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Design of PI Regulators for Dynamic Systems with Constrained Control and Fixed Endpoints of Trajectories

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Abstract. The problem of optimal control for time-varying linear systems with fixed endpoints of trajectories is considered. A corresponding quadratic objective functional depends on the control, the state of the object and on its integral. New technique of designing the PI controller for the automatic control systems with box constraints on values of control is proposed. The problem is solved by using Lagrange multipliers of a special type.

1. Introduction

The construction of the automatic operation control of real and complex systems requires the use of new information technologies, it is actually necessary to develop new principles of design the systems with a high level of complexity. Actually, there are well-known two task statements for the optimal control problem.

According to one of them, the optimal control is determined as a function of time and the initial state of the system (programmed control). Another formulation of the problem involves the synthesis of optimal control with feedback, i.e. control is determined as a function of the current state of the system and time. The solving optimal control problem in the first statement is based on the Pontryagin maximum principle (the solution is reduced to the corresponding two-point boundary value problem). The solving of the same problem with the second task statement is based on the dynamic programming method (the problem reduces to the solution of the Bellman equation). Works by Pontryagin and Bellman constitute the mathematical basis of optimal control theory [3, 4, 9]. In the field of automatic control published works one can find various examples of mathematical formulation and methods of solving optimal control problems [1, 2, 5, 6]. But still, the development of various methods of constructing the PI and PID controllers with the necessary properties is an urgent problem [8, 10].

In this paper, we propose a new approach of designing PI controller for the automatic control systems based on feedback principle, with the constraints on the values of control.

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2. Problem Statement

Consider the controlled linear system which is described by the vector differential equation:

$$\dot{x} = A(t)x + B(t)u + f(t), \quad t \in [t_0, T], \quad x(t_0) = x_0, \quad x(T) = 0, \tag{1}$$

$$u(t) \in U(t) = \{u \mid \alpha(t) \leq u(t) \leq \beta(t), \;\; t \in [t_0, T]; \;\; \alpha(\cdot), \; \beta(\cdot) \in C[t_0, T]\}, \tag{2}$$

where $x = x(t)$ is a n-vector of state of the controlled object; $u(t)$ is a m-vector of piecewise continuous controls; $A(t)$, $B(t)$ are $(n \times n)$ and $(n \times m)$-matrices, respectively; $f(t)$ is a piecewise continuous and $\alpha(t), \beta(t)$ are continuous functions on the time interval $[t_0, T]$; x_0 is a given n-vector; t_0, T are given initial and final time values.

We shall assume that system (1) is controllable. Denote $\Delta(t_0, T)$ the set of all permissible controls $u(t)$, satisfying the conditions $u(t) \in U(t)$, $t \in [t_0, T]$, and the corresponding trajectories $x(u, t)$ of system (1) defined on the interval $t_0 \leq t \leq T$.

Suppose that on the set $\Delta(t_0, T)$ the functional J is given, which depends on the control, on the state of the object and its integral:

$$J(y, x, u) = \frac{1}{2} \int_{t_0}^{T} [y^*(t)D(t)y(t) + x^*(t)Q(t)x(t) + u^*(t)R(t)u(t)]\, dt, \tag{3}$$

$$\dot{y} = C(t)x, \quad t \in [t_0, T], \quad y(t_0) = y_0, \quad y(T) = 0, \tag{4}$$

where $y(t)$ is a p-vector; $C(t)$ is a $(p \times n)$-matrix; $R(t)$ is a positive definite $(m \times m)$-matrix; $Q(t)$ and $D(t)$ are positive semidefinite $(n \times n)$ and $(p \times p)$-matrices, respectively. Symbol * denotes a transposition. Here y_0 is a given p-vector which characterizes the integrated property of the system trajectory $x(t)$:

$$\int_{t_0}^{T} C(t)x(t)\, dt = -y_0. \tag{5}$$

It is required to find a synthesizing control $u(y, x, t)$ such that the corresponding pair $(x(t), u(t))$ gives the minimum to the functional (3), where the trajectory $x(t)$, the solution of the differential equation (1), satisfies the integrated restriction (5) and the constraints (2) are hold for the control $u(t) = u(y(t), x(t), t)$.

A method based on the Lagrange multipliers of special type [7] is used to solve the optimal control problem (1)-(4).

3. Solving the Problem

Consider the following functional to solve the problem

$$\begin{aligned} L(y, x, u) = \int_{t_0}^{T} \Big\{ & \frac{1}{2} y^* D(t) y + \frac{1}{2} x^* Q(t) x + \frac{1}{2} u^* R(t) u + [K_1(t)y + K_2(t)x + q_1(t)]^* [C(t)x - \dot{y}] \\ & + \left[K_2^*(t)y + K_3(t)x + q_2(t)\right]^* [A(t)x + B(t)u + f(t) - \dot{x}] + \lambda_1^*(t)\,[\alpha(t) - u] + \lambda_2^*(t)\,[u - \beta(t)] \\ & + \lambda_3^* \left[x - W_2^*(t, T)q_1(t) - W_3(t, T)q_2(t) - z_2(t)\right] + \lambda_4^*(t)\,[y - W_1(t, T)q_1(t) - W_2(t, T)q_2(t) - z_1(t)] \Big\}\, dt, \end{aligned} \tag{6}$$

where $q_1(t)$ and $q_2(t)$ are p and n-vectors, respectively; $K_1(t)$, $K_2(t)$ and $K_3(t)$ are $(p \times p)$, $(p \times n)$ and $(n \times n)$-matrices, respectively. Multipliers $[K_2^*(t)y + K_3(t)x + q_2(t)]$ and $[K_1(t)y + K_2(t)x + q_1(t)]$ eliminate the constraints on (y, x, u) in the form of system of differential equations (1) and (4); multipliers $\lambda_1(t)$ and $\lambda_2(t)$ eliminate the constraints (2) on control u; multipliers $\lambda_3(t)$ and $\lambda_4(t)$ are used to determine boundary conditions for vector-functions $q_1(t)$ and $q_2(t)$. Such design of the functional (6) allows to reduce the original conditional extremum problem to the type of an unconditional extremum.

Using the optimality conditions, we obtain :

$$u = -R^{-1}(t)B^*(t)[K_2^*(t)y + K_3(t)x + q_2(t)] + R^{-1}(t)[\lambda_1(t) - \lambda_2(t)]. \quad (7)$$

Suppose that matrices $K_1(t)$, $K_2(t)$, $K_3(t)$ are solutions of Riccati equations:

$$\dot{K}_1(t) - K_2(t)B(t)R^{-1}(t)B^*(t)K_2^*(t) + D(t) = 0. \quad K_1(t_0) = K_{10}, \quad (8)$$

$$\dot{K}_2(t) - K_2(t)B(t)R^{-1}B^*(t)K_3(t) + K_2(t)A(t) + K_1(t)C(t) = 0, \quad K_2(t_0) = K_{20}, \quad (9)$$

$$\begin{aligned} &\dot{K}_3(t) + K_3(t)A(t) + A^*(t)K_3(t) - K_3(t)B(t)R^{-1}(t)B^*(t)K_3(t) \\ &\quad + K_2^*(t)C(t) + C^*(t)K_2(t) + Q(t) = 0, \quad K_3(t_0) = K_{30}, \end{aligned} \quad (10)$$

and the vector-functions $q_1(t)$, $q_2(t)$, $q_3(t)$ satisfy the differential equations

$$\dot{q}_1 = K_2(t)B(t)R^{-1}(t)B^*(t)q_2 - W_1^{-1}(t,T)W_2(t,T)[W_3(t,T) - W_2^*(t,T)W_1^{-1}(t,T)W_2(t,T)]^{-1}B(t)\varphi(t), \quad (11)$$

$$\dot{q}_2 = -C^*(t)q_1 - \left[A(t) - B(t)R^{-1}(t)B^*(t)K_3(t)\right]^* q_2 + \left[W_3(t,T)W_2^*(t,T) - W_1^{-1}(t,T)W_2(t,T)\right]^{-1} B(t)\varphi(t), \quad (12)$$

where $\varphi(t) = R^{-1}(t)[\lambda_1(t) - \lambda_2(t)]$, and multipliers $\lambda_3(t)$, $\lambda_4(t)$ are determined as follows:

$$\lambda_3(t) = -K_3(t)f(t) - [K_2(t) - W_1^{-1}(t,T)W_2(t,T)][W_3(t,T) - W_2^*(t,T)W_1^{-1}(t,T)W_2(t,T)]^{-1}B(t)\varphi(t),$$

$$\lambda_4(t) = -K_2(t)f(t) - \{K_2(t) - W_1^{-1}(t,T)W_2(t,T)[W_3(t,T) - W_2^*(t,T)W_1^{-1}(t,T)W_2(t,T)]^{-1}\}B(t)\varphi(t).$$

Note that we determine the initial conditions for the differential equations (11) and (12) from the following equations:

$$y(t) = W_1(t,T)q_1(t) + W_2(t,T)q_2(t) + z_1(t), \quad (13)$$

$$x(t) = W_2^*(t,T)q_1(t) + W_3(t,T)q_2(t) + z_2(t), \quad (14)$$

where matrices $W_1(t,T)$, $W_2(t,T)$, $W_3(t,T)$ satisfy to the following matrix differential equations:

$$\dot{W}_1(t,T) = W_2(t,T)C^*(t) + C(t)W_2^*(t,T), \quad W_1(T,T) = 0,$$

$$\dot{W}_2(t,T) = -W_1(t,T)K_2(t)B_1(t) + W_2(t,T)A_1^*(t) + C(t)W_3(t,T), \quad W_2(T,T) = 0,$$

$$\dot{W}_3(t,T) = -W_2^*(t,T)K_2(t)B_1(t) - B_1(t)K_2^*(t)W_2(t,T) + W_3(t,T)A_1^*(t) + A_1(t)W_3(t,T) - B_1(t), \quad W_3(T,T) = 0,$$

and vector-functions $z_1(t)$, $z_2(t)$ are determined as the solution of the differential equations:

$$\dot{z}_1 = C(t)z_2, \quad z_1(T) = 0,$$

$$\dot{z}_2 = -B_1(t)K_2^*(t)z_1 + A_1(t)z_2 + f(t), \quad z_2(T) = 0,$$

from which we find $z_1(T_0)$, $z_2(t_0)$, and then determine the initial conditions using (13) and (14):

$$\begin{aligned} q_1(t_0) = {} & W_1^{-1}(t_0,T)\{I + W_2(t_0,T)\left[W_3(t_0,T) - W_2^*(t_0,T)W_1^{-1}(t_0,T)W_2(t_0,T)\right]^{-1} W_2^*(t_0,T)W_1^{-1}(t_0,T)\} \\ & \times [y(t_0) - z_1(t_0)] - W_1^{-1}(t_0,T)W_2(t_0,T)\left[W_3(t_0,T) - W_2^*(t_0,T)W_1^{-1}(t_0,T)W_2(t_0,T)\right]^{-1}[x(t_0) - z_2(t_0)], \end{aligned} \quad (15)$$

$$q_2(t_0) = \left[W_3(t_0,T) - W_2^*(t_0,T)W_1^{-1}(t_0,T)W_2(t_0,T)\right]^{-1}[x(t_0) - z_2(t_0)] - W_2^*(t_0,T)W_1^{-1}(t_0,T)[y(t_0) - z_1(t_0)]. \quad (16)$$

Thus a differential equation which determines the law of motion of the system (1) with the control (7) is as follows

$$\dot{x} = A_1(t)x - B_1(t)[K_2^*(t)y + q_2(t)] + B(t)\varphi(t) + f(t), \quad x(t_0) = x_0, \quad x(T) = 0,$$

$$\dot{y} = C(t)x, \quad y(t_0) = y_0, \quad y(T) = 0,$$

where $A_1(t) = A(t) - B(t)R^{-1}(t)B^*(t)K_3(t)$, $B_1(t) = B(t)R^{-1}(t)B^*(t)$.

Let

$$\omega(y, x, t) = -R^{-1}(t)B^*(t)[K_2^*(t)y + K_3(t)x + q_2(t)].$$

Multipliers $\lambda_1(t) \geq 0$, $\lambda_2(t) \geq 0$, and control $u(t)$ we determine so that they satisfy the following conditions:

$$R(t)[u(t) - \omega(y, x, t)] - \lambda_1(t) + \lambda_2(t) = 0,$$

$$\lambda_1^*(t)[\alpha(t) - u(t)] = 0, \quad \lambda_2^*(t)[u(t) - \beta(t)] = 0.$$

Hence we obtain the following

Theorem 3.1. *A pair $(x(t), u(t)) \in \Delta(t_0, T)$ is optimal if and only if*

1) the state vector $x(t)$ satisfies the following differential equation:

$$\dot{x} = A_1(t)x - B_1(t)[K_2^*(t)y + q_2(t)] + B(t)\varphi(y, x, t) + f(t), \quad x(t_0) = x_0, \quad x(T) = 0;$$

2) the control vector $u(t)$ is determined as

$$u(y, x, t) = -R^{-1}(t)B^*(t)[K_2^*(t)y + K_3(t)x + q_2(t)] + \varphi(y, x, t),$$

where matrices $K_2(t)$, $K_3(t)$ are solutions of equations (8)-(10), vector-function $q_2(t)$ satisfies equations (11), (12) with initial conditions (15), (16), and $\varphi(y, x, t)$ is determined as follows:

$$\varphi(y, x, t) = R^{-1}(t)[\lambda_1(y, x, t) - \lambda_2(y, x, t)].$$

The theorem can be proved by using of a special type Lagrange multipliers [7].

4. Example

We consider as an example the optimal control problem: minimize the functional

$$J(y, x, u) = \frac{1}{2}\int_{t_0}^{T} [y^2 + 3x_1^2 + 3x_2^2 + u^2]\, dt$$

subject to:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad \dot{y} = x_1, \quad x_1(t_0) = 2, \quad x_1(T) = 0, \quad x_2(t_0) = 2, \quad x_2(T) = 0,$$

$$\int_{t_0}^{T} x_1(t)\, dt = 8, \quad y(t_0) = -8, \quad y(T) = 0, \quad \alpha \leq u(t) \leq \beta, \quad t \in [t_0, T], \quad \alpha = -1.5, \quad \beta = 1.5, \quad t_0 = 0, \quad T = 7.$$

The desired optimal control can be written as

$$u(y, x, t) = \omega(y, x, t) + \varphi(y, x, t),$$

where

$$\omega(y, x, t) = -y - 3x_1 - 3x_2 - q_3(t), \quad \varphi(y, x, t) = \max\{0;\ \alpha - \omega(y, x, t)\} - \max\{0;\ \omega(y, x, t) - \beta\}.$$

Optimal trajectories $y(t)$, $x_1(t)$, $x_2(t)$ in the time interval $[t_0, T]$ are determined by the system of differential equations

$$\dot{y}(t) = x_1(t), \quad y(t_0) = -8; \quad \dot{x}_1(t) = x_2(t), \quad x_1(t_0) = 2;$$
$$\dot{x}_2(t) = -y(t) - 3x_1(t) - 3x_2(t) - q_3(t) + \varphi(y, x, t), \quad x_2(t_0) = 2;$$
$$\dot{q}_1(t) = q_3(t) + m_3(t)\varphi(y, x, t), \quad q_1(t_0) = q_{10},$$
$$\dot{q}_2(t) = -q_1(t) + 3q_3(t) + m_5(t)\varphi(y, x, t), \quad q_2(t_0) = q_{20},$$
$$\dot{q}_3(t) = -q_2(t) + 3q_3(t) + m_6(t)\varphi(y, x, t), \quad q_3(t_0) = q_{30}.$$

and the optimal control $u(y,x,t)$ is obtained as following

$$u(y,x,t) = \begin{cases} -1.5, & t \le t \le t_1, \\ -y(t) - 3x_1(t) - 3x_2(t) - q_3(t), & t_1 < t \le T, \end{cases}$$

where control switching is performed at time $t_1 \approx 2.52$, which is defined from the condition

$$-y(t_1) - 3x_1(t_1) - 3x_2(t_1) - q_3(t_1) = -1.5.$$

Graphs of solutions $y = y(t)$, $x_1 = x_1(t)$, $x_2 = x_2(t)$ and $u = u(y, x_1, x_2, t)$ are shown on Figs. 1, 2. The obtained control u provides a fairly accurate fulfillment of the final conditions $y(T) = 0$, $x_1(T) = 0$, $x_2(T) = 0$ (in the numerical calculations were obtained the values: $y(T) \approx -0.314 \cdot 10^{-8}$, $x_1(T) \approx 0.244 \cdot 10^{-8}$, $x_2(T) \approx -0.180 \cdot 10^{-8}$).

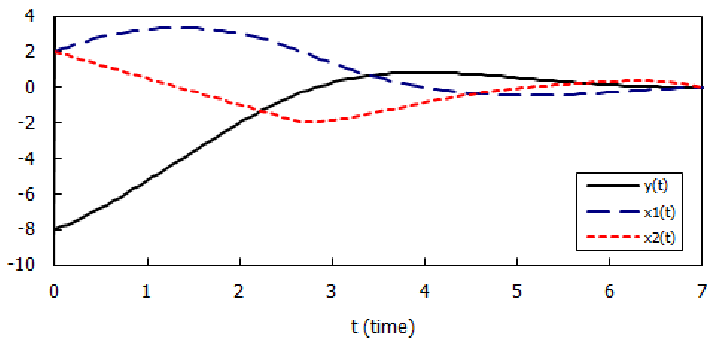

Figure 1: Graph of the optimal trajectories

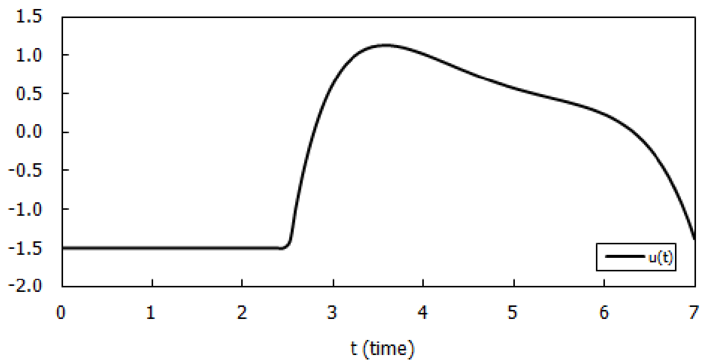

Figure 2: Graph of the optimal control

5. Conclusion

In this paper, we propose a constructive method PI controller design based on the principle of feedback and transfers the system from the initial state to the desired final state during a given time interval with constraints on the control values. The problem is solved using Lagrange multipliers of a special type, depending on the phase coordinates and time. The proposed method for solving the optimal control problem with fixed endpoints of trajectories and constraints on the values of the control was implemented on the computer using the application package Maple 15 and tested using the model task example.

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