$ and $G(\mathbf{y}) = b$. Hence $G(\mathbf{x}, \mathbf{y}) = (a, b)$ and $(a, b) \in [A \times B]_G$. As a result we obtain that $[A \times B]_G = [A]_G \times [B]_G$.

2. This is a result of (1).

□

For the proof of the following theorem we use Theorem 3.12.

Theorem 3.16. *Let G be a regular method preserving the G-convergence of subsequences; and A and B subsets of X. Then we have the following.*

1. $(A \times B)_G = (A)_G \times (B)_G$.
2. *If A and B are G-open, then $A \times B$ is G-open.*

Proof. 1. Let $(a, b) \in (A \times B)_G$. If $\mathbf{x} = (x_n)$ and $\mathbf{y} = (y_n)$ are the sequences of the points in X such that $G(\mathbf{x}) = a$ and $G(\mathbf{y}) = b$, then $G(\mathbf{x}, \mathbf{y}) = (a, b)$ and since $(a, b) \in (A \times B)_G$, by Theorem 3.12 the sequence $(\mathbf{x}, \mathbf{y})$ stays almost in $A \times B$. Hence the sequences $\mathbf{a}$ and $\mathbf{b}$ are respectively almost in A and B. Hence by Theorem 3.12 $(a, b) \in (A)_G \times (B)_G$.

On the other hand, if $(a, b) \in (A)_G \times (B)_G$ and $(\mathbf{x}, \mathbf{y}) = (x_n, y_n)$ is a sequence of the points in $X \times X$ such that $G(\mathbf{x}, \mathbf{y}) = (a, b)$, then $G(\mathbf{x}) = a$ and $G(\mathbf{y}) = b$. Hence the sequences $\mathbf{x}$ and $\mathbf{y}$ are respectively almost in A and B since $a \in (A)_G$ and $b \in (B)_G$. Therefore the sequence $(\mathbf{x}, \mathbf{y}) = (x_n, y_n)$ is almost in $A \times B$ and by Theorem 3.12 $(a, b) \in (A \times B)_G$.

2. If A and B are G-open, then by [29, Theorem 3.5] and [29, Corollaries 3.6 and 3.7] we have $A \subseteq (A)_G$ and $B \subseteq (B)_G$. Hence $A \times B \subseteq (A)_G \times (B)_G = (A \times B)_G$ which means that $A \times B$ is G-open. □

Theorem 3.17. *If $f \colon X \to X$ is G-continuous and A is a G-closed subgroup with operations of X, then the graph set $G_A = \{(a, f(a)) \mid a \in A\}$ is a G-closed subgroup with operations of $X \times X$.*

Proof. Let $\mathbf{x} = (\mathrm{a}, \mathrm{b}) = (a_n, b_n)$ be a sequence of the points in G_A with $G(\mathbf{x}) = (G(\mathbf{a}), G(\mathbf{b})) = (u, v)$. Then $G(\mathbf{a}) = u$ and $G(\mathbf{b}) = v$. Since $f(\mathrm{a}) = \mathbf{b}$, by G-continuity of f we have

$$f(u) = f(G(\mathrm{a})) = G(f(\mathrm{a})) = G(\mathrm{b}) = v$$

On the other hand since $G(\mathbf{a}) = u$ and A is G-closed we have $u \in A$ and $(u, v) \in G_A$. This proves that G_A is G-closed.

If $(a, f(a)), (b, f(b)) \in G_A$, then for $\star \in \Omega_2$ we have

$$(a, f(a)) \star (b, f(b)) = (a \star b, f(a) \star f(b)) = (a \star b, f(a) \star f(b)) = (a \star b, f(a \star b))$$

and since A is a subgroup with operations $a \star b \in A$ and hence $(a \star b, f(a \star b)) \in G_A$. Further for $a \in A$ and $\omega \in \Omega_1$ we have

$$\omega(a, f(a)) = (\omega(a), \omega(f(a))) = (\omega(a), f(\omega(a)) \in A.$$

Hence G_A is a G-closed subgroup with operations of $X \times X$ as required. □

As a result of Theorem 3.17 we state the following corollary.

Corollary 3.18. *If $f \colon X \to X$ is G-continuous, then the graph set $G_X = \{(x, f(x)) \mid x \in X\}$ is a G-closed subgroup with operations of $X \times X$.*

Theorem 3.19. *$\pi_1 \colon X \times X \to X, (x, y) \mapsto x$ and $\pi_2 \colon X \times X \to X, (x, y) \mapsto y$ projection maps are G-continuous morphisms of topological groups with operations.*

Proof. If $(\mathbf{x}, \mathbf{y}) = (x_n, y_n)$ is a sequence of the points of $X \times X$ such that $G(\mathbf{x}, \mathbf{y}) = (G(\mathbf{x}), G(\mathbf{y})) = (u, v)$, then

$$G(\pi_1(\mathbf{x}, \mathbf{y})) = G(\mathbf{x}) = u = \pi_1(u, v)$$

and hence π_1 becomes G-continuous.

Similarly one can prove that π_2 is G-continuous. □

Theorem 3.20. *Let G be a method preserving the G-convergence of subsequences. Then $\pi_1 \colon X \times X \to X, (x, y) \mapsto x$ and $\pi_2 \colon X \times X \to X, (x, y) \mapsto y$ projection maps are G-open morphisms of topological groups with operations.*

Proof. Let $A \subseteq X \times X$ be a G-open subset. To prove that $\pi_1(A)$ is G-open we use Theorem 3.13. Let $a \in \pi_1(A)$ and $\mathbf{a} = (a_n)$ a sequence of the points in X such that $G(\mathbf{a}) = a$. Choose a point $b \in X$ with $(a, b) \in A$. Then we have $G(\mathbf{a}, \mathbf{b}) = (a, b) \in A$, where $\mathbf{b}$ is the constant sequence $\mathbf{b} = (b, b, \dots)$. Since A is G-open by Theorem 3.13, the sequence $(\mathbf{a}, \mathbf{b})$ is almost in A and hence the sequence $\mathbf{a} = (a_n)$ is almost in $\pi_1(A)$.

Similarly one can prove that π_2 is also G-open. □

Theorem 3.21. *A map $f : X \to X \times X$ is G-continuous if and only if the compositions $\pi_1 f$ and $\pi_2 f$ for the projection maps π_1 and π_2 are G-continuous.*

Proof. Since the projection maps π_1 and π_2 are G-continuous, the compositions $\pi_1 f$ and $\pi_2 f$ are G-continuous.

On the other hand, if the compositions $\pi_1 f$ and $\pi_2 f$ are G-continuous and $\mathbf{x}$ is a sequence of points of X with $G(\mathbf{x}) = u$, then we have that

$$G(\pi_1 f(\mathbf{x})) = (\pi_1 f)(u) = \pi_1(f(u))$$

$$G(\pi_2 f(\mathbf{x})) = (\pi_2 f)(u) = \pi_2(f(u))$$

and hence

$$G(f(\mathbf{x})) = G(\pi_1 f(\mathbf{x}), \pi_2 f(\mathbf{x})) = (G(\pi_1 f(\mathbf{x})), G(\pi_2 f(\mathbf{x}))) = (\pi_1(f(u)), \pi_2(f(u))) = f(u).$$

Hence f is G-continuous. □

Theorem 3.22. *Let $f, g : X \to X$ be morphism of groups with operations. Then we have the following:*

1. *$(f, g) : X \to X \times X, x \mapsto (f(x), g(x))$ is G-continuous if and only if f and g are G-continuous.*
2. *$(f \times g) : X \times X \to X \times X, (x, y) \mapsto (f(x), g(y))$ is G-continuous if and only if f and g are G-continuous.*

Proof. 1. If f and g are G-continuous, and $\mathbf{x} = (x_n)$ is a sequence of point of X with $G(\mathbf{x}) = u$, then by the G-continuity of f and g, we have $G(f(\mathbf{x})) = f(u)$ and $G(g(\mathbf{x})) = g(u)$. That concludes

$$G((f, g)(\mathbf{x})) = G(f(\mathbf{x}), g(\mathbf{x})) = (G(f(\mathbf{x})), G(g(\mathbf{x}))) = (f(G(\mathbf{x})), g(G(\mathbf{x}))) = (f(u), g(u)) = (f, g)(u)$$

and hence the map (f, g) becomes G-continuous.

The sufficiency is obvious by Theorem 3.19.

2. If f and g are G-continuous and $(\mathbf{x}, \mathbf{y})$ is a sequence of the points of $X \times X$ such that $G(\mathbf{x}, \mathbf{y}) = (G(\mathbf{x}), G(\mathbf{y})) = (u, v)$, Then $G(\mathbf{x}) = u$ and $G(\mathbf{y}) = v$. By the G-continuities of f and g we have that $G(f(\mathbf{x})) = f(u)$ and $G(f(\mathbf{y})) = f(v)$. Hence we have the following

$$G((f \times g)(\mathbf{x}, \mathbf{y})) = G(f(\mathbf{x}), g(\mathbf{y})) = (G(f(\mathbf{x}), G(g(\mathbf{y}))) = (f(G(\mathbf{x})), g(G(\mathbf{y}))) = (f(u), g(u))) = (f \times g)(u, v)$$

which completes the G-continuity of $f \times g$.

The proof of converse way is obvious by Theorem 3.19 and 1. □

Theorem 3.23. *Let G be a method preserving the G-convergence of subsequences. Then for the projection map $\pi_1 : X \times X \to X, (x, y) \mapsto x$ if $A \subseteq X$ is a G-open subset, then ${\pi_1}^{-1}(A)$ is a G-open subset in $X \times X$.*

Proof. Let $A \subseteq X$ be a G-open subset. To prove that ${\pi_1}^{-1}(A)$ is G-open we use Theorem 3.13 . Let $(u, v) \in {\pi_1}^{-1}(A)$ and $(\mathbf{x}, \mathbf{y})$ a sequence in $X \times X$ such that $G(\mathrm{x}, \mathrm{y}) = (G(\mathrm{x}), G(\mathrm{y})) = (u, v)$. Then $u \in A$ and $G(\mathbf{x}) = u$. Since A is G-open the sequence $\mathbf{x}$ is almost in A. Hence the sequence $(\mathbf{x}, \mathbf{y})$ is almost in ${\pi_1}^{-1}(A)$. Therefore ${\pi_1}^{-1}(A)$ is G-open. □

4. *G*-Connected Topological Groups with Operations

G-connectedness of a topological group with operations is adapted from [11] as follows.

Definition 4.1. A non-empty subset A of X is called *G-connected* if there are no non-empty disjoint G-closed subsets F and K of A such that $A = F \cup K$. Particularly X is called G-connected, if there are no non-empty, disjoint G-closed subsets of X whose union is X.

Theorem 4.2. *If one of the subsets A and B is a G-connected neighbourhood of $0 \in X$, then $A \star B$ for Ω_2' is G-connected.*

Proof. Let B be a G-connected neighbourhood of $0 \in X$. Since by Theorem [11, Theorem 1], the image of a G-connected subset under a G-continuous function is G-connected and by Theorem 3.7 for $a \in A$ the function $f_a \colon X \to X, x \mapsto a \star x$ is G-continuous, we obtain that the set $a \star B$, is G-connected. By the fact that $a \star 0 = 0$, each subset $a \star B$ includes 0. Hence by Theorem [11, Theorem 3] $A \star B = \bigcup_{a \in A} a \star B$ is G-connected. □

Theorem 4.3. *The G-connected component of the identity $0 \in X$ of an additive group of X is a G-closed, subgroup with operations of X.*

Proof. Write K_0 for the G-connected component of the point 0. By Theorem [10, Theorem 5], K_0 is G-closed. To prove that K_0 is a subgroup with operations, we initially need prove that $K_0 \star K_0 \subseteq K_0$, where $K_0 \star K_0$ is the set of all points $x \star y$ for $x, y \in K_0$. Here

$$K_0 \star K_0 = \bigcup_{x \in K_0} (x \star K_0)$$

and each $x \star K_0$ is G-connected subset including $0 \in K_0$. Hence $K_0 \times K_0$ is G-connected as a union of G-connected subsets which include 0 as a common point. But the largest G-connected subset in X including 0 is K_0. Hence $K_0 \star K_0 \subseteq K_0$. Further since each $\omega \in \Omega_1$ is G-continuous, $\omega(K)$ is a G-connected subset including $0 \in G$. Hence $\omega(K_0) \subseteq K_0$. Hence by Definition 3.4 K_0 is a subgroup with operations of X. □

Theorem 4.4. *The G-connected component of $0 \in X$ is a G-closed, normal subgroup of X.*

Proof. By Theorem [10, Theorem 6] $(K_0, +)$ is a normal subgroup of X. Moreover by Theorem 4.2 for $\star \in \Omega_2{}'$ the subset $X \star K_0$ is G-connected. By the fact that K_0 is the largest G-connected subset including $0 \in X$, it follows that $X \star K_0 \subseteq K_0$. Hence K_0 is a nomal subgroup with operations of X. □

Theorem 4.5. *Let $0 \in X$ be the identity of additive operation. Writing K_a for the G-connected component of a point $a \in X$, $K_a = K_0 + a$.*

Proof. Since the function $f_a \colon X \to X, x \mapsto x + a$ is G-continuous, $K_0 + a$ is G-connected and $a \in K_0 + a$. Hence $K_0 + a \subseteq K_a$ since K_a is the largest G-connected subset including a.

On the other hand, if $b \in K_a$, then $b - a \in K_a - a \subseteq K_0$ since $K_a - a$ is a G-connected subset including $0 \in X$. Hence $b - a \in K_0$ and so $b \in K_0 + a$. That means $K_a \subseteq K_0 + a$. Consequently we obtain that $K_a = K_0 + a$. □

Theorem 4.6. *If $f, g \colon X \times X \to X$ are G-continuous, then $B = \{(x, y) \mid f(x, y) = g(x, y)\}$ is a G-closed subset of $X \times X$.*

Proof. If $(\mathbf{x}, \mathbf{y}) = (x_n, y_n)$ is a sequence of the points in B such that $(G(\mathbf{x}), G(\mathbf{y})) = (u, v)$, then $f(\mathbf{x}, \mathbf{y}) = g(\mathbf{x}, \mathbf{y})$ and $G(f(\mathbf{x}, \mathbf{y})) = G(g(\mathbf{x}, \mathbf{y}))$. By the G-continuities of f and g it follows that $f(G(\mathbf{x}, \mathbf{y})) = g(G((\mathbf{x}, \mathbf{y}))$ which implies that $f(u, v) = g(u, v)$. Hence $(u, v) \in B$, i.e., B is G-closed. □

Theorem 4.7. *In a topological group X, the G-hull of an abelian subgroup is still an abelian subgroup.*

Proof. Let A be an abelian subgroup of X. By Theorem 3.10 we know that $[A]_G$ is a subgroup of X. Hence it is sufficient just to prove that it is abelian. Let $f, g \colon X \times X \to X$ be the functions defined by $f(x, y) = x + y$ and $g(x, y) = y + x$. Then f and g are G-continuous and by Theorem 4.6 $B = \{(x, y) \mid f(x, y) = g(x, y)\}$ is G-closed. Since A is abelian we have $A \times A \subseteq B$ and and hence $[A \times A]_G \subseteq [B]_G$. Since B is G-closed $[B]_G \subseteq B$ and by Proposition 3.15 $[A \times A]_G = [A]_G \times [A]_G$. Hence we conclude that $[A]_G \times [A]_G \subseteq B$ and therefore $[A]_G$ is abelian. □

Theorem 4.8. *If X is G-connected, then $X \times X$ is still G-connected.*

Proof. If X is G-connected, then for an $a \in X$, the subset $A = \{a\} \times X$ is G-connected as the image of a G-connected set under G-continuous map $f_a \colon X \to X \times X, x \mapsto (a, x)$. Similarly for each $x \in X$ the subset $B_x = X \times \{x\}$ is G-connected and $A \cap B_x$ has a common point (a, x). Since $X \times X = \bigcup_{x \in X} A \cup B_x$ by [11, Corollary 3] $X \times X$ is G-connected. □

5. Fundamental System of G-Open Neighbourhoods

Let X be a topological group with operations and $a \in X$. A class $\mathcal{B}_a$ of G-open neighbourhoods of a is called a *fundamental system of G-open neighbourhoods* of a if for each G-open neighbourhood U of a, there is a $V \in \mathcal{B}_a$ such that $V \subseteq U$.

Theorem 5.1. *Let $a \in X$. If $\mathcal{B}_0$ is a fundamental system of G-open neighbourhoods of* 0*, then the subsets $a + U$ for $U \in \mathcal{B}_0$ constitute a fundamental system of G-open neighbourhoods of a.*

Proof. Let U be a G-open neighbourhood of a. Since the map $f_a \colon X \to X, x \mapsto x - a$ is G-open, $U - a$ is a G-open neighbourhood of 0. Since $\mathcal{B}_0$ is a fundamental system of G-open neighbourhoods of 0, there exists a G-open neighbourhood V of 0 such that $V \subseteq U - a$. Hence $V + a$ is a G-open neighbourhood of a and $V + a \subseteq U$ as required. □

Theorem 5.2. *A fundamental system $\mathcal{B}_0$ of G-open neighbourhoods of* 0 *satisfies the following conditions:*

1. *If $a \in U \in \mathcal{B}_0$, then there exists $V \in \mathcal{B}_0$ such that $V + a \subseteq U$.*
2. *If $U \in \mathcal{B}_0$ and $a \in X$, then there exists $V \in \mathcal{B}_0$ such that $a + V - a \subseteq U$.*
3. *If U is a G-open neighbourhood of* 0*, then there is a $V \in \mathcal{B}_0$ such that $V \subseteq U - U$.*
4. *If U is a G-open neighbourhood of* 0*, then there is a $V \in \mathcal{B}_0$ such that $V \subseteq U + U$.*

Proof. 1. If $a \in U \in \mathcal{B}_0$, then $U - a$ is a G-open neighbourhood of 0. Since $\mathcal{B}_0$ is a fundamental system of G-open neighbourhoods of 0, there exists $V \in \mathcal{B}_0$ such that $V \subseteq U - a$. Hence we have $V + a \subseteq U$ as required.

2. If $U \in \mathcal{B}_0$ and $a \in X$, then $-a + U + a$ is a G-open neighbourhood of $0 \in X$. Since $\mathcal{B}_0$ is a fundamental system of G-open neighbourhoods of 0, there is a G-open neighbourhood V in $\mathcal{B}_0$ such that $V \subseteq -a + U + a$. It follows that $a + V - a \subseteq U$ to complete the proof.

3. If U is a G-open neighbourhood of 0, then by [34, Theorem 29] $U - U$ is a G-open neighbourhood of 0. Since $\mathcal{B}_0$ is a fundamental system of G-open neighbourhoods of 0, there is a G-open neighbourhood V in $\mathcal{B}_0$ such that $V \subseteq U - U$.

4. The proof is similar to that of 5. □

Theorem 5.3. *For a subgroup with operations A of X, the following are equivalent:*

1. *A is a G-neighbourhood of* 0.
2. *A is a G-open neighbourhood of* 0.
3. *A is a G-closed neighbourhood of* 0.

Proof. (1) ⇒ (2) If A is a G-neighbourhood of 0 and $a \in A$, then $A + a$ is a G-neighbourhood of a because the map $f_a \colon X \to X, x \mapsto x + a$ is G-open and since A is a subgroup with operations $A + a = A$. Hence A is a G-neighbourhood of each a. Therefore A is a G-open neighbourhood of 0.

(2) ⇒ (3): If A is a G-open neighbourhood of 0, then by [11, Theorem 5] it is also G-closed which completes the proof.

(3) ⇒ (1): The proof is obvious. □

Proposition 5.4. *For any point $a \in X$, the subset $\{a\}$ is G-closed.*

Proof. If $\mathbf{x}$ is a sequences in $\{a\}$ which is G-convergent to u, then $\mathbf{x}$ is a constant sequence $\mathbf{x} = (a, a, \dots)$ and by the regularity of G, we conclude that $G(\mathbf{x}) = u = a$. Hence $\{a\}$ is G-closed. □

Theorem 5.5. *For a fundamental system $\mathcal{B}$ of G-open neighbourhoods of* 0 *we have the following:*

1. $\bigcap_{B\in\mathcal{B}} B = \{0\}$.
2. *The intersection of all G-open neighbourhoods of* $0 \in X$ *is* $\{0\}$.

Proof. 1. If $\mathcal{B}$ be a fundamental system of G-open neighbourhoods of 0, then for a non-zero element a of X, $0 \notin \{a\}$ and since by Proposition 5.4 $\{a\}$ is G-closed, $X\backslash\{a\}$ is G-open. Since $\mathcal{B}$ is a fundamental system of G-open neighbourhoods of 0, there is a $B \in \mathcal{B}$ such that $0 \in B \subseteq X\backslash\{a\}$. Hence $a \notin B$ and therefore $a \notin \bigcap_{B\in\mathcal{B}} B$. We conclude that $\bigcap_{B\in\mathcal{B}} B = \{0\}$ as required.

2. By (1), the proof is obvious. □

6. Conclusion

In this paper we consider G-continuity, G-hull, G-sequential connectedness and fundamental system of G-open neighbourhoods for a category of topological groups with operations which include topological groups. Some of the results, especially those on G-fundamental system, are even new in topological group case.

To generalize the results of this paper to more general case of topological $\mathbb{T}$ algebras, we first recall a fact on semi-abelian categories: The notion of semi-abelian category as proposed in [28] (see also [40] and [46]) has typical categorical properties such as possessing finite products, coproducts, a zero object and hence kernels, pullbacks of monomorphisms and coequalizers of kernel pairs. Groups, rings, algebras and all abelian categories are semi-abelian, say.

In [6] for a certain algebraic theory the term 'algebraic model' is used for the objects of the semi-abelian category. Let $\mathbb{T}$ be an algebraic theory whose category is semi-abelian. A *topological model* of $\mathbb{T}$ is a model of the theory of $\mathbb{T}$ with a topology which makes all the operations of the theory continuous. The category $\mathsf{Top}^{\mathbb{T}}$, for a semi-abelian theory $\mathbb{T}$, is generally no longer semi-abelian because it is not Bar exact. But in [6] the category $\mathsf{Top}^{\mathbb{T}}$ of the topological models $\mathbb{T}$ is studied and some classical results in topological groups is generalized to this category $\mathsf{Top}^{\mathbb{T}}$. For example when $\mathbb{T}$ is the theory of groups, then $\mathsf{Top}^{\mathbb{T}}$ becomes the category of topological groups and we obtain the results for topological groups.

Hence the methods of the paper [6] could be be useful to deal with $\mathsf{Top}^{\mathbb{T}}$ and obtain more general results for topological $\mathbb{T}$ algebras.

Acknowledgment

We would like to thank the referee for careful reading and useful comments, especially on the modifications of Theorems 3.12 and 3.16, which improve the paper.

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