



Skew-Rank of an Oriented Graph in Terms of the Rank and Dimension of Cycle Space of its Underlying Graph

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Abstract. Let G^σ be an oriented graph and $S(G^\sigma)$ be its skew-adjacency matrix, where G is called the underlying graph of G^σ . The skew-rank of G^σ , denoted by $sr(G^\sigma)$, is the rank of $S(G^\sigma)$. Denote by $d(G) = |E(G)| - |V(G)| + \theta(G)$ the dimension of cycle spaces of G , where $|E(G)|$, $|V(G)|$ and $\theta(G)$ are the edge number, vertex number and the number of connected components of G , respectively. Recently, Wong, Ma and Tian [European J. Combin. 54 (2016) 76–86] proved that $sr(G^\sigma) \leq r(G) + 2d(G)$ for an oriented graph G^σ , where $r(G)$ is the rank of the adjacency matrix of G , and characterized the graphs whose skew-rank attain the upper bound. However, the problem of the lower bound of $sr(G^\sigma)$ of an oriented graph G^σ in terms of $r(G)$ and $d(G)$ of its underlying graph G is left open till now. In this paper, we prove that $sr(G^\sigma) \geq r(G) - 2d(G)$ for an oriented graph G^σ and characterize the graphs whose skew-rank attain the lower bound.

1. Introduction

In this paper, we only consider simple graphs without multiple edges and loops. Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. The *adjacency matrix* of G of order n is defined as the $n \times n$ symmetric square matrix $A = A(G) = (a_{ij})$, where $a_{ij} = 1$ if $v_i v_j \in E(G)$, otherwise $a_{ij} = 0$. The *rank* $r(G)$ of G is defined to be the rank of $A(G)$, and the *nullity* $\eta(G)$ of G is defined to be the multiplicity of 0 as an eigenvalue of $A(G)$. Obviously, $|V(G)| = r(G) + \eta(G)$. We use Bondy and Murty [2] for terminologies and notations not defined here.

An oriented graph G^σ is a digraph which assigns each edge of G with a direction σ , where G is called the *underlying graph* of G^σ . The *skew-adjacency matrix* associated to G^σ is the $n \times n$ matrix $S(G^\sigma) = (s_{ij})$, where $s_{ij} = -s_{ji} = 1$ if (v_i, v_j) is an arc of G^σ , otherwise $s_{ij} = s_{ji} = 0$. The *skew-rank* $sr(G^\sigma)$ of an oriented graph G^σ is defined as the rank of the skew-adjacency matrix $S(G^\sigma)$. Since $S(G^\sigma)$ is skew-symmetric, every eigenvalue of $S(G^\sigma)$ is a pure imaginary number or 0, and the skew-rank of an oriented graph is even.

Let $C_n^\sigma = v_1 v_2 \cdots v_n v_1$ be an even oriented cycle. Denote by $\text{sgn}(C_n^\sigma)$ the *sign* of C_n^σ , which is defined as the sign of $\prod_{i=1}^n s_{v_i, v_{i+1}}$ with $v_{n+1} = v_1$. An even oriented cycle C_n^σ is called *evenly-oriented* (resp., *oddly-oriented*) if its sign is positive (resp., negative). G^σ is called *evenly-oriented* if every even cycle in G^σ is evenly-oriented.

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Sometimes we use the notation $G^\sigma - H^\sigma$ instead of $G^\sigma - V(H^\sigma)$ if H^σ is an *induced subgraph* of G^σ , where $G^\sigma - H^\sigma$ is the subgraph obtained from G^σ by deleting all vertices of H^σ and all incident edges. For an induced subgraph H^σ and a vertex x outside H^σ , the induced subgraph of G^σ with vertex set $V(H^\sigma) \cup \{x\}$ is simply written as $H^\sigma + x$. For a vertex $v \in V(G^\sigma)$, let $G^\sigma - v$ denote the oriented graph obtained from G^σ by removing the vertex v and all edges incident with v . A vertex $x \in V(G)$ is called a *cut-point* of a connected graph G if the resultant graph $G - x$ has at least two components. A vertex of G^σ is called a *pendant* vertex if its degree is 1 in G , and is called a *quasi-pendant* vertex if it is adjacent to a pendant vertex. An induced subgraph H (resp., H^σ) of a graph G (resp., G^σ) is called a *pendant cycle* (resp., *pendant oriented cycle*) of G (resp., G^σ) if H is a cycle such that H has a unique vertex of degree 3 in G . Denote by $d(G) = |E(G)| - |V(G)| + \theta(G)$ the *dimension of cycle spaces* of G , where $|E(G)|$, $|V(G)|$ and $\theta(G)$ are the edge number, vertex number and the number of connected components of G , respectively. Obviously, when G is connected, then G is a tree if $d(G) = 0$, and G is a unicyclic graph if $d(G) = 1$. A *matching* in a graph G is a set of pairwise nonadjacent edges. A *maximum matching* is one that contains as many edges of G as possible. The *matching number* of G , denoted by $m(G)$, is the size of a maximum matching in G . Denote by P_n , C_n a path and a cycle of order n , respectively. A graph is called *empty* if it has some vertex and no edges.

In 1957, Collatz and Sinogowitz [5] first posed the problem of characterizing all graphs G with $\eta(G) > 0$. This problem is of great interest in both chemistry and mathematics. For a bipartite graph G which corresponds to an alternant hydrocarbon in chemistry, if $\eta(G) > 0$, it is indicated that the corresponding molecule is unstable. The nullity of a graph is also meaningful in mathematics since it is related to the singularity of adjacency matrix.

Till now, many scholars investigated the nullity of graphs, they focused on special graph classes, such as trees, unicyclic graphs, bicyclic graphs, bipartite graphs and so on. There are also some papers focused on the study of the connection between the nullity (or rank) of graphs G in terms of certain structural parameters, such as matching number, dimension of cycle spaces and so on. Recently, Wang and Wong [16] obtained the bounds for the matching number of G in terms of the $r(G)$ and $d(G)$, that is:

$$\left\lceil \frac{r(G) - d(G)}{2} \right\rceil \leq m(G) \leq \left\lfloor \frac{r(G) + 2d(G)}{2} \right\rfloor.$$

The bounds for the matching number can be rewritten in an equivalent form as bounds for the nullity of G , that is:

$$|V(G)| - 2m(G) - d(G) \leq \eta(G) \leq |V(G)| - 2m(G) + 2d(G).$$

In 2015, Song, Song and Tam [14] characterized the graphs G that satisfy the equality $\eta(G) = |V(G)| - 2m(G) + 2d(G)$. The lower bound $|V(G)| - 2m(G) - d(G)$ of $\eta(G)$ was characterized by Wang [15] and independently by Rula, Chang and Zheng in [13].

In 2016, Ma, Wong and Tian [9] proved that

$$\eta(G) \leq 2d(G) + p(G),$$

where $p(G)$ is the number of pendant vertices of G , they also proved that the equality is attained if and only if every component of G is a cycle with size a multiple of 4.

Recently, the skew-rank of skew-adjacency matrix of an oriented graph has received a lot of attentions. Li and Yu [7] studied the skew-rank of oriented graphs and characterized oriented unicyclic graphs attaining the minimum value of the skew-rank among oriented unicycle graphs of order n with girth k . Qu and Yu [11] characterized the bicyclic oriented graphs with skew-rank 2 or 4. Lu, Wang and Zhou [8] characterized the bicyclic oriented graphs with skew-rank 6. Qu, Yu and Feng [12] obtained more results about the minimum skew-rank of graphs. They also characterized the unicyclic graphs with skew-rank 4 or 6, respectively.

In [3], Chen and Tian proved that $sr(G^\sigma) \geq \sum_{i=1}^k q_i - 2k$ if G^σ is a connected oriented graph with k pairwise edge-disjoint cycles of orders q_1, q_2, \dots, q_k . In [10], Ma, Wong and Tian characterized the bounds of skew-rank of an oriented connected graph G^σ in terms of matching number, that is:

$$2m(G) - 2\beta(G) \leq sr(G^\sigma) \leq 2m(G),$$

where $\beta(G) = |E(G)| - |V(G)| + 1$. The oriented graphs satisfying $sr(G^\sigma) = 2m(G) - 2\beta(G)$ are characterized definitely.

In 2016, Wong, Ma and Tian [17] proved that

$$sr(G^\sigma) \leq r(G) + 2d(G)$$

for an oriented graph G^σ . They characterized the oriented graphs G^σ whose skew-rank can attain the upper bound.

A natural problem is : How about the lower bound of the skew-rank of an oriented graph G^σ in terms of the rank and the dimension of cycle spaces $d(G)$ of its underlying graph G ? In this paper, we will prove that

$$sr(G^\sigma) \geq r(G) - 2d(G)$$

for an oriented graph G^σ and characterize the oriented graphs G^σ whose skew-rank can attain the lower bound. Our main results are Theorems 1.1 and 1.3.

Theorem 1.1. *Let G^σ be a finite oriented graph without loops and multiple arcs. Then*

$$sr(G^\sigma) \geq r(G) - 2d(G).$$

Combining with the upper bound of the skew-rank of an oriented graph G^σ in [17] and our result in Theorem 1.1, we have

$$r(G) - 2d(G) \leq sr(G^\sigma) \leq r(G) + 2d(G).$$

Definition 1.2. ([17]) Let G be a graph with at least one pendant vertex. The operation of deleting a pendant vertex and its adjacent vertex from G is called δ -transformation.

The resultant subgraph G_0 of G without pendant vertices, obtained from G by applying a series of δ -transformations, is called a *crucial subgraph* of G . An oriented graph G^σ will be called *lower-optimal* if the skew-rank of G^σ attains the lower bound $r(G) - 2d(G)$. The cycles of G are called *pairwise vertex-disjoint* if distinct cycles (if any) of G have no common vertices.

Theorem 1.3. *Let G^σ be a finite oriented graph without loops and multiple arcs of order n . Then G^σ is lower-optimal if and only if the following conditions all hold.*

- (1) *Cycles (if any) of G^σ are pairwise vertex-disjoint.*
- (2) *Each cycle C_q^σ of G^σ is evenly-oriented with order $q \equiv 2 \pmod{4}$.*
- (3) *A series of δ -transformations can switch G to a crucial subgraph G_0 , which is the disjoint union of $d(G)$ cycles together with some isolated vertices.*

The rest of this paper is organized as follows: in Section 2, some necessary lemmas are introduced. In Section 3, we will prove Theorem 1.1. In Section 4, we will give some useful lemmas and theorems, and prove Theorem 1.3.

2. Preliminaries

In this section, we introduce some elementary lemmas and known results.

Lemma 2.1. ([7])

- (a) *Let H^σ be an induced subgraph of G^σ . Then $sr(H^\sigma) \leq sr(G^\sigma)$.*
- (b) *Let $G^\sigma = G_1^\sigma \cup G_2^\sigma \cup \dots \cup G_t^\sigma$, where $G_1^\sigma, G_2^\sigma, \dots, G_t^\sigma$ are connected components of G^σ . Then $sr(G^\sigma) = \sum_{i=1}^t sr(G_i^\sigma)$.*
- (c) *Let G^σ be an oriented graph on n vertices. Then $sr(G^\sigma) = 0$ if and only if G^σ is an empty graph.*

Note that the results of Lemma 2.1 also hold for the underlying graph G of G^σ .

Lemma 2.2. ([7]) Let T^σ be an oriented acyclic graph with matching number $m(T)$. Then $r(T) = sr(T^\sigma) = 2m(T)$.

Lemma 2.3. ([7]) Let C_n^σ be an oriented cycle of order n . Then we have

$$sr(C_n^\sigma) = \begin{cases} n, & C_n^\sigma \text{ is oddly-oriented,} \\ n - 2, & C_n^\sigma \text{ is evenly-oriented,} \\ n - 1, & \text{otherwise.} \end{cases}$$

Lemma 2.4. ([6]) Let G be a graph containing a pendant vertex, and H be the induced subgraph of G obtained by deleting this pendant vertex together with the vertex adjacent to it. Then $r(G) = r(H) + 2$.

Lemma 2.5. ([7]) Let G^σ be an oriented graph containing a pendant vertex, and H^σ be the induced subgraph of G^σ obtained by deleting this pendant vertex together with the vertex adjacent to it. Then $sr(G^\sigma) = sr(H^\sigma) + 2$.

Lemma 2.6. ([6]) Let x be a cut-point of a graph G and G_1, G_2, \dots, G_t be all components of $G - x$. If there exists a component, say G_1 , such that $r(G_1) = r(G_1 + x) - 2$, then $r(G) = r(G - x) + 2$. If $r(G_1) = r(G_1 + x)$, then $r(G) = r(G_1) + r(G - G_1)$.

Lemma 2.7. ([17]) Let x be a vertex of G^σ . Then $sr(G^\sigma - x)$ is equal either to $sr(G^\sigma)$ or to $sr(G^\sigma) - 2$.

Lemma 2.8. ([1]) If x is a vertex of a graph G , then $r(G) - 2 \leq r(G - x) \leq r(G)$.

Lemma 2.9. ([4]) Let C_q be a cycle of order q . Then $r(C_q) = q - 2$ if $q \equiv 0 \pmod{4}$, and $r(C_q) = q$ otherwise. Let P_q be a path of order q , then $r(P_q) = q$ if q is even, and $r(P_q) = q - 1$ if q is odd.

3. Proof for Theorem 1.1

In this section, we will prove Theorem 1.1. First, we will introduce the following lemma that will be useful for later.

Lemma 3.1. ([17]) Let G be a graph with a vertex x . Then

- (a) $d(G) = d(G - x)$ if x lies outside any cycle of G .
- (b) $d(G - x) \leq d(G) - 1$ if x lies on a cycle of G .
- (c) $d(G - x) \leq d(G) - 2$ if x is a common vertex of distinct cycles of G .
- (d) If the cycles of G are pairwise vertex-disjoint, then $d(G)$ precisely equals the number of cycles in G .

From [17], we know that a similar result as Lemma 3.1 holds for an oriented graph G^σ . Now, we will prove Theorem 1.1.

Proof of Theorem 1.1. We shall apply induction on $d(G)$ to prove $sr(G^\sigma) \geq r(G) - 2d(G)$.

Case 1. If $d(G) = 0$, then the result follows from Lemma 2.2.

Case 2. If $d(G) \geq 1$, then G^σ has at least one cycle. Let x be a vertex of a cycle of G^σ . By Lemma 3.1,

$$d(G - x) \leq d(G) - 1. \tag{1}$$

The induction hypothesis to $G^\sigma - x$ allows us to assume

$$sr(G^\sigma - x) \geq r(G - x) - 2d(G - x). \tag{2}$$

By Lemmas 2.1 and 2.8,

$$sr(G^\sigma) \geq sr(G^\sigma - x), r(G - x) \geq r(G) - 2. \tag{3}$$

Combining with inequalities (1)–(3), we have

$$sr(G^\sigma) \geq sr(G^\sigma - x) \geq r(G - x) - 2d(G - x) \geq r(G) - 2 - 2d(G) + 2 = r(G) - 2d(G). \tag{4}$$

This completes the proof. \square

Combining with the upper bound of the skew-rank of an oriented graph G^σ in [17] and our result in Theorem 1.1, we have

$$r(G) - 2d(G) \leq sr(G^\sigma) \leq r(G) + 2d(G).$$

Now, we will prove the Theorem 1.3 to characterize the graphs whose skew-rank can attain the lower bound.

4. Proof for Theorem 1.3

In this section, we will give some useful lemmas and theorems, and prove the Theorem 1.3.

Lemma 4.1. *Let x be a vertex lying on a cycle of G^σ . If G^σ is lower-optimal, then*

(a) $sr(G^\sigma) = sr(G^\sigma - x), r(G) = r(G - x) + 2, d(G) = d(G - x) + 1.$

(b) $G^\sigma - x$ is lower-optimal.

(c) x lies on only one cycle of G and x is not a quasi-pendant vertex of G .

Proof. From Theorem 1.1 and G^σ is lower-optimal, we have $r(G) - 2d(G) = sr(G^\sigma) \geq r(G) - 2d(G)$, which forces inequalities (1)–(3) in the proof of Theorem 1.1, all turn into equalities. So, (a) and (b) of this lemma are all derived.

By Lemma 3.1 and (a) of this lemma, we know that x cannot be a common vertex of two distinct cycles in G^σ .

Suppose that x is a quasi-pendant vertex adjacent to a pendant vertex y , by Lemma 2.5, we have $sr(G^\sigma) = sr(G^\sigma - x - y) + 2 = sr(G^\sigma - x) + 2$, which contradicts to (a) of this lemma.

This completes the proof. \square

From Lemma 2.6 of [8] and Lemma 4.3 of [17], we have the following lemma.

Lemma 4.2. ([8]) *Let C_q^σ be a pendant oriented cycle of G^σ with x the unique vertex of C_q of degree 3, and let $H^\sigma = G^\sigma - C_q^\sigma, K^\sigma = H^\sigma + x$. Then*

$$sr(G^\sigma) = \begin{cases} q - 2 + sr(K^\sigma), & C_q^\sigma \text{ is evenly-oriented,} \\ q + sr(H^\sigma), & C_q^\sigma \text{ is oddly-oriented,} \\ q - 1 + sr(K^\sigma), & \text{otherwise.} \end{cases}$$

Theorem 4.3. *Let C_q^σ be a pendant oriented cycle of G^σ with x the unique vertex of C_q of degree 3, and let $H^\sigma = G^\sigma - C_q^\sigma, K^\sigma = H^\sigma + x$. If G^σ is lower-optimal, then*

(a) $q \equiv 2 \pmod{4}$ and C_q^σ is evenly-oriented.

(b) $s(G^\sigma) = q - 2 + sr(K^\sigma), sr(H^\sigma) = sr(K^\sigma), r(G) = q + r(K)$ and $r(H) = r(K).$

(c) both H^σ and K^σ are lower-optimal.

Proof. Assertion (a) of this theorem will be derived after three claims.

Claim 1. q is even.

Suppose that q is odd, by Lemma 4.2,

$$sr(G^\sigma) = q - 1 + sr(K^\sigma). \tag{5}$$

Further, since G^σ is lower-optimal, $r(G) = sr(G^\sigma) + 2d(G) = q - 1 + sr(K^\sigma) + 2d(G) \geq q - 1 + r(K) - 2d(K) + 2d(G) = q - 1 + r(K) + 2 = q + 1 + r(K)$, where the inequality follows from Theorem 1.1.

Since x lies on the cycle C_q , by (a) of Lemma 4.1, we have

$$r(G) = r(G - x) + 2 = q - 1 + r(H) + 2 = q + 1 + r(H). \tag{6}$$

So, $r(G) = q + 1 + r(H) \geq q + 1 + r(K)$, i.e., $r(H) \geq r(K)$.

From Lemma 2.8, we know that $r(H) \leq r(K)$. So,

$$r(H) = r(K). \tag{7}$$

Let $A(G)$ be the adjacency matrix of G , where

$$A(G) = \begin{pmatrix} A & \alpha & 0 \\ \alpha^T & 0 & \beta \\ 0 & \beta^T & B \end{pmatrix},$$

where A is the adjacency matrix of $C_q - x$, B is the adjacency matrix of H , α^T refers to the transpose of α . From the process of the proof in Lemma 4.4 in [17], we have

$$r(G) = r \begin{pmatrix} A & 0 & 0 \\ 0 & a & \beta \\ 0 & \beta^T & B \end{pmatrix},$$

where $a = -\alpha^T A^{-1} \alpha$. So,

$$r(G) = r(A) + r \begin{pmatrix} a & \beta \\ \beta^T & B \end{pmatrix} \leq r(A) + r \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + r \begin{pmatrix} 0 & \beta \\ \beta^T & B \end{pmatrix}.$$

From Equation (7) of this theorem, we have

$$r(K) = r \begin{pmatrix} 0 & \beta \\ \beta^T & B \end{pmatrix} = r(H) = r(B).$$

That is

$$r(G) \leq q - 1 + 1 + r(H) = q + r(H). \tag{8}$$

Combining with Equations (6) and (8) of this theorem, we know have $r(G) = q + 1 + r(H) \leq q + r(H)$, this is a contradiction. So, q is even.

Let z be a vertex of C_q adjacent to x . By (a) of Lemma 4.1 and Lemmas 2.4 and 2.5, we have

$$sr(G^\sigma) = sr(G^\sigma - z) = q - 2 + sr(K^\sigma). \tag{9}$$

$$r(G) = r(G - z) + 2 = q - 2 + 2 + r(K) = q + r(K). \tag{10}$$

Since z lies on C_q , so

$$d(G) = d(K) + 1 = d(H) + 1. \tag{11}$$

Combining with Equations (9)–(11) and G^σ is lower-optimal, we have $sr(G^\sigma) = r(G) - 2d(G) = q + r(K) - 2d(K) - 2 = q - 2 + sr(K^\sigma)$, so

$$sr(K^\sigma) = r(K) - 2d(K). \tag{12}$$

By (a) of Lemma 4.1 and Lemmas 2.4 and 2.5, we also have

$$sr(G^\sigma) = sr(G^\sigma - x) = q - 2 + sr(H^\sigma). \tag{13}$$

$$r(G) = r(G - x) + 2 = q - 2 + 2 + r(H) = q + r(H). \tag{14}$$

Combining with Equations (10) and (14), we have

$$r(H) = r(K). \tag{15}$$

Combining with Equations (11), (13) and (14), we have $sr(G^\sigma) = r(G) - 2d(G) = q + r(H) - 2d(H) - 2 = q - 2 + sr(H^\sigma)$, so,

$$sr(H^\sigma) = r(H) - 2d(H). \tag{16}$$

Combining with Equations (9), (10), (12), (13), (15) and (16), we obtain (b) and (c) of this theorem.

Claim 2. $q \equiv 2 \pmod{4}$.

Suppose to the contrary that $q = 2m$, where m is an even integer. Let $G_1 = C_q - x$, by Lemma 2.9, we have $r(G_1) = r(C_q)$. By Lemma 2.6, we have $r(G) = r(G_1) + r(G - G_1) = q - 2 + r(K)$, which contradicts to Equation (10).

Claim 3. C_q^σ is evenly-oriented.

Suppose to the contrary that C_q^σ is oddly-oriented, by Lemma 4.2, then we have

$$sr(G^\sigma) = q + sr(H^\sigma), \tag{17}$$

which contradicts to Equation (13).

This completes the proof. \square

Theorem 4.4. Let y be a pendant vertex of G^σ adjacent to x , and let $H^\sigma = G^\sigma - y - x$. If G^σ is lower-optimal, then x does not lie on any cycle of G and H^σ is lower-optimal.

Proof. By (c) of Lemma 4.1, we know that x does not lie on any cycle of G .

By Lemmas 2.4 and 2.5, we have $r(H) = r(G) - 2$ and $sr(H^\sigma) = sr(G^\sigma) - 2$, respectively. Since x does not lie on any cycle of G , we have $d(G) = d(H)$. So,

$$sr(H^\sigma) = sr(G^\sigma) - 2 = r(G) - 2d(G) - 2 = r(H) + 2 - 2d(H) - 2 = r(H) - 2d(H).$$

This completes the proof. \square

The next paragraph is from [17], which will be useful for later.

In Section 4 of [17], let G be a graph with pairwise vertex-disjoint cycles, and let $C(G)$ denote the set of cycles in G . By compressing each cycle O of G into a vertex t_O , we obtain an acyclic graph T_G from G . More definitely, the vertex set $V(T(G))$ is taken to be $U \cup C_G$, where U consists of all vertices of G that do not lie on any cycle and C_G consists of vertex t_O that is obtained by compressing a cycle O , i.e., $C_G = \{t_O : O \in C(G)\}$. Two vertices in U are adjacent in T_G if and only if they are adjacent in G ; A vertex $u \in U$ is adjacent to a vertex $t_O \in C_G$ if and only if u is adjacent (in G) to a vertex in the cycle O ; Vertices t_{O_1}, t_{O_2} are adjacent in T_G if and only if there exists an edge in G joining a vertex of $O_1 \in C(G)$ to a vertex of $O_2 \in C(G)$. It is clear that T_G is always acyclic. Observe the graph $T_G - C_G$ (obtained from T_G by deleting vertices in C_G and the incident edges) is the same as the graph obtained from G by deleting all cycles and the incident edges, the resultant graph is denoted by Γ_G .

Theorem 4.5. Let G^σ be an oriented graph of order n . If G^σ is lower-optimal, then

- (a) cycles (if any) of G^σ are pairwise vertex-disjoint, each cycle C_q^σ of G^σ is evenly-oriented with order $q \equiv 2(\pmod 4)$.
- (b) $r(G) = r(T_G) + \sum_{O \in \mathcal{C}(G)} |V(O)|$ and $r(T_G) = r(\Gamma_G)$.

Proof. If G has no cycle, then the theorem holds naturally. Suppose G has cycles, let x be a vertex of any cycle. By Lemma 4.1, we know that x lies on only one cycle of G , so the first assertion of (a) follows.

We now proceed by induction on the order n to prove the left assertions.

If $n = 1$, then all left assertions hold naturally. Suppose the left assertions all hold for any lower-optimal oriented graph of order smaller than n , and suppose G^σ is a lower-optimal oriented graph of order $n \geq 2$.

Case 1. If T_G has no edges, i.e., G consists of disjoint cycles and some isolated vertices, then the left assertions follow from the following two claims.

Claim 1. G^σ is lower-optimal if and only if each component of G^σ is lower-optimal.

Claim 2. A single oriented cycle C_q^σ is lower-optimal if and only if C_q^σ is evenly-oriented with $q \equiv 2(\pmod 4)$ (by Lemmas 2.3 and 2.9).

Case 2. If T_G has at least one edge, then T_G has at least one pendant vertex y . If $y \in U$, then y is also a pendant vertex of G . If $y = t_O \in C_G$, then G has a pendant cycle.

Subcase 2.1. G has a pendant vertex y .

Let x be the vertex of G adjacent to y , $H^\sigma = G^\sigma - x - y$. By Theorem 4.4, we know that x is not a vertex of any cycle and H^σ is lower-optimal. The induction hypothesis to H^σ implies that

- (1) each cycle C_p^σ of H^σ is evenly-oriented with order $p \equiv 2(\pmod 4)$.
- (2) $r(H) = r(T_H) + \sum_{O \in \mathcal{C}(H)} |V(O)|$ and $r(T_H) = r(\Gamma_H)$.

Since all cycles of G belong to H , we have each cycle C_q^σ of G^σ is evenly-oriented with order $q \equiv 2(\pmod 4)$, and $\sum_{O \in \mathcal{C}(H)} |V(O)| = \sum_{O \in \mathcal{C}(G)} |V(O)|$. Noting that y is also a pendant vertex of T_G (resp., Γ_G) adjacent to x and $T_H = T_G - x - y$ (resp., $\Gamma_H = \Gamma_G - x - y$), combining with (2) of Subcase 2.1 and Lemma 2.4, then we have

$$r(G) = r(H) + 2 = r(T_H) + \sum_{O \in \mathcal{C}(H)} |V(O)| + 2 = r(T_G) + \sum_{O \in \mathcal{C}(G)} |V(O)|,$$

and

$$r(T_G) = r(T_H) + 2 = r(\Gamma_H) + 2 = r(\Gamma_G).$$

Subcase 2.2. G has a pendant cycle C_q .

Let x be the unique vertex of C_q of degree 3, $H^\sigma = G^\sigma - C_q^\sigma$ and $K^\sigma = H^\sigma + x$. By (c) of Theorem 4.3, we know that both H^σ and K^σ are lower-optimal. The induction hypothesis to K^σ implies that

- (i) each cycle C_p^σ of K^σ is evenly-oriented with order $p \equiv 2(\pmod 4)$.
- (ii) $r(K) = r(T_K) + \sum_{O \in \mathcal{C}(K)} |V(O)|$ and $r(T_K) = r(\Gamma_K)$.

Combining with (a) of Theorem 4.3, assertion (i) of Subcase 2.2 and $\mathcal{C}(G) = \mathcal{C}(K) \cup \{C_q\}$ imply that each cycle of G^σ is evenly-oriented with order $q \equiv 2(\pmod 4)$. Applying (b) of Theorem 4.3 and assertion (ii) of Subcase 2.2, we have

$$r(G) = q + r(K) = q + r(T_K) + \sum_{O \in \mathcal{C}(K)} |V(O)|. \tag{18}$$

Since T_K is isomorphic to T_G , and $q + \sum_{O \in \mathcal{C}(K)} |V(O)| = \sum_{O \in \mathcal{C}(G)} |V(O)|$, it follows from Equation (18) that

$$r(G) = r(T_G) + \sum_{O \in \mathcal{C}(G)} |V(O)|, \tag{19}$$

which proves the first assertion of (b) of this theorem.

By (b) of Theorem 4.3, we have

$$r(G) = q + r(H). \tag{20}$$

Noting that $C(G) = C(H) \cup \{C_q\}$, then from Equations (19) and (20), we have

$$r(T_G) = r(G) - \sum_{O \in C(G)} |V(O)| = q + r(H) - \sum_{O \in C(G)} |V(O)| = r(H) - \sum_{O \in C(H)} |V(O)|. \tag{21}$$

Since H^σ is also lower-optimal, the first assertion of (b) of this theorem applying to H implies that

$$r(H) = r(T_H) + \sum_{O \in C(H)} |V(O)|. \tag{22}$$

Equations (21) and (22) implies that

$$r(T_G) = r(T_H). \tag{23}$$

The induction hypothesis to H^σ implies that

$$r(T_H) = r(\Gamma_H). \tag{24}$$

Since $\Gamma_G = \Gamma_H$, combining with Equations (23) and (24), we have $r(T_G) = r(\Gamma_G)$.

This completes the proof. \square

Let T be an acyclic graph with at least one edge, we denote by \tilde{T} the subgraph obtained from T by deleting all pendant vertices of T .

Lemma 4.6. ([10]) *Let T be an acyclic graph with at least one edge. Then*

- (a) $r(\tilde{T}) < r(T)$.
- (b) if $r(T - W) = r(T)$ for a subset W of $V(T)$, then there is a pendant vertex v such that $v \notin W$.

Proof of Theorem 1.3.

Sufficiency: Suppose that G^σ meets all the conditions (1)–(3) in Theorem 1.3 and k steps of δ -transformations can switch G to a crucial subgraph G_0 , which is the disjoint union of $d(G)$ cycles together with l isolated vertices. By Lemmas 2.4 and 2.5, we have

$$sr(G^\sigma) = 2k + sr(G_0^\sigma), \quad r(G) = 2k + r(G_0). \tag{25}$$

Since each cycle C_q of the crucial subgraph G_0 of G is evenly-oriented with order $q \equiv 2 \pmod{4}$, by Lemmas 2.3 and 2.9, we have

$$sr(G_0^\sigma) = \sum_{O \in C(G)} sr(O^\sigma) = \sum_{O \in C(G)} |V(O)| - 2d(G) = \sum_{O \in C(G)} r(O) - 2d(G) = r(G_0) - 2d(G). \tag{26}$$

By Equalities (25) and (26), we have

$$sr(G^\sigma) = 2k + sr(G_0^\sigma) = 2k + r(G_0) - 2d(G) = 2k + r(G) - 2k - 2d(G) = r(G) - 2d(G).$$

This completes the proof of sufficiency.

Necessity: Let G^σ be a lower-optimal oriented graph. By (a) of Theorem 4.5, we can obtain the (1) and (2) of Theorem 1.3. Thus G has precisely $d(G)$ vertex-disjoint cycles, and the acyclic graph T_G respect to G is well defined. Now, we will proceed by induction on the order n of G^σ to prove (3) of Theorem 1.3.

Case 1. If $n = 1$, then the assertion holds naturally.

Case 2. Suppose the assertion holds for all lower-optimal oriented graphs with order smaller than n , and let G^σ be a lower-optimal oriented graph of order n .

Subcase 2.1. If T_G has no edges, then G is the disjoint union of $d(G)$ cycles along with some isolated vertices, and the assertion holds naturally.

Subcase 2.2. If T_G has at least one edge, by (b) of Theorem 4.5, we have

$$r(T_G) = r(\Gamma_G) = r(T_G - C_G).$$

(b) of Lemma 4.6 shows that there is a pendant vertex of T_G not in C_G . Thus G has at least one pendant vertex. Let y be a pendant vertex of G adjacent to a vertex x of G , by Theorem 4.4, x does not lie on any cycle of G and the graph $H^\sigma = G^\sigma - x - y$ is also lower-optimal, and also has $d(G)$ cycles. The induction hypothesis applying to H^σ implies that a series of δ -transformations can switch H to a crucial subgraph G_0 consisting of $d(G)$ disjoint union cycles together with some isolated vertices. Combining with the first step of δ -transformation applying to G and all the other δ -transformations done latter, we can switch G to the crucial subgraph G_0 .

This completes the proof. \square

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