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One-Dimensional Schrodinger Operator with a Negative Parameter and Its Applications to the Study of the Approximation Numbers of a Singular Hyperbolic Operator

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Abstract. In this paper we use the one-dimensional Schrödinger operator with a negative parameter to the study of the approximation numbers of a hyperbolic type singular operator. Estimates for the distribution function of the approximation numbers are obtained.

1. Introduction

This paper is the continuation of the paper [8]. The questions on existence of the resolvent and discreetness of the spectrum for a class of singular differential operators of hyperbolic type are studied in [8]. In the paper we give estimates of the distribution function of approximation numbers (s-numbers) of the above mentioned operators. It is well-known [1, 13] that the estimates of the approximation numbers (s-numbers) give a possibility to estimate eigenvalues.

The questions of qualitative description of operators in bounded domains by using approximation numbers have been well studied and quite comprehensive bibliography is given, for example, in [1, 13]. In the case of an unbounded domain the issues of the properties of approximation numbers of differential operators are studied in [4, 10]. This research is devoted only to elliptic and pseudo-differential operators.

A review of literature shows that the problems of qualitative descriptions of the resolvents of hyperbolic type differential operators in unbounded domains using the approximation numbers are insufficiently studied.

Consider the differential operator of hyperbolic type

$$A_0 u = u_{xx} - u_{yy} + a(y)u_x + c(y)u$$

in the space $L_2(\Omega)$ with the domain $C^{\infty}_{0,\pi}(\overline{\Omega})$ of infinitely differentiable functions satisfying the conditions $u(-\pi,\ y) = u(\pi,\ y)$ and $u_x(-\pi,\ y) = u_x(\pi,\ y)$ and compactly supported with respect to the variable y, where $\Omega = \{(x, y) : -\pi < x < \pi, -\infty < y < \infty)\}$.

Further, we assume that the coefficients $a(y), c(y)$ satisfy the conditions:

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i) $|a(y)| \geq \delta_0 > 0$, $c(y) \geq \delta > 0$ are continuous functions in $\mathbb{R}$ ($\mathbb{R} = (-\infty, \infty)$);

ii) $\mu_0 = \sup\limits_{|y-t|\leq 1} \frac{c(y)}{c(t)} < \infty, \mu_1 = \sup\limits_{|y-t|\leq 1} \frac{a(y)}{a(t)} < \infty$.

iii) $\lim\limits_{|y|\to\infty} c(y) = \infty$.

It is easy to show that the operator A_0 admits a closure in the metric of the space $L_2(\Omega)$, which will be denoted by A.

From the results of [8] it follows that

1) there exists the resolvent of the operator A when the condition *i)* is fulfilled;

2) the spectrum of the operator A is discrete when the conditions *i)*-*iii)* are fulfilled.

Further, the question of discretness of the spectrum is associated with such problems as the estimations of approximation numbers (s-numbers) and their distribution function.

The nonzero s-numbers of the operator $(A + \mu I)^{-1}$ for $\mu \geq 0$ will be ordered according to decreasing magnitude and observing their multiplicities

$$s_k((A + \mu I)^{-1}) = \lambda_k \left(\sqrt{[(A + \mu I)^{-1}]^*(A + \mu I)^{-1}} \right), \quad k = 1, 2, 3, ...,$$

where λ_k are eigenvalues of the operator $\sqrt{[(A + \mu I)^{-1}]^*(A + \mu I)^{-1}}$.

For all s_k greater than $\lambda > 0$, we introduce the counting functions $N(\lambda) = \sum\limits_{(s_k>\lambda)} 1$.

The main result of this paper is the following theorem.

Theorem 1.1. *Let the conditions i)-iii) be satisfied. Then the estimate*

$$c^{-1} \sum_{n=-\infty}^{\infty} \lambda^{-\frac{1}{2}} mes(y \in \mathbb{R} : Q_n(y) \leq c^{-1}\lambda^{-1}) \leq N(\lambda) \leq c \sum_{n=-\infty}^{\infty} \lambda^{-1} mes(y \in \mathbb{R} : K_n^{\frac{1}{2}}(y) \leq c\lambda^{-1}),$$

holds, where $Q_n(y) = \left|n^2 + ina(y) + c(y) + \mu\right|$, $K_n(y) = \left(\left|na(y)\right| + c(y) + \mu\right)$ *and the constant* $c > 0$ *is independent of* $Q_n(y), K_n(y)$ *and* λ.

2. Some Properties of the One-Dimensional Schrodinger Operator

In this section we study the Sturm-Liouville operator with a negative parameter in $L_2(\mathbb{R})$

$$(l_n + \mu I)\, u = -u'' + \left(-n^2 + ina(y) + c(y) + \mu\right) u$$

with the domain $C_0^\infty(\mathbb{R})$ of infinitely differentiable and compactly supported functions, where the parameter $n = 0, \pm1, \pm2, ..., \ i^2 = -1, \ \mu \geq 0$.

The operator $l_n + \mu I$ admits a closure which we also denote by $l_n + \mu I$.

We note previously that the following cases:

I) $l_0 u = -u'' + c(y)u, \ u \in D(l_0)$ (a very comprehensive bibliography is contained in [2, 3, 5, 11, 12],

II) $lu = -u'' + (c_1(y) + ic_2(y))u, \ u \in D(l)$

have been well studied when $n = 0$, where $c_1(y) \geq 0, \ c_2(y) \geq 0, i^2 = -1$. The issues of discreetness of the spectrum and estimates of the approximation numbers (s-numbers) are well studied in [2, 4, 10] for this case.

It is easy to see that $-n^2 \to -\infty$ for $|n| \to \infty$ in the operator l_n. Hence, we note that the operator l_n is not a semi-bounded operator. Therefore there is a completely different situation in this case. Because, in particular, the methods used in the study of cases I) and II) are not quite suitable to the study of spectral problems of the operator l_n. Therefore the spectral problems of the operator l_n require special approach.

Lemma 2.1. *Let the condition i) be satisfied. Then there exist a continuous inverse operator* $(l_n + \mu I)^{-1}$ *to the operator* $l_n + \mu I$ *defined on the whole* $L_2(\mathbb{R})$ *for* $\mu \geq 0$.

Lemma 2.2. *Let the conditions i)-ii) be satisfied. Then the operator $(l_n + \mu I)^{-1}$ is compact if and only if*

$$\lim_{|y|\to\infty} c(y) = \infty.$$

Proofs of Lemmas 2.1 and 2.2 follow from Lemmas 2.9-2.13 of [8] and from the following inequality

$$\mu_0^{-1} \cdot w \cdot c(y) \leq \int_y^{y+w} c(t)dt \leq \mu_0 \cdot w \cdot c(y),$$

where $w \leq 1$.

The proof of the last inequality follows from the condition *ii*).

Lemma 2.3. *Let the condition i) be satisfied and $\mu \geq 0$. Then the estimate*

$$\| \sqrt{c(y)}u\|_2 + \| \sqrt{|na(y)|}u\|_2 + \|u'\|_2 \leq c_0\|(l_n + \mu I)u\|_2,$$

holds for all $u \in D(l_n)$, where $c > 0$ is a constant and $\| \cdot \|_2$ is the norm in the space $L_2(\mathbb{R})$.

Proof. Using Lemma 2.9 of the paper [8], we have

$$\sqrt{c(y)}u = \sqrt{c(y)}(l_n + \mu I)^{-1}f = \sqrt{c(y)}(l_{n,\gamma} + \mu I)^{-1}(I - A_{\mu,\gamma})^{-1}f$$

where $(l_n + \mu I)u = f$, $f \in L_2(\mathbb{R})$. Here

$$(l_{n,\gamma} + \mu I)u = -u'' + (-n^2 + in(a(y) + \gamma) + c(y) + \mu)u, \ \ u \in D(l_{n,\gamma}),$$

where γ is a number such that $a(y) \cdot \gamma > 0$ for $y \in \mathbb{R}$. It is not difficult to see that

$$(l_n + \mu I)u = (l_{n,\gamma} + \mu I)u - in\gamma u = f \in L_2(\mathbb{R}).$$

Let us denote $(l_{n,\gamma} + \mu I)u = v$, $u = (l_{n,\gamma} + \mu I)^{-1}v$ and $in\gamma u = in\gamma(l_{n,\gamma} + \mu I)^{-1}v = A_{\mu,\gamma}v$. Then the equality

$$(l_n + \mu I)u = v - A_{\mu,\gamma}v = f$$

holds, i.e. the equation $(l_n + \mu I)u = f$ is equivalent to the equation $v - A_{\mu,\gamma}v = f$. This implies that

$$(l_n + \mu I)^{-1}f = (l_{n,\gamma} + \mu I)^{-1}(I - A_{\mu,\gamma})^{-1}f.$$

The properties of the operator $l_{n,\gamma} + \mu I$ are stated in [8].

Hence

$$\begin{aligned} &\| \sqrt{c(y)}u\|_2 = \| \sqrt{c(y)}(l_n + \mu I)^{-1}f\|_2 = \| \sqrt{c(y)}(l_{n,\gamma} + \mu I)^{-1}(I - A_{\mu,\gamma})^{-1}f\|_2 \leq \\ &\leq \left\| \sqrt{c(y)}\left(l_{n,\gamma} + \mu I\right)^{-1} \right\|_{2\to 2} \left\| \left(I - A_{\mu,\gamma}\right)^{-1} f \right\|_2 \leq c_1 \left\| \sqrt{c(y)}\left(l_{n,\gamma} + \mu I\right)^{-1} \right\|_{2\to 2} \left\|f\right\|_2 \end{aligned} \tag{1}$$

where $c_1 = \|(I - A_{\mu,\gamma})^{-1}\|_{2\to 2}$.

From (1) and Lemmas 2.6 and 2.7 of the paper [8] we find, that

$$\begin{aligned} &\| \sqrt{c(y)}u\|_2 \leq c_1\| \sqrt{c(y)}(l_{n,\gamma} + \mu I)^{-1}\|_{2\to 2} \cdot \|f\|_2 \leq \\ &\leq c_1\| \sqrt{c(y) + \mu}(l_{n,\gamma} + \mu I)^{-1}\|_{2\to 2} \cdot \|f\|_2 \leq \\ &\leq c_1 \sup_{j\in\mathbb{Z}} \| \sqrt{c(y) + \mu}\varphi_j(l_{n,\gamma,j} + \mu I)^{-1}\|_{L_2(\Delta_j)\to L_2(\Delta_j)} \cdot \|f\|_2. \end{aligned}$$

From this relation and from *a*) of Lemma 2.2 of [8] we have

$$\| \sqrt{c(y)}u\|_2 \leq c_3\|f\|_2, \tag{2}$$

where $c_3 = c_1 \cdot c_2$, $c_2 = \sup_{j\in\mathbb{Z}} \| \sqrt{c(y)+\mu}\varphi_j(l_{n,\gamma,j}+\mu I)^{-1}\|_{L_2(\Delta_j)\to L_2(\Delta_j)}$. From (2) it follows that

$$\| \sqrt{c(y)}u\|_2 \le c_3\|(l_n+\mu I)u\|_2, \tag{3}$$

where $c_3 > 0$ is a constant.

Similarly, by reproducing the computations and argument used in the proof of the inequality (3), we obtain the following inequalities:

$$\left\| \sqrt{na(y)}u\right\|_2 \le c_4 \left\|(l_n+\mu I)u\right\|_2, \tag{4}$$

$$\|u'\|_2 \le c_5 \left\|(l_n+\mu I)\, u\right\|_2, \tag{5}$$

where $c_4 > 0$ and $c_5 > 0$ are constants.

The proof of Lemma 2.3 follows from the inequalities (3), (4) and (5). Here $c_0 = \max\{c_3, c_4, c_5\}$. □

We introduce the following sets:

$$M = \left\{u \in L_2(\mathbb{R}) : \ \|l_n u\|_2^2 + \|u\|_2^2 \le 1\right\},$$

$$\tilde{M}_{c_0} = \left\{u \in L_2(\mathbb{R}) : \ \|u'\|_2^2 + \left\| \sqrt{|na(y)|}u\right\|_2^2 + \left\| \sqrt{c(y)}u\right\|_2^2 \le c_0\right\},$$

$$\tilde{\tilde{M}}_{c_0^{-1}} = \left\{u \in L_2(\mathbb{R}) : \ \|-u''\|_2^2 + \left\|n^2u\right\|_2^2 + \left\|ina(y)u\right\|_2^2 + \left\|c(y)u\right\|_2^2 \le c_0{}^{-1}\right\},$$

where $c_0 > 1$ is a constant.

Lemma 2.4. *Let the condition i) be fulfilled. Then the inclusions*

$$\tilde{\tilde{M}}_{c_0^{-1}} \subseteq M \subseteq \tilde{M}_{c_0}$$

hold, where $c_0 > 1$ is a constant independent of u and n.

Proof. This lemma can be proved by methods in [7, 9]. We give the proof for completeness. Let $u \in \tilde{\tilde{M}}_{c_0^{-1}}$. Then

$$\|l_n u\|_2^2 + \|u\|_2^2 \le \|-u''\|_2^2 + \|n^2u\|_2^2 + \|ina(y)u\|_2^2 + \|c(y)u\|_2^2 + \|u\|_2^2 \le$$
$$\le c_0(\|-u''\|_2^2 + \|n^2u\|_2^2 + \|ina(y)u\|_2^2 + \|c(y)u\|_2^2) \le c_0 \cdot c_0^{-1} \le 1, c_0 = c_0(\delta).$$

This implies that $u \in M$, i.e. $\tilde{\tilde{M}}_{c_0^{-1}} \subseteq M$.

Now, let $u \in M$. Then, by virtue of Lemma 2.3, we have

$$\|u'\|_2^2 + \| \sqrt{|na(y)|}u\|_2^2 + \| \sqrt{c(y)}u\|_2^2 \le c_0 \cdot \|(l_n+\mu I)u\|_2^2 \le c_0(\|l_n u\|_2^2 + \|u\|_2^2) \le c_0.$$

Therefore $u \in \tilde{M}_{c_0}$, i.e. $M \subseteq \tilde{M}_{c_0}$. □

The Kolmogorov k-width of the set M is defined as follows

$$d_k = \inf_{\{G_k\}} \sup_{u\in M} \inf_{v\in G_k} \|u - v\|_2,\ k\ =\ 0,\ 1,\ 2,\ ...,$$

where G_k is a set of all subspaces of $L_2(\mathbb{R})$ whose dimension does not exceed k.

Lemma 2.5. *Let the condition i) be fulfilled. Then the estimate*

$$c_0^{-1}\tilde{\tilde{d}}_k \le s_{k+1} \le c_0\tilde{d}_k,\ \ k = 1,\ 2,\ ...,$$

holds, where $c_0 > 1$ is a constant, s_{k+1} are the s-numbers of the operator l^{-1}, and $d_k, \tilde{d}_k$ and $\tilde{\tilde{d}}_k$ are the Kolmogorov widths of the sets M, $\tilde{M}_{c_0}$ and $\tilde{\tilde{M}}_{c_0^{-1}}$, respectively.

Proof. From Lemma 2.4 and properties of the widths it follows that

$$c_0^{-1}\tilde{\tilde{d}}_k \le d_k \le c_0\tilde{d}_k.$$

Hence, taking the equality $s_{k+1} = d_k$ ([1]) into account, we obtain the proof of Lemma 2.5. □

Lemma 2.6. *Let the condition i) be fulfilled. Then the estimate*

$$\tilde{\tilde{N}}(c_0\lambda) \le N(\lambda) \le \tilde{N}(c_0^{-1}\lambda)$$

holds, where $N(\lambda) = \sum\limits_{s_{k+1}>\lambda} 1$ *is a counting function for* s_{k+1}, $\tilde{N}(\lambda) = \sum\limits_{\tilde{d}_k>\lambda} 1$ *is a counting function for* $\tilde{d}_k$, $\tilde{\tilde{N}}(\lambda) = \sum\limits_{\tilde{\tilde{d}}_k>\lambda} 1$ *is a counting function for* $\tilde{\tilde{d}}_k$.

Proof. According to Lemma 2.5, we have

$$N(\lambda) = \sum_{s_{k+1}>\lambda} 1 \le \sum_{c_0\tilde{d}_k>\lambda} 1 = \sum_{\tilde{d}_k>c_0^{-1}\lambda} 1 = \tilde{N}(c_0^{-1}\lambda).$$

Similarly

$$\tilde{\tilde{N}}(c\lambda) = \sum_{\tilde{\tilde{d}}_k>c\lambda} 1 = \sum_{c^{-1}\tilde{\tilde{d}}_k>\lambda} 1 \le \sum_{s_{k+1}>\lambda} 1 = N(\lambda).$$

Lemma 2.6 is proved. □

Lemma 2.7. *Let the condition i)-ii) be fulfilled and* $\mu \ge 0$*. Then the estimate*

$$c^{-1}\lambda^{-\frac{1}{2}} mes(y \in \mathbb{R} : Q_n(y) \le c^{-1}\lambda^{-1}) \le N(\lambda) \le c\lambda^{-1} mes(y \in \mathbb{R} : K_n^{\frac{1}{2}}(y) \le c\lambda^{-1})$$

holds for approximate numbers (s-numbers) of the operator $(l_n + \mu I)^{-1}$*, where* $N(\lambda) = \sum_{s_k>\lambda} 1$ *is a counting function for* s_k*,* s_k *are the singular numbers of the operator* $(l_n + \mu I)^{-1}$.

Proof. By $L_2^2(\mathbb{R}, Q_n(y))$, $L_2^1(\mathbb{R}, K_n(y))$ we denote the space obtained by replenishment of $C_0^\infty(\mathbb{R})$ with respect to the norms

$$||u||_{L_2^2(\mathbb{R},Q_n(y))} = \left(\int_{-\infty}^{\infty} (|u''|^2 + Q_n^2(y)|u|^2)dy\right)^{\frac{1}{2}},$$

$$||u||_{L_2^1(\mathbb{R},K_n(y))} = \left(\int_{-\infty}^{\infty} (|u'|^2 + K_n(y)|u|^2)dy\right)^{\frac{1}{2}},$$

where the functions $Q_n(y)$ and $K_n(y)$ are the same as in Theorem 1.1.

It is easy to verify that

$$\tilde{\tilde{M}} \subset L_2^2(\mathbb{R}, Q_n(y)), \tilde{M} \subset L_2^1(\mathbb{R}, K_n(y)).$$

From this relation and from Lemmas 2.4-2.6 it follows that in order to obtain estimates for the widths of the sets $\tilde{M}_c$, $\tilde{\tilde{M}}_{c^{-1}}$ it is sufficient to find the estimates for the widths of the subspaces

$$L_2^2(\mathbb{R}, Q_n(y)) \hookrightarrow L_2(\mathbb{R})$$

and

$$L_2^1(\mathbb{R}, K_n(y)) \hookrightarrow L_2(\mathbb{R}),$$

Hence, the proof of Lemma 2.7 follows from the results of [11] and the inequalities [6]

$$c_1^{-1} K_n^{\frac{1}{2}}(y) \le K_n^*(y) \le c_1 K_n^{\frac{1}{2}}(y),$$

$$c_2^{-1} Q_n^{\frac{1}{2}}(y) \le Q_n^*(y) \le c_2 Q_n^{\frac{1}{2}}(y)$$

are valid under the conditions *i)*-*ii)*, where $K_n^*(y)$, $Q_n^*(y)$ are special averaging of functions $K_n(y)$, $Q_n(y)$ [11]. □

Now we give proof of Theorem 1.1.

Proof. From Theorem 1.1 and (3.4) of the paper [8] it follows that the operator $(A + \mu I)^{-1}$ has the following representation

$$u\,(x, y) = (A + \mu I)^{-1} f\,(x, y) = \sum_{\mathbf{n}=-\infty}^{\infty} (l_n + \mu I)^{-1} f_n\,(y) \cdot e^{inx}, \tag{6}$$

where $f(x, y) \in L_2(\Omega)$, $f(x, y) = \sum\limits_{n=-\infty}^{\infty} f_n(y) \cdot e^{inx}$, $(i^2 = -1)$, $(l_n + \mu I)^{-1}$ is the inverse operator to the operator $l_n + \mu I$.

Therefore, from (6) it follows that if s is a singular point of the operator $(A + \mu I)^{-1}$, then s is a singular number of one of the operator $(l_n + \mu I)^{-1}$ and vice versa, if s is a singular number of one of the operators $(l_n + \mu I)^{-1}$, then s is a singular point of the operator $(A + \mu I)^{-1}$. Hence the proof of Theorem 1.1 easily follows from Lemma 2.7. □

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