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A Kantorovich Type Integral Modification of $q-$ Bernstein-Schurer Operators

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Abstract. The $q-$ Bernstein-Schurer summation type operators are modified in order to make them applicable for approximation of integrable functions. The aim of the paper is twofold. Firstly, to find refined error estimates, $|\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x) - f(x)|$ without using Schwarz's inequality. Secondly, to obtain a generalized Voronovskaya type asymptotic formula. The rate of approximation in terms of modulus of smoothness are also established.

1. Introduction

In the last two decades approximation methods of linear positive operators has been studied using the notion of quantum calculus. Since, the integer $[n]_q$ is a generalization of the ordinary integer n, the linear positive operators can be modified accordingly. In this direction, firstly, Phillips [19] modified the classical Bernstein operators in order to propose the generalized Bernstein operators,

$$B_{n,q}(f,x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) b_{n,k}(q;x), \;\; f \in C[0,1],$$

where $b_{n,k}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{r=0}^{n-k-1}(1-q^r x)$. These operators have been studied by several authors (cf.[16]-[22]). In the sequel q-analogue of many well known positive linear operators e.g. Baskakov, modified-beta and Szász operators have been introduced and studied (cf. [2],[6],[12],[13]). Agrawal et al. [1] introduced a $q-$analogue of Bernstein-Schurer-Stancu operators:

$$S^{(\alpha,\beta)}_{n,p}(f,q,x) = \sum_{k=0}^{n+p} b^{q}_{n+p,k}(x) f\left(\frac{[k]_q+\alpha}{[n]_q+\beta}\right), x \in [0,1], \; f \in C[0,1],$$

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where $b^q_{n+p,k}(x) = \left[{n+p \atop k}\right]_q x^k (1-x)_q^{n+p-k}$. The authors of [1] proved uniform convergence, Voronovskaya type theorem and contraction property for the operators $S^{(\alpha,\beta)}_{n,p}(f,q,x)$. However, these operators are not suitable for a larger class of functions, the class of integrable functions. For this reason we apply the technique of integral modification (see [15]) to the operators $S^{(\alpha,\beta)}_{n,p}(f,q,x)$.
Let I_0 denote the interval $\left[\frac{\alpha}{[n]+\beta}, \frac{[n+p+1]+\alpha}{[n]+\beta}\right]$, $f \in C[0,1]$ we define

$$\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x) := \begin{cases} ([n]+\beta)\displaystyle\sum_{k=0}^{n+p} b^q_{n+p,k}(x) q^{-k} \int_{\frac{[k]+\alpha}{[n]+\beta}}^{\frac{[k+1]+\alpha}{[n]+\beta}} f(t)\, d_q t, & x \in I_0 \\ f(x), & [0,1] \setminus I_0. \end{cases} \tag{1}$$

The formula (1) is called the Kantorovich type integral modification.

Remark 1.1. *When $\alpha = \beta = 0$, the operators $\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x)$ reduce to the $q-$ Bernstein-Schurer operators $S_{n,p}(f,q,x)$.*

The operators $\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x)$ can be expressed by

$$\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x) = \begin{cases} \displaystyle\sum_{k=0}^{n+p} b^q_{n+p,k}(x) \int_0^1 f\left(\frac{[k]+\alpha+q^k t}{[n]+\beta}\right) d_q t, & x \in I_0 \\ f(x), & [0,1] \setminus I_0. \end{cases}$$

Following definitions of $q-$calculus are required to present the results.
Let $0 < q \leq 1$ and $n \in \mathbb{N}$,(the set of positive integers). Then the $q-$integer $[n]_q$, the $q-$factorial $[n]_q!$ and the $q-$ binomial coefficients $\left[{n \atop k}\right]_q$ are given by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q} = 1 + q + ... + q^{n-1}, & q \neq 1 \\ n, & q = 1 \end{cases}$$

$$[n]_q! = \begin{cases} \prod_{j=0}^{n-1}[n-j]_q, & n = 1, 2, ... \\ 1, & n = 0 \end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{[n]_q!}{[k]_q![n-k]_q!}, & 0 \leq k \leq n \\ 0, & k > n \end{cases}$$

respectively. The $q-$rising product $(1-x)_q^n$ is defined by

$$(1-x)_q^n = \prod_{j=0}^{n-1}(1-q^j x),$$

where it is assumed that $(1-x)_q^0 = 1$. The $q-$Jackson integral is given by

$$\int_0^a f(x)\, d_q x = (1-q)\, a \sum_{n=0}^{\infty} f(aq^n)\, q^n \tag{2}$$

provided the sums are absolutely convergent. For the details of $q-$calculus we refer to the monograph [14] by Kac and Cheung.
For $f \in C[0,1]$ the Peetre K-functional is defined by

$$K_2(f,t^2) := \inf_{g \in C^2[0,1]} \{||f-g|| + t^2 ||g''||\},$$

$C^2[0,1] = \{g \in C[0,1] : g'' \in C[0,1]\}$. The equivalence of $K_2(f,t^2)$ and $\omega^2(f,t)$ ([3], pp.11) implies that there exist absolute constants $C_1, C_2 > 0$ such that

$$C_1\, K_2(f,t^2) \le \omega^2(f,t) \le C_2\, K_2(f,t^2), \tag{3}$$

where $\omega^2(f,t) = \sup\{\left|\Delta_h^r f(x)\right| : x, x+2h \in [0,1], 0 < h \le t\}$ is the second order modulus of smoothness of f and $\Delta_h^2 f(x) = \sum_{k=0}^{2} \binom{2}{k}(-1)^k f(x+(2-k)h)$.
Denote $L_\rho([0,1])$ the class of measurable functions f in $[0,1]$ such that

$$\int_0^1 |f(t)|^\rho\, dt < \infty, \quad 1 \le \rho < \infty.$$

Let $f \in L_\rho([a,b]), 1 \le \rho < \infty$. Then the Hardy-Littlewood majorant (see [23, pp. 244]) $\theta(x;f)$ of the function f is defined as

$$\theta(x;f) := \sup_{\xi \ne x} \frac{1}{\xi - x} \int_x^\xi f(t)\, dt. \quad (a \le \xi \le b).$$

In order to find the error $|L(f,x) - f(x)|$, the estimate of the first order absolute moment, $L(|e_1 - xe_0|)(x)$ (L being a linear positive operator) is very important. It is customary to use the Schwarz's inequality, $L(f \cdot g)(x) \le (L(f^2)(x))^{1/2}(L(g^2)(x))^{1/2}$ for this purpose. However, application of Schwarz's inequality yields an upper bound than that of $L(|e_1 - xe_0|)(x)$ and also involves unknown constants. Theorem 4.1 is an improvement over such estimates as we find the exact expression of $L(|e_1 - xe_0|)(x)$ in case the operator L is $S_{n,p}^{*(\alpha,\beta)}$. Thus, error estimates in Theorems 4.1 and 4.4 are sharp than similar results for operators studied in [1],[6] and [11] since the unknown constants are not involved. Moreover, we have the most general condition on the sequence (q_n) in theorem 3.1 that include the case $q_n \to 1, q_n^n \to 0$ that is most common case studied for other operators (see [1],[4],[5],[6],[8], [11], [17] and [18]).
From now on, we use the simple notation $[n]$ instead of $[n]_q$ as long as q is fixed. In case $\rho = \infty$, $L_\rho[0,1]$ will be the class $C[0,1]$.

2. Preliminaries

The central moments are calculated using the expressions of $S_{n,p}^{(\alpha,\beta)}(e_i,q,x)$ for the monomials $e_i(t) = t^i, i = 0,1,2,\ldots$·

Lemma 2.1. *There holds*

$$\mathbb{S}_{n,p,q}^{*(\alpha,\beta)}(e_m)(x) = \sum_{r=0}^{m} C_{m,r,q}\binom{m}{r}\left(\frac{1+(1-q)\alpha}{[n]+\beta}\right)^{m-r} S_{n,p}^{(\alpha,\beta)}(e_r,q,x),$$

where $C_{m,r,q} = \sum_{i=0}^{r}\binom{r}{i}\frac{(q-1)^i}{[1+i+m-r]}$.

Proof. We have $q^k = ([k]+\alpha)(1+(q-1)t) + (1+(1-q)\alpha)t$. This means

$$\left(\frac{[k]+\alpha+q^k t}{[n]+\beta}\right)^m = \sum_{r=0}^{m}\binom{m}{r}\left(\frac{1+(1-q)\alpha}{[n]+\beta}\right)^{m-r}(1+(q-1)t)^r\left(\frac{[k]+\alpha}{[n]+\beta}\right)^r t^{m-r}.$$

Therefore,

$$\mathbb{S}_{n,p,q}^{*(\alpha,\beta)}(e_m)(x) = \sum_{r=0}^{m}\binom{m}{r} S_{n,p}^{(\alpha,\beta)}(e_r,q,x)\left(\frac{1+(1-q)\alpha}{[n]+\beta}\right)^{m-r}\int_0^1 (1+(q-1)t)^r\, t^{m-r}\, d_q t.$$

By the definition $q-$integral (2)

$$\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(e_m)(x) = \sum_{r=0}^{m}\binom{m}{r}S^{(\alpha,\beta)}_{n,p}(e_r,q,x)\left(\frac{1+(1-q)\alpha}{[n]+\beta}\right)^{m-r}(1-q)\sum_{j=0}^{\infty}q^j(1+(q-1)q^j)^r q^{j(m-r)}$$
$$= \sum_{r=0}^{m}\binom{m}{r}S^{(\alpha,\beta)}_{n,p}(e_r,q,x)\left(\frac{1+(1-q)\alpha}{[n]+\beta}\right)^{m-r}(1-q)\sum_{i=0}^{r}\sum_{j=0}^{\infty}\binom{r}{i}(q-1)^i q^{j(1+i+m-r)}$$
$$= \sum_{r=0}^{m}\binom{m}{r}S^{(\alpha,\beta)}_{n,p}(e_r,q,x)\left(\frac{1+(1-q)\alpha}{[n]+\beta}\right)^{m-r}\sum_{i=0}^{r}\binom{r}{i}(q-1)^i\frac{1}{[1+i+m-r]}.$$

□

Corollary 2.2.

$$\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(e_0)(x) = 1,$$
$$\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(e_1)(x) = \frac{(1+[2]\alpha)+2q[n+p]_q x}{[2]([n]+\beta)}.$$

And

$$\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(e_2)(x) = \frac{(1+(1-q)\alpha)\Big((1+(1-q)\alpha)+2q(1+2q)([n+p]x+\alpha)\Big)}{[3]!([n]+\beta)^2}$$
$$+\frac{\Big([3]!+2[3](q-1)+[2](q-1)^2\Big)\Big([n+p]\{([n+p]-1)x^2+(1+2\alpha)x\}+\alpha^2\Big)}{[3]!([n]+\beta)^2}.$$

Proof. Using Lemma 2.1, we easily find $\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(e_i)(x)$ for $i = 0, 1, 2$. □

Lemma 2.3. *There exists absolute constant $C > 0$ such that*

$$\left|\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}((e_1-xe_0)^m)(x)\right| \le C\left|S^{(\alpha,\beta)}_{n,p}((e_1-xe_0)^m,q,x)\right|.$$

Proof. From the definition we have $\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}$

$$\mathbb{S}^{*(\alpha,\beta)}_{n,p}((e_1-xe_0)^m)(x) = \sum_{k=0}^{n+p} b^q_{n,k}(x)\int_0^1\left(\frac{[k]_q+\alpha+q^k t}{[n]_q+\beta}-x\right)^m d_q t$$
$$= \sum_{k=0}^{n+p}\sum_{j=0}^{m} b^q_{n,k}(x)\binom{m}{j}\left(\frac{[k]_q+\alpha}{[n]_q+\beta}-x\right)^j\left(\frac{q^k}{[n]+\beta}\right)^{m-j}\frac{1}{[m+1-j]}. \quad (4)$$

The proof easily follows from (4) and Cor. 1 of Lemma 2 of [1]. □

Corollary 2.4. *We have*

$$\left|\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}((e_1-xe_0)^2)(x)\right| \le C\frac{[n+p]}{([n]+\beta)^2}\left(\varphi^2(x)+\frac{[p]^2}{[n+p]}\right),$$

$\varphi^2(x) = x(1-x), x \in [0,1]$. *And*

$$\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}((e_1-xe_0)^m)(x) = O\left(\frac{1}{([n]+\beta)^{\lfloor\frac{m+1}{2}\rfloor}}\right), m \in \mathbb{N}\cup\{0\}.$$

where $\lfloor x\rfloor$ is the integer part of $x \ge 0$.

Lemma 2.5. *Let $\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x)$ be defined by (1) and $\varphi^2(x) = x(1-x)$. Then, there exists a real number $\hat{q}$ in $(0,1)$ such that for all q in $(0,\hat{q})$*

$$\left|\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}((e_1 - xe_0)^2)(x)\right| \leq C\frac{[n+p]}{([n]+\beta)^2}\varphi^2(x).$$

Proof. The proof is similar to Lemma 5 of [7]. □

Lemma 2.6. *[23]If $1 < \rho < \infty$, $f \in L_\rho[0,a]$, then $\theta(x;f) \in L_\rho[0,a]$ and*

$$\left\|\theta(f)\right\|_{L_\rho[0,a]} \leq 2^{1/\rho}\frac{\rho}{\rho-1}\|f\|_{L_\rho[0,a]}.$$

3. Convergence

Following theorem provides a criterion for the uniform convergence of sequence $\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)$ to the function f.

Theorem 3.1. *Let $f \in C[0,1]$ and (q_n) be a sequence in $(0,1)$. Then $\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x)$ converges to $f(x)$ uniformly if and only if $q_n \uparrow 1$, $q_n^n \downarrow 0$.*

Proof. From the conditions $q_n \uparrow 1$, $q_n^n \downarrow 0$ on the sequence (q_n) we have that $\lim_{n\to\infty}[n] = \infty$. This means $\lim_{n\to\infty} \mathbb{S}^{*(\alpha,\beta)}_{n,p,q_n}(e_m)(x) = 0$ uniformly for $m = 0,1,2$. An application of Korovkin's theorem then leads to the implication $\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x) \to f(x)$ uniformly.
For the converse we use the contradiction argument. Suppose $q_n \uparrow q_0$ for some $q_0 \neq 1$. Then, we have that $q_0 < 1$ and $\lim_{n\to\infty}[n] = \frac{1}{1-q_0}$. This means

$$\lim_{n\to\infty} \mathbb{S}^{*(\alpha,\beta)}_{n,p,q_n}(e_1)(x) = \frac{(1-q_0)(1+(1+q_0)\alpha) + 2q_0 x}{(1+q_0)(1+(1-q_0)\beta)}.$$

Since $q_0 < 1$, $\lim_{n\to\infty} \mathbb{S}^{*(\alpha,\beta)}_{n,p,q_n}(e_1)(x) \neq x$. Therefore, from Korovkin's theorem it follows that $\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x)$ does not converge to $f(x)$. This completes the proof. □

Theorem 3.2 (Vorvonskaya Type Formula). *Let $f \in C^2[0,1]$ and (q_n) be a sequence in $(0,1)$ such that $q_n \to \ell$, $q_n^n \to \mu$. Then*

$$\begin{aligned}
&\lim_{n\to\infty}([n]_{q_n} + \beta)\left(S^{*(\alpha,\beta)}_{n,p,q_n}(f)(x) - f(x)\right) \\
&= \frac{1}{[2]_\ell}\Big[1 + [2]_\ell\alpha + (2N\ell - [2]_\ell B)\,x\Big]f'(x) + \frac{1}{2B[3]_\ell!}\Big[(1+\alpha)^2 + \alpha\ell(\alpha + 4\ell + 4\alpha\ell) \\
&+ \Big(-2B(1+\alpha+\alpha\ell)[3]_\ell + N\ell\Big(3 + 5\ell + 4\ell^2 + 4\alpha[3]_\ell\Big)\Big)x \\
&+ \Big(-4BN\ell[3]_\ell + N(N-1)\ell\Big(1 + \ell + 4\ell^2\Big) + B^2[3]_\ell!\Big)x^2\Big]f''(x), \\
&+ O\Big(\frac{1}{B}\Big),
\end{aligned}$$

where $N = \frac{1-\mu\ell^p}{1-\ell}$, and $B = \frac{1-\mu+(1-\ell)\beta}{1-\ell}$.

Proof. We have

$$f(t) - f(x) = (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(x) + \varepsilon(t,x)(t-x)^2,$$

where $\varepsilon(t,x) \in C^2[0,1]$ and $\lim_{t\to x}\varepsilon(t,x) = 0$. An application of $S^{*(\alpha,\beta)}_{n,p}$ on both sides yields

$$\lim_{n\to\infty}([n]_{q_n}+\beta)\left(S^{*(\alpha,\beta)}_{n,p,q_n}(f)(x) - f(x)\right) = \lim_{n\to\infty}(g_1(n,x)f'(x) + g_2(n,x)f''(x)) + \lim_{n\to\infty}([n]_{q_n}+\beta)S^{*(\alpha,\beta)}_{n,p}\left(\varepsilon(t,x)(t-x)^2, q_n, x\right),$$

where $g_i(n,x) = \frac{1}{i!}\lim_{n\to\infty}([n]+\beta)S^{*(\alpha,\beta)}_{n,p}((e_1 - xe_0)^i, q_n, x), i = 1,2.$
Now

$$\begin{aligned} g_1(n,x) &= \lim_{n\to\infty}([n]_{q_n}+\beta)S^{*(\alpha,\beta)}_{n,p}(e_1 - xe_0, q_n, x)\\ &= \lim_{n\to\infty}\left(\frac{(1+[2]_{q_n}\alpha) + 2q[n+p]_{q_n}x}{[2]_{q_n}} - ([n]_{q_n}+\beta)x\right)\\ &= \left\{\frac{1+[2]_\ell\alpha + (2N\ell - [2]_\ell B)\,x}{[2]_\ell}\right\}. \end{aligned}$$

From linearity of the operator $S^{*(\alpha,\beta)}_{n,p}$ and Cor. 2.2 we have

$$\begin{aligned} g_2(n,x) &= \lim_{n\to\infty}([n]_{q_n}+\beta)\frac{1}{2}\left(S^{*(\alpha,\beta)}_{n,p}((e_2,q_n,x) - 2xS^{*(\alpha,\beta)}_{n,p}(e_1,q_n,x) + x^2S^{*(\alpha,\beta)}_{n,p}(e_0,q_n,x)\right)\\ &= \frac{1}{2}\left\{\frac{1+\alpha\left(2+4\ell^2\right)+\alpha^2\left(1+\ell+4\ell^2\right)}{B[3]_\ell!}\right.\\ &\quad + \left(\frac{-2B(1+\alpha+\alpha\ell)[3]_\ell + N\ell\left(3+5\ell+4\ell^2+4\alpha[3]\right)}{B[3]_\ell!}\right)x\\ &\quad \left. + \left(\frac{-4BN\ell[3]_\ell + (-1+N)N\ell\left(1+\ell+4\ell^2\right) + B^2\left(1+2\ell+2\ell^2+\ell^3\right)}{B[3]_\ell!}\right)x^2\right\}. \end{aligned}$$

Since $\varepsilon(t,x)$ is continuous, for arbitrary $\epsilon > 0$ there exists a $\delta > 0$ such that $|\varepsilon(t,x)| < \epsilon$ if $|t-x| \le \delta$. Therefore, $|\varepsilon(t,x)(t-x)^2| \le \epsilon\delta^2 + M(t-x)^2$, $M = \sup_{t\in[0,1]}|\varepsilon(t,x)|$. Using Lemma 2.3 we obtain

$$\begin{aligned} \left|\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}\left(\varepsilon(t,x)(t-x)^2\right)(x)\right| &\le \epsilon\delta^2\left|\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}\left((t-x)^2\right)(x)\right| + M\left|\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}\left((t-x)^4\right)(x)\right|\\ &\le C_1\epsilon\delta^2\left(\frac{1}{[n]+\beta}\right) + C_2\left(\frac{1}{[n]+\beta}\right)^2. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty}([n]_{q_n}+\beta)\mathbb{S}^{*(\alpha,\beta)}_{n,p,q_n}\left(\varepsilon(t,x)(t-x)^2)(x)\right) = \lim_{n\to\infty}C_1\epsilon\delta^2 + C_2\left(\frac{1}{[n]+\beta}\right).$$

In view of arbitrariness of ϵ and the fact that $\lim_{n\to\infty}\frac{1}{[n]+\beta} = \frac{1-\ell}{1-\mu+(1-\ell)\beta} = \frac{1}{B}$ the proof is completed. □

Remark 3.3. *If the sequence (q_n) is such that $q_n \to 1$ and $q_n^n \to \mu$, then we obtain*

$$\lim_{n\to\infty}([n]+\beta)\left(\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x) - f(x)\right) = \frac{1}{2}(1+2\alpha+(-1-2\beta+\mu+2\mu p)x)f'(x) + \frac{1}{2}x^2f''(x).$$

A particular case is obtained by a special type of sequence (q_n) that satisfy $q_n \to 1$ and $q_n^n \to 0$, that is

$$\lim_{n\to\infty}([n]+\beta)\left(\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x) - f(x)\right) = \frac{1}{2}(1+2\alpha-(1+2\beta)x)f'(x) + \frac{1}{2}x^2f''(x).$$

4. Point wise Approximation

Theorem 4.1. *Let* $f \in C[0,1], h > 0$ *and the polynomial* $\Pi_N(x) = \sum_{j=0}^{N}[j]!\left(\frac{1-q}{q}\right)^j x^j$. *Then,*

$$\left|\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x) - f(x)\right| \leq \Bigg(1 + h^{-1}\Bigg[2\alpha\left(\frac{\alpha}{[2]([n]+\beta)} - x\right)\Pi_{n+p}(x) + 2(xq^{-1})[n+p]\left(\frac{2\alpha}{[2]([n]+\beta)} - x\right)\Pi_{n+p-1}(x) + \frac{2[n+p][n+p-1]}{[2]([n]+\beta)}(xq^{-1})^2\Pi_{n+p-2}(x) + \left(\frac{[n+p](3+q)}{[2]([n]+\beta)} - 1\right)x + \frac{1+2\alpha}{[2]([n]+\beta)}\Bigg]\Bigg)\omega_1(f,h).$$

Proof. Since

$$|f(t) - f(x)| \leq \left(1 + \frac{|t-x|}{h}\right)\omega_1(f,h), \tag{5}$$

the positivity of $\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x)$, and inequality (5) together imply that

$$\left|\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x) - f(x)\right| \leq \left(1 + \frac{S^{*(\alpha,\beta)}_{n,p}(|e_1 - xe_0|, q, x)}{h}\right).\omega_1(f,h). \tag{6}$$

for every $f \in C[0,1], x \in [0,1]$ and for every $h > 0$. We need to calculate the exact moment of first order $S^{*(\alpha,\beta)}_{n,p}(|e_1 - xe_0|, q, x)$. The inequality $\frac{[k]+\alpha}{[n]+\beta} \leq x \leq \frac{[k+1]+\alpha}{[n]+\beta}$ means

$$\begin{aligned}
&([n]+\beta)\sum_{k=0}^{n+p} b^q_{n,k}(x)q^{-k}\int_{\frac{[k]+\alpha}{[n]+\beta}}^{\frac{[k+1]+\alpha}{[n]+\beta}} |e_1(t) - xe_0(t)|\, d_qt \\
&= -([n]+\beta)\sum_{k=0}^{n+p} b^q_{n,k}(x)q^{-k}\int_{\frac{[k]+\alpha}{[n]+\beta}}^{x} |t-x|\, d_qt + ([n]+\beta)\sum_{k=0}^{n+p} b^q_{n,k}(x)q^{-k}\int_{x}^{\frac{[k+1]+\alpha}{[n]+\beta}} |t-x|\, d_qt \\
&= ([n]+\beta)\sum_{k=0}^{n+p} b^q_{n,k}(x)q^{-k}\Bigg(\frac{1}{[2]}\left(\frac{[k]+\alpha}{[n]+\beta} + \frac{[k+1]+\alpha}{[n]+\beta}\right)^2 \\
&- x\left(\frac{[k]+\alpha}{[n]+\beta} + \frac{[k+1]+\alpha}{[n]+\beta}\right) + \frac{2qx^2}{[2]}\Bigg) = T_1 + T_2 + T_3, \text{ say.}
\end{aligned} \tag{7}$$

Now $[k+1] = [k] + q^k$ so we have that

$$\begin{aligned}
T_1 &= ([n]+\beta)\sum_{k=0}^{n+p} b^q_{n,k}(x)\frac{q^{-k}}{[2]}\left(2\left(\frac{[k]+\alpha}{[n]+\beta}\right)^2 + 2\frac{q^k}{[n]+\beta}\left(\frac{[k]+\alpha}{[n]+\beta}\right) + \left(\frac{q^k}{[n]+\beta}\right)^2\right) \\
&= \frac{1}{[2]([n]+\beta)}\sum_{k=0}^{n+p} b^q_{n,k}(x)2q^{-k}\left([k][k-1] + q^k[k]\frac{(3+q)}{2} + \left(\alpha + \frac{1}{2}\right)q^k + \alpha^2\right) \\
&= \sum_{i=1}^{4} E_i, \text{ say.}
\end{aligned} \tag{8}$$

For E_1 we calculate as

$$E_1 = \frac{2}{[2]([n]+\beta)}\sum_{k=0}^{n+p} b^q_{n,k}(x)q^{-k}[k][k-1] = \frac{2[n+p][n+p-2]x^2}{q^2[2]([n]+\beta)}\Pi_{n+p-2}(x). \tag{9}$$

In a similar way, we obtain

$$E_2 = \frac{(3+q)[n+p]x}{[2]([n]+\beta)}, E_3 = \frac{2\alpha+1}{[2]([n]+\beta)} \text{ and } E_4 = \frac{2\alpha^2}{[2]([n]+\beta)}\Pi_{n+p}(x).$$

Here we have used the identity $\sum_{k=0}^{n+p} b^q_{n,k}(x)[k] = [n+p]x$. Therefore,

$$T_1 = \frac{2[n+p][n+p-2]x^2q^{-2}\Pi_{n+p-2}(x) + (3+q)[n+p]x + 2\alpha + 1 + 2\alpha^2\Pi_{n+p}(x)}{[2]([n]+\beta)}. \tag{10}$$

Next

$$\begin{aligned} T_2 &= -x\sum_{k=0}^{n+p} q^{-k}\left(2([k]+\alpha)+q^k\right)b^q_{n,k}(x) \\ &= -2x\sum_{k=0}^{n+p} q^{-k}[k]b^q_{n,k}(x) - 2x\alpha\sum_{k=0}^{n+p} q^{-k}b^q_{n,k}(x) - x\sum_{k=0}^{n+p} b^q_{n,k}(x) \\ &= -2[n+p]x^2q^{-1}\Pi_{n+p-1}(x) - 2x\alpha\Pi_{n+p}(x) - x. \end{aligned} \tag{11}$$

And

$$T_3 = ([n]+\beta)\frac{2q}{[2]}x^2\Pi_{n+p}(x). \tag{12}$$

Combining expressions (10)-(12), we obtain

$$\begin{aligned} S^{*(\alpha,\beta)}_{n,p}(|e_1 - xe_0|, q, x) = \Bigg[2\alpha\left(\frac{\alpha}{[2]([n]+\beta)} - x\right)\Pi_{n+p}(x) + 2(xq^{-1})[n+p]\left(\frac{2\alpha}{[2]([n]+\beta)} - x\right)\Pi_{n+p-1}(x) \\ + \frac{2[n+p][n+p-1]}{[2]([n]+\beta)}(xq^{-1})^2\Pi_{n+p-2}(x) + \left(\frac{[n+p](3+q)}{[2]([n]+\beta)} - 1\right)x + \frac{1+2\alpha}{[2]([n]+\beta)}\Bigg]. \end{aligned} \tag{13}$$

The proof now follows from (6) and (13). □

Remark 4.2. *If we apply Schwarz's inequality in (5) for the operators $S^{*(\alpha,\beta)}_{n,p}$ and choose $h^2 = |S^{*(\alpha,\beta)}_{n,p}((e_1 - xe_0)^2, q, x)|$ together with Cor.2.4 then following less precise error estimate is found:*

$$\left|\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x) - f(x)\right| \le 2\omega_1\left(f, \sqrt{\frac{[n+p]}{([n]+\beta)^2}\left(\varphi^2(x) + \frac{[p]^2}{[n+p]}\right)}\right).$$

The need to incorporate non differentiable function lead to the estimates in terms of second order modulus of smoothness $\omega_2(f, h)$.

Theorem 4.3. *Let $f \in C[0,1]$. For the sequence of operators (1), there exists a positive absolute constant C such that*

$$\left|\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x) - f(x)\right| \le C\,\omega_2\left(f, \frac{[n+p]}{([n]+\beta)^2}\delta_n(x)\right) + \omega\left(f, \frac{(2p-\beta)(1+2\alpha)}{([n]+\beta)}\right),$$

where $\delta_n(x) = \varphi^2(x) + \frac{[p]^2}{[n+p]}$.

Proof. We choose $g \in C^2[0,1]$ so as to write

$$\begin{aligned}\left|\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x) - f(x)\right| &\le \left|S^{*(\alpha,\beta)}_{n,p}(f-g,q,x)\right| + \left|S^{*(\alpha,\beta)}_{n,p}(g,q,x) - g(x)\right| + \left|f(x)-g(x)\right| \\ &\le 2\|f-g\| + \left|S^{*(\alpha,\beta)}_{n,p}(g,q,x) - g(x)\right|.\end{aligned}$$

Next, we define an auxiliary operator

$$T^{*(\alpha,\beta)}_{n,p,q}(f)(x) = \mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x) - f(z) + f(x),$$

where $z = \frac{(1+[2]\alpha)+2q[n+p]_q x}{[2]([n]+\beta)}$. In view of Cor.2.2, we get

$$T^{*(\alpha,\beta)}_{n,p,q}(e_0)(x) = 1,$$

$$T^{*(\alpha,\beta)}_{n,p,q}(e_1)(x) = \mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x) - z + x = x.$$

Since $g \in C^2[0,1]$, so we can write

$$g(\cdot) = g(x) + (\cdot - x)g'(x) + \int_x^{\cdot} (\cdot - u)g''(u)\,du.$$

Therefore,

$$\begin{aligned}\left|T^{*(\alpha,\beta)}_{n,p,q}(g)(x) - g(x)\right| &= \left|T^{*(\alpha,\beta)}_{n,p,q}\left(\int_x^{\cdot}(\cdot - u)g''(u)\,du\right)(x)\right| \\ &\le \left|S^{*(\alpha,\beta)}_{n,p}\left(\int_x^{\cdot}(\cdot - u)g''(u)\,du, q, x\right)\right| + \left|\int_x^{z}(z-u)g''(u)\,du\right| \\ &\le S^{*(\alpha,\beta)}_{n,p}((e_1-x)^2,q,x)\|g''\|_\infty + (z-x)^2\|g''\|_\infty.\end{aligned}$$

By straight forward calculations, it can be shown that

$$\begin{aligned}(z-x)^2 &= \left(\frac{(1+[2]\alpha)+2q[n+p]_q x}{[2]([n]+\beta)} - x\right)^2 \\ &= \left(\frac{(1+[2]\alpha)+\big(2q[n+p]_q - [2]([n]+\beta)\big)x}{[2]([n]+\beta)}\right)^2 \\ &\le C\,\frac{[n+p]}{([n]+\beta)^2}\left(\varphi^2(x) + \frac{[p]^2}{[n+p]}\right).\end{aligned}$$

Therefore,

$$\left|T^{*(\alpha,\beta)}_{n,p,q}(g)(x) - g(x)\right| \le C\frac{[n+p]}{([n]+\beta)^2}\delta_n^2(x)\|g''\|_\infty.$$

Hence,

$$\begin{aligned}\left|\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x) - f(x)\right| &\le 2\|f-g\|_\infty + \left|T^{*(\alpha,\beta)}_{n,p,q}(g)(x) - g(x)\right| + |f(z)-f(x)| \\ &\le C\left\{\|f-g\|_\infty + \frac{[n+p]}{([n]+\beta)^2}\delta_n^2(x)\|g''\|_\infty\right\} + \omega(f,|z-x|).\end{aligned}$$

Taking the infimum for g over $C^2[0,1]$ and using

$$|z-x| \le \frac{(2p-\beta)(1+2\alpha)}{([n]+\beta)},$$

we find that

$$\left|\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x) - f(x)\right| \le C\,K_2\left(f, \frac{[n+p]}{([n]+\beta)^2}\delta_n(x)\right) + \omega\left(f, \frac{(2p-\beta)(1+2\alpha)}{([n]+\beta)}\right).$$

From (3), the proof follows. □

In following theorem we establish a global rate of approximation for the operators $S^{*(\alpha,\beta)}_{n,p}$ which is free from the constant in front of the right hand side.

Theorem 4.4. *Let* $f \in C[0,1]$. *Then,*

$$\left|\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x) - f(x)\right| \le \frac{2}{h}\|S^{*(\alpha,\beta)}_{n,p}((e_1 - xe_0), q, x)\|_\infty \omega_1(f,h)$$
$$+ 3\left(\frac{1}{2} + \|S^{*(\alpha,\beta)}_{n,p}((e_1 - xe_0), q, x)\|_\infty + \|S^{*(\alpha,\beta)}_{n,p}((e_1 - xe_0)^2, q, x)\|_\infty\right)\omega_2(f,h).$$

Proof. We choose the function $Z_h f$ defined by

$$Z_h f(x) = \frac{1}{h}\int_{-h}^{h}\left(1 - \frac{|t|}{h}\right) f_h(x+t)\,dt, \;\; x \in [0,1],$$

where f_h is the extension of f in the interval $[-h, 1+h]$ defined as

$$f_h(x) = \begin{cases} P_-(x), & -h \le x \le 0 \\ f(x), & 0 \le x \le 1, \\ P_+(x), & 1 < x \le 1+h, \end{cases},$$

$P_-(x)$ and $P_+(x)$ are linear polynomials of the best approximation on $[-h,0]$ and $[1,h]$ respectively. This function is called the Zhuk's function and has some of following properties

(a) $\|f - Z_h f\|_\infty \le \frac{3}{4}\,\omega_2(f;h)$,
(b) $\|(Z_h f)'\|_\infty \le \frac{1}{h}\left[2\omega_1(f;h) + \frac{3}{2}\omega_2(f;h)\right]$,
(c) $\|(Z_h f)''\|_\infty \le \frac{3}{2h^2}\,\omega_2(f;h)$,
(d) $\|(Z_h f)\|_\infty \le \|f\|_\infty + \frac{3}{2}\omega_2(f;h)$.

For the proof see Lemma 2.4 in [10]. We have from $\left\|S^{*(\alpha,\beta)}_{n,p}(f)\right\|_\infty \le 1$

$$\left\|\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x) - f(x)\right\|_\infty \le 2\left\|Z_h f - f\right\|_\infty + \left\|S^{*(\alpha,\beta)}_{n,p}(Z_h f, q, x) - Z_h f(x)\right\|_\infty \tag{14}$$

It is known that $(Z_h f)'$ exists and is absolutely continuous on $[0,1]$ (see [10]). From this we can write

$$Z_h f(t) = Z_h f(x) + (t-x)(Z_h f)'(x) + \int_x^t (t-u)(Z_h f)''(u)\,du.$$

An application of $S^{*(\alpha,\beta)}_{n,p}$ on both sides together with the linearity of $S^{*(\alpha,\beta)}_{n,p}$ and $S^{*(\alpha,\beta)}_{n,p}(e_0, q, x) = 1$ we obtain that

$$\left\|S^{*(\alpha,\beta)}_{n,p}(Z_h f, q, x) - Z_h f(x)\right\|_\infty = \left\|(Z_h f)'(x) S^{*(\alpha,\beta)}_{n,p}(e_1 - e_0 x, q, x)\right\|_\infty + \left\|S^{*(\alpha,\beta)}_{n,p}\left(\int_x^t (t-u)(Z_h f)''(u)\,du, q, x\right)\right\|_\infty$$
$$\le \frac{1}{h}\left[2\omega_1(f;h) + \frac{3}{2}\omega_2(f;h)\right]\left\|S^{*(\alpha,\beta)}_{n,p}(e_1 - e_0 x, q, x)\right\|_\infty$$
$$+ \frac{3}{4h^2}\,\omega_2(f;h)\|S^{*(\alpha,\beta)}_{n,p}((e_1 - xe_0)^2, q, x)\|_\infty. \tag{15}$$

Here we have used (b) and (c) properties of $Z_h f(x)$. Finally, combining inequalities (14) and (15) together with property (a) completes the proof. □

5. L_ρ– Approximation

Theorem 5.1. *If* $f \in L_\rho[0,1], 1 \le \rho < \infty$ *and* (q_n) *be a sequence in* $(0,1]$ *such that* $q_n \uparrow 1$, *then*

$$\lim_{n\to\infty} \left\| S_{n,p}^{*(\alpha,\beta)}(e_m, q_n, x) - f(x) \right\|_{L_\rho(I_0)} = 0.$$

Proof. From Luzin's theorem, for any positive integer $n \in \mathbb{N}$ there exists a function $g_n \in C[0,1]$ such that

$$\|f(x) - g_n(x)\|_{L_\rho[0,1]} < \frac{1}{3n}.$$

From Theorem 3.1, it follows that there exists a positive integer N with the property

$$\|S_{n,p}^{*(\alpha,\beta)}(g_n, q_n, x) - g_n(x)\|_{C[0,1]} < \frac{1}{3n} \text{ for all } n > N.$$

We have

$$\begin{aligned} &\|S_{n,p,q_n}^{*(\alpha,\beta)}(f)(x) - f(x)\|_{L_\rho(I_0)} \\ &\le \|S_{n,p}^{*(\alpha,\beta)}(f - g_n, q_n, x)\|_{L_\rho(I_0)} + \|S_{n,p}^{*(\alpha,\beta)}(g_n, q_n, x) - g_n(x)\|_{C[0,1]} + \|f(x) - g_n(x)\|_{L_\rho(I_0)}. \end{aligned} \tag{16}$$

In order to complete the proof we need to show that $\|S_{n,p}^{*(\alpha,\beta)}\| < C$ and C is independent of f, x and n. Applying Jensen's inequality, we obtain

$$\begin{aligned} \int_{\frac{\alpha}{[n]+\beta}}^{\frac{[n+p+1]+\alpha}{[n]+\beta}} |S_{n,p,q_n}^{*(\alpha,\beta)}(f)(x)|^\rho \, d_{q_n}x &= \int_{\frac{\alpha}{[n]+\beta}}^{\frac{[n+p+1]+\alpha}{[n]+\beta)}} \left| \sum_{k=0}^{n+p} b_{n,k}^{q_n}(x) \int_0^1 f\left(\frac{[k]+\alpha+{q_n}^k t}{[n]+\beta}\right) d_{q_n}t \right|^\rho d_{q_n}x \\ &\le \int_{\frac{\alpha}{[n]+\beta}}^{\frac{[n+p+1]+\alpha}{[n]+\beta}} \sum_{k=0}^{n+p} b_{n,k}^{q_n}(x) \left(\int_0^1 \left| f\left(\frac{[k]+\alpha+q_n^k t}{[n]+\beta}\right) \right| d_{q_n}t \right)^\rho d_{q_n}x \\ &\le \int_{\frac{\alpha}{[n]+\beta}}^{\frac{[n+p+1]+\alpha}{[n]+\beta}} \sum_{k=0}^{n+p} b_{n,k}^{q_n}(x) \, d_{q_n}x \left\| f \right\|_{L_\rho(I_0)}^\rho . \end{aligned}$$

From the positivity of $b_{n,k}^{q_n}(x)$ and the identity $\sum_{k=0}^{n+p} b_{n,k}^{q_n}(x) = 1$ we get

$$\left\| S_{n,p,q_n}^{*(\alpha,\beta)}(f)(x) \right\|_{L_\rho(I_0)} \le \frac{[n+p+1]}{[n]+\beta} \left\| f \right\|_{L_\rho(I_0)} \le \left\| f \right\|_{L_\rho(I_0)} .$$

Since we know that $\lim_{q\to 1} \|\cdot\|_{L_\rho,q} = \|\cdot\|_{L_\rho}$, and q_n is arbitrary in $(0,1]$ it follows that

$$\left\| S_{n,p,q_n}^{*(\alpha,\beta)}(f)(x) \right\|_{L_\rho(I_0)} \le \left\| f \right\|_{L_\rho(I_0)} . \tag{17}$$

Moreover, $\left\| S_{n,p}^{*(\alpha,\beta)} \right\|_{L_\rho([0,1]\setminus I_0)} = \left\| f \right\|_{L_\rho([0,1]\setminus I_0)}$ Therefore, $\left\| S_{n,p}^{*(\alpha,\beta)} \right\|_{L_\rho[0,1]} \le 1$ for all $f \in L_\rho[0,1]$. Finally combining these estimates and using in (16) we obtain

$$\|S_{n,p,q_n}^{*(\alpha,\beta)}(f)(x) - f(x)\|_{L_\rho[0,1]} \le \frac{1}{n} \to 0, \text{ as } n \to \infty.$$

The proof is completed. □

Theorem 5.2. *Let $1 \leq \rho < \infty$. If $f, f, f'' \in L_\rho([0,1])$, then*

$$\|S^{*(\alpha,\beta)}_{n,p}(f,x) - f(x)\|_{L_\rho([0,1])} \leq C\left(\frac{1}{[n]_q + \beta}\right)\left(\|f\|_{L_\rho([0,1])} + \|f''\|_{L_\rho([0,1])}\right),$$

where $C = C(p,q,\alpha,\beta)$ is free from n.

Proof. Using the representation

$$f(t) - f(x) = (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(\xi)$$

and linearity of $S^{*(\alpha,\beta)}_{n,p}(f)$, we can write

$$\begin{aligned}\mathbb{S}^{*(\alpha,\beta)}_{n,p,q}(f)(x) - f(x) &= S^{*(\alpha,\beta)}_{n,p}(t-x,q,x)f'(x) + \frac{1}{2}S^{*(\alpha,\beta)}_{n,p}((t-x)^2 f''(\xi),q,x)\\ &= A_1 + A_2, \text{say.}\end{aligned}$$

$$\begin{aligned}|A_1| &= |f'(x)|\left|\frac{1+[2]\alpha+2q[n+p]x}{[2]([n]_q+\beta)} - x\right|\\ &= |f'(x)|\left|\frac{1+[2]\alpha+(2q[n+p]-[2]([n]+\beta))\,x}{[2]([n]_q+\beta)}\right|\\ &= |f'(x)|\frac{\left|(1-q)(1+[2]\alpha)+(2q(1-q^{n+p})-(1+q)(1-q^n+(1-q)\beta))\,x\right|}{[2](1-q)([n]_q+\beta)}\end{aligned}$$

The coefficient of x

$$\begin{aligned}&= 2q(1-q^{n+p}) - (1+q)(1-q^n+(1-q)\beta)\\ &= (1+q-2q^{p+1})q^n + (q-1) + (q^2-1)\beta\\ &\leq (1+q-2q^{p+1}) + (q-1) + (q^2-1)\beta\\ &= -2q^{p+1} + (q^2-1)\beta\\ &\leq 0.\end{aligned}$$

Also we have

$$|2q(1-q^{n+p}) - (1+q)(1-q^n+(1-q)\beta)| \leq 2.$$

Therefore,

$$\|A_1\|_{L_\rho([0,1])} \leq \frac{C(\alpha,\beta)}{([n]+\beta)}\left\|f'\right\|_{L_\rho([0,1])}.$$

From [9] we get

$$\|A_1\|_{L_\rho([0,1])} \leq C(p,q,\alpha,\beta)\frac{1}{[n+2]_q}\left(\|f\|_{L_\rho([0,1])} + \|f''\|_{L_\rho([0,1])}\right).$$

Next,

$$
\begin{aligned}
|A_2| &\le \sum_{k=0}^{n+p} b_{n,k}^{q}(x) \int_0^1 \left| \int_x^{\frac{[k]_q+\alpha+q^k t}{[n]_q+\beta}} \left(\frac{[k]_q+\alpha+q^k t}{[n]_q+\beta} - u \right) f''(u)\, du \right| d_q t \\
&\le \sum_{k=0}^{n+p} b_{n,k}^{q}(x) \int_0^1 \left(\frac{[k]_q+\alpha+q^k t}{[n]_q+\beta} - x \right)^2 \left|\theta(x;f'')\right| d_q t \\
&\le C(p,q) \left|\theta(x;f'')\| S_{n,p}^{*(\alpha,\beta)}((t-x)^2,q,x)\right|
\end{aligned}
$$

Here $\theta(x;f'')$ is the Hardy-Littlewood majorant of $f''(x)$.
Using Lemma 2.4 and 2.6 we obtain that

$$
\|A_2\|_{L_\rho([0,1])} \le C(p,q,\alpha,\beta)\left(\frac{1}{[n]_q+\beta}\right)\|\theta(x;f'')\|_{L_\rho([0,1])} \le C(p,q,\alpha,\beta)\left(\frac{1}{[n]_q+\beta}\right)\|f''\|_{L_\rho([0,1])}.
$$

Finally, we have for $\rho > 1$

$$
\|\mathbb{S}_{n,p,q}^{*(\alpha,\beta)}(f)(x) - f(x)\|_{L_\rho([0,1])} \le C\left(\frac{1}{[n]_q+\beta}\right)\left(\left\|f\right\|_{L_\rho([0,1])} + \left\|f''\right\|_{L_\rho([0,1])}\right).
$$

For $\rho = 1$ the calculations are straight forward and similar to the case $\rho > 1$, however we do not need the maximal function. □

Remark 5.3. *The error estimates and Vornovskaya type asymptotic results for the $q-$analogue of Bernstein–Schurer–Stancu operators [1], Durrmeyer operators [11],[12] and Baskakov-Durrmeyer operators [6] can be refined and generalized by the methods in Theorem 4.1 and Theorem 3.2.*

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