Filomat 32:4 (2018), 1379–1393
https://doi.org/10.2298/FIL1804379B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Bernstein Polynomials Method and it's Error Analysis for Solving Nonlinear Problems in the Calculus of Variations: Convergence Analysis via Residual Function

Ahmad Sami Bataineh[a]

[a] *Department of Mathematics, Faculty of Science, Al-Balqa Applied University, 19117 Al Salt, Jordan*

Abstract. In this paper, Bernstein polynomials method (BPM) and their operational matrices are adopted to obtain approximate analytical solutions of variational problems. The operational matrix of differentiation is introduced and utilized to reduce the calculus of variations problems to the solution of system of algebraic equations. The solutions are obtained in the form of rapidly convergent finite series with easily computable terms. Comparison between the present method and the homotopy perturbation method (HPM), the non-polynomial spline method and the B-spline collocation method are made to show the effectiveness and efficiency for obtaining approximate solutions of the calculus of variations problems. Moreover, convergence analysis based on residual function is investigated to verified the numerical results.

1. Introduction

Calculus of variations is a field of mathematical analysis that deals with maximizing or minimizing functionals. Functionals are real valued functions on some vector space. They are often expressed as definite integrals involving functions and their derivatives. *variational problems* appear in engineering and science applications where minimization of functionals, such as, the Lagrangian, strain, potential, and total energy, etc. give the laws governing a system's behavior. In optimal control theory, minimization of certain functionals give control functions for optimum performance of the system [1]. The brachistochrone, geodesics and isoperimetric problems have played an important role in the development of the calculus of variations (cf. , [2, 3]). Several methods have been used to solve variational problems in the literature (cf. ,[4–15]).

Let us consider the simplest form of the variational problems

$$v[u(x)] = \int_{x_0}^{x_1} F(x, u(x), u'(x))\,\mathrm{d}x, \tag{1}$$

where v is the functional that its extremum must be found. In order to find the extreme value of v, the boundary points of the admissible curves are given by

$$u(x_0) = \alpha, \quad u(x_1) = \beta. \tag{2}$$

2010 *Mathematics Subject Classification*. 49M30, 11C08, 11C20, 15Axx, 65Fxx.
Keywords. Calculus of variations; Variational problems; Euler-Lagrange equation; Bernstein polynomials, Operational matrix of differentiation
Received: 01 May 2017; Accepted: 01 March 2018
Communicated by Predrag Stanimirović
Email address: a_s_bataineh@yahoo.com (Ahmad Sami Bataineh)

The necessary condition for the solution of the problem (1) is to satisfy the Euler-Lagrange equation:

$$F_u - \frac{\mathrm{d}}{\mathrm{d}x}F_{u'} = 0, \tag{3}$$

with the boundary conditions given in (2). The Euler-Lagrange equation (3) is in general a nonlinear differential equation of the second order which does not always have analytic solution.

In this paper, we will adopt the Bernstein polynomials method (BPM) for solving the Euler-Lagrange equation (3) which arises from problems in calculus of variations. BPM has been applied to solve a variety of linear and nonlinear problems in mathematics and physics, (cf. , [16**?** –40]).

2. Orthonormal Bernstein polynomials and their properties

Definition 1. *The Bernstein polynomials of degree m defined on the interval* $[a, b]$ *are given as [23, 24]*

$$\mathrm{B}_{i,m}(x) = \binom{m}{i}\frac{(x-a)^i(b-x)^{m-i}}{(b-a)^m}, \quad i = 0, 1, \ldots, m,$$

where the binomial coefficient is

$$\binom{m}{i} = \frac{m!}{i!\,(m-i)!}.$$

For convenience, we usually set $\mathrm{B}_{i,m} = 0$ *if* $i < 0$ *or* $i > m$*.*

Some useful properties for $\mathrm{B}_{i,m}(x)$ on $[a, b]$:

1. **Recurrence formula:** The Bernstein polynomial of degree $m-1$ in terms of a linear combination of Bernstein polynomials of degree m on the interval $[a, b]$ is given as

$$(b-a)\mathrm{B}_{i,m-1}(x) = \left(\frac{m-i}{m}\right)\mathrm{B}_{i,m}(x) + \left(\frac{i+1}{m}\right)\mathrm{B}_{i+1,m}(x).$$

2. **Derivative formula:** The derivatives of the mth degree Bernstein polynomials are given by

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{B}_{i,m}(x)\right) = \frac{1}{(b-a)^n}\frac{m!}{(m-n)!}\sum_{k=\max(0,i+n-m)}^{\min(i,n)}(-1)^{k+n}\binom{n}{k}\mathrm{B}_{i-k,m-n}(x).$$

3. Bernstein polynomials method

In this section the basic ideas of the Bernstein polynomials method are introduced by considered Eq. (3) subject to the boundary conditions (2).

3.1. Approximating functions by Bernstein polynomials

Let $u(x)$ be the exact solution of (3). We want to approximate $u(x)$ and it's $n^{th}-$derivatives by $u_m(x)$ and $u_m^{(n)}(x)$ in the matrix form as

$$\begin{aligned} u(x) \simeq u_m(x) &= \sum_{i=0}^{m} c_i \mathrm{B}_{i,m}(x), \\ &= c_0\mathrm{B}_{0,m}(x) + c_1\mathrm{B}_{1,m}(x) + \ldots + c_m\mathrm{B}_{m,m}(x), \\ &= \mathrm{C}\Phi(x), \end{aligned} \tag{4}$$

$$\begin{aligned} u'(x) \simeq u'_m(x) &= \sum_{i=0}^{m} c_i \mathrm{B}'_{i,m}(x), \\ &= c_0\mathrm{B}'_{0,m}(x) + c_1\mathrm{B}'_{1,m}(x) + \ldots + c_m\mathrm{B}'_{m,m}(x), \\ &= \mathrm{C}\Phi'(x), \end{aligned} \tag{5}$$

$$\vdots$$

$$\begin{aligned} u^{(n)}(x) \simeq u^{(n)}_m(x) &= \sum_{i=0}^{m} c_i \mathrm{B}^{(n)}_{i,m}(x), \\ &= c_0\mathrm{B}^{(n)}_{0,m}(x) + c_1\mathrm{B}^{(n)}_{1,m}(x) + \ldots + c_m\mathrm{B}^{(n)}_{m,m}(x), \\ &= \mathrm{C}\Phi^{(n)}(x), \end{aligned} \tag{6}$$

where C is an unknown constant matrix of size $1 \times (m+1)$ to be determining and $\Phi(x), \Phi'(x), \ldots \Phi^{(n)}(x)$ are the $1 \times (m+1)$ matrices of Bernstein polynomials with it's derivatives defined as

$$\begin{aligned} \mathrm{C} = [\mathrm{c}_0, \mathrm{c}_1, \ldots, \mathrm{c}_m], \quad \Phi(x) &= [\mathrm{B}_{0,m}(x), \mathrm{B}_{1,m}(x), \ldots, \mathrm{B}_{m,m}(x)]^{\mathrm{T}}, \\ \Phi'(x) &= [\mathrm{B}'_{0,m}(x), \mathrm{B}'_{1,m}(x), \ldots, \mathrm{B}'_{m,m}(x)]^{\mathrm{T}}, \\ &\vdots \\ \Phi^{(n)}(x) &= [\mathrm{B}^{(n)}_{0,m}(x), \mathrm{B}^{(n)}_{1,m}(x), \ldots, \mathrm{B}^{(n)}_{m,m}(x)]^{\mathrm{T}}. \end{aligned}$$

4. Applications of approximating functions and operational matrices

To solve Eq.(3) subject to the boundary cognitions (2) by means of the Bernstein polynomials method we employ Eqs. (4) and then we define the residual $\Re(x)$ for Eq. (3) as

$$\Re(x) \simeq F_{\mathrm{C}\Phi(x)} - \frac{\mathrm{d}}{\mathrm{d}x}\left[F_{\mathrm{C}\Phi'(x)}\right]. \tag{7}$$

Now, to find the solution $u_m(x)$ given in Eq. (4) we generate $(m-1)$ set of linear or nonlinear algebraic equations by applying the collocation nodes of the roots of Chebyshev polynomials

$$x_i = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{(2i+1)\pi}{2(m-1)}\right), \; i = 0, 1, \ldots, m-2. \tag{8}$$

Also, by imposing the boundary conditions (2) into Eq. (4) we have

$$u_m(x_0) = \mathrm{C}\Phi(x_0) = \alpha, \quad u_m(x_1) = \mathrm{C}\Phi(x_1) = \beta. \tag{9}$$

Eq's. (7) and (9) generate $(m+1)$ set of linear or nonlinear algebraic equations respectively. These algebraic equations can be solved for the unknown coefficients of the vector C. Consequently, $u_m(x)$ given in Eq. (4) can be easily calculated using Maple 15 with Digits= 100.

5. Error estimates and residual correction procedure

In this section, we follow [48,49] and give two different error estimations for the orthonormal Bernstein approximate solution of Eq. (3). For this purpose, we constitute two different error estimation procedures which involve residual correction and Runge-Kutta Fehlberg Method (RK45).

Let $u(x)$, and $u_m(x)$ be the exact solution and the approximate solution of (3), respectively. Now, let us constitute a residual correction procedure for the method. This procedure is based on the residual function. First, by attaining $e(x) := u(x) - u_m(x)$ yield $u(x) := e(x) + u_m(x)$, therefor Eq. (3) will be

$$F_{e(x)+u_m(x)} - \frac{\mathrm{d}}{\mathrm{d}x} F_{e'(x)+u'_m(x)} = 0, \tag{10}$$

where

$$e(x) \simeq e^*_n(x) = C^e \Phi(x), \quad C^e = [c^e_0, c^e_1, \ldots, c^e_m], \quad n \geq m.$$

Applying the method introduced in section (4) to the Eq.(10), subject to the boundary conditions

$$e^*_n(x_0) = 0, \quad e^*_n(x_1) = 0, \tag{11}$$

gives the new approximate solution of the absolute error, namely $e^*_n(x)$.

Corollary 2. *If $u_m(x)$ is the approximate solutions of (3), then $u_m(x) + e^*_n(x)$ is also approximate solution for (3).*

Thus, the approximate solutions $u_m(x) + e^*_n$ is better approximation than $u_m(x)$. We will call the approximate solutions $u_m(x) + e^*_n(x)$ as the corrected approximate solution.

As a second error estimation of the absolute error, we can use any two elements of the sequence of approximations. This estimation is similar to the error analysis of RK45. In the exact arithmetic, the absolute errors can be bounded by

$$\left| \|u(x) - u_m(x)\| - \|u(x) - u_n(x)\| \right| = C \|u(x) - u_k(x)\| \leq \|u_n(x) - u_m(x)\|, \quad n \geq m,$$

where

$$\|u(x) - u_k(x)\| = \max \{\|u(x) - u_m(x)\|, \|u(x) - u_n(x)\|\}, C = \frac{\|u_n(x) - u_m(x)\|}{\|u(x) - u_k(x)\|}.$$

Thus, if the error sequence is monotone, then we can use $\|u_{m+1}(x) - u_m(x)\|$ to may estimate the absolute error $\|u(x) - u_m(x)\|$.

6. Investigating convergence via residual function

Here, we follow the investigation of convergence process given in [41] to the convergence of the Bernstein polynomials solution by using the residual function of model problems in Banach space for the different values of m. Therefore, we can determine the behavior of our solutions. The residual function $\Re_m(x)$ is obtained, when our solution $u_m(x)$ is written into Eq. (3) as

$$\Re_m(x) \simeq F_{u_m(x)} - \frac{\mathrm{d}}{\mathrm{d}x} \left[F_{u'_m(x)} \right], \tag{12}$$

where $\Re_m(x)$ can be defined on the interval $[a, b]$ as $\Re_m(x) : [a, b] \to \mathrm{R}$ and $\Re_m(x)$ can be written in the Taylor series expansion as

$$\Re_m(x) = \sum_{i=0}^{m} r_i x^i. \tag{13}$$

Now, let us give a required theorem for our investigation.

Theorem 1. *. Let B be a Banach space. The residual function sequence* $\{\Re_m(x)\}_{m=0}^{\infty}$ *is convergent in B and the following inequality is satisfied so that* $0 < \gamma_m < 1$*. Here,* γ_m *is constant in B:*

$$\|\Re_{m+1}(x)\| \leq \gamma_m \|\Re_m(x)\|. \tag{14}$$

Proof 1. *We want to show that* $\{\Re_{m+1}(x)\}_{m=0}^{\infty}$ *is a Cauchy sequence in the Banach space B. For this purpose, consider,*

$$\begin{aligned} \left\|\Re_{m+1}(x)\right\| &= \left\|\sum_{i=0}^{m+1} r_i x^i\right\| \\ &= \sup\left\{\left|\sum_{i=0}^{m+1} r_i x^i\right| : x \in [a,b]\right\} \\ &\leq \sup\left\{\sum_{i=0}^{m+1} \left|r_i x^i\right| : x \in [a,b]\right\} \\ &= \left|\Re_{m+1}(b)\right|, \end{aligned}$$

so, the inequality (14) can be written at the point b as

$$\left|\Re_{m+1}(b)\right| \leq \gamma_m |\Re_m(b)|. \tag{15}$$

Starting from the inequality (15)

$$\left|\Re_{m+1}(b)\right| \leq \left|\Re_{m+1}(b) - \Re_m(b)\right| \leq (\gamma_m - 1) \left|\Re_m(b)\right|, \tag{16}$$

and by generalizing the inequality (16)

$$\left|\Re_{m+1}(b) - \Re_m(b)\right| \leq (\gamma_m - 1) \left|\Re_m(b)\right| \leq (\gamma_m - 1)^2 \left|\Re_{m-1}(b)\right| \leq \cdots \leq (\gamma_m - 1)^{m+1} \left|\Re_0(b)\right|.$$

For every $m, n \in N$, $m \geq n$, *we have*

$$\begin{aligned} \left|\Re_m(b) - \Re_n(b)\right| &\leq \left|(\Re_m(b) - \Re_{m-1}(b)) + (\Re_{m-1}(b) - \Re_{m-2}(b)) + \cdots + (\Re_{n+1}(b) - \Re_n(b))\right| \\ &\leq \left|(\Re_m(b) - \Re_{m-1}(b))\right| + \left|(\Re_{m-1}(b) - \Re_{m-2}(b))\right| + \cdots + \left|(\Re_{n+1}(b) - \Re_n(b))\right| \\ &\leq (\gamma_m - 1)^m \left|\Re_0(b)\right| + (\gamma_m - 1)^{m-1} \left|\Re_0(b)\right| + \cdots + (\gamma_m - 1)^{n+1} \left|\Re_0(b)\right| \\ &= \left(\tfrac{1-(\gamma_m - 1)^{m-n}}{1-(\gamma_m - 1)}\right) (\gamma_m - 1)^{n+1} \left|\Re_0(b)\right|, \end{aligned}$$

and since $0 < \gamma_m < 1$, *we get,*

$$\lim_{m,n\to\infty} \left|\Re_m(b) - \Re_n(b)\right| = 0. \tag{17}$$

Therefore, $\{\Re_m(b)\}_{m=0}^{\infty}$ *is a Cauchy sequence in the Banach space B and so it is convergent. This completes the proof of Theorem 1.*

In other words, the parameter γ_m can be defined as, for every $m \geq 0$, we have

$$\gamma_m = \begin{cases} \dfrac{\left\|\Re_{m+1}(x)\right\|}{\left\|\Re_m(x)\right\|} = \frac{\left|\Re_{m+1}(b)\right|}{\left|\Re_m(b)\right|}, & \left|\Re_m(b)\right| \neq 0 \\ 0 & \left|\Re_m(b)\right| = 0, \end{cases} \tag{18}$$

then the residual function sequence $\left\{\left|\Re_m(x)\right|\right\}_{m=0}^{\infty}$ approach to zero as m is increased. The main advantage of this algorithm is that it works successfully in handling nonlinear equations directly with a minimum size of calculations.

7. Numerical experiments

To demonstrate the effectiveness of the BPM algorithm discussed above, several examples of variational problems will be studied in this section.

7.1. Problem 1

We consider the following variational problem:

$$\min v = \int_0^1 [u(x) + u'(x) - 4e^{(3x)}]^2 \, dx, \tag{19}$$

subject to the boundary conditions

$$u(0) = 1, \quad u(1) = e^3. \tag{20}$$

The corresponding Euler-Lagrange equation is given by

$$u'' - u - 8e^{(3x)} = 0, \tag{21}$$

To solve (20)–(21) via the BPM approach, we take $m = 5$ and approximate the solution as

$$u_5(x) = \sum_{i=0}^{5} c_i B_{i,5}(x) = C\Phi(x).$$

Now according to (7) and (9) we have

$$\begin{aligned} \Re(x) &\simeq C\Phi''(x) - C\Phi(x) - 8e^{3x}, && (22)\\ C\Phi(0) &= 1, \quad C\Phi(1) = e^3. && (23) \end{aligned}$$

Eq's (22) and (23) using the collocation nodes (8) gives system of linear algebraic equations depend on the unknown coefficients $\{c_0, \ldots, c_5\}$ that can be easily to solve using Maple 15 with Digits= 100. Thus

$$u_5(x) = 1. + 2.974218455x + 4.05881680x^2 + 9.12544718x^3 - 7.22074889x^4 + 10.14780338x^5.$$

Therefore, the $\infty-$norm of the absolute error function for $m = 5$ is obtained as follows:

$$||u - u_5||_\infty = 0.027$$

Moreover, the approximate solution $u_5(x)$ can be corrected using our procedure giving in section (5). According to (10) and (11) we have

$$\begin{aligned} &C^e(\Phi''(x) + u_5(x)) - C^e(\Phi(x) + u_5(x)) - 8e^{3x} = 0,\\ &C^e\Phi(0) = 0, \quad C^e\Phi(1) = 0. \end{aligned}$$

The above Eqs. for $n = 14$ using (8) gives system of linear algebraic equations depend on the unknown coefficients $\{c_0^e, \ldots, c_{14}^e\}$ that can be easily to solve. Thus

$$\begin{aligned} e_{14}^*(x) = \; & 0.02578122855x + 0.4411815511x^2 - 4.625261062x^3 + 10.59061766x^4 \\ & -8.05703178x^5 + 0.5329292x^6 + 2.6266857x^7 - 6.4629679x^8 + 13.63326283x^9 \\ & -18.9679561x^{10} - 10.768213x^{12} + 17.8387073x^{11} + 3.7766341x^{13} \\ & -0.58436953x^{14}. \end{aligned}$$

Finally the corrected approximate solution is $u_5(x) + e^*_{14}(x)$.

The maximum absolute errors for the present method and the Non-Polynomial spline method of order sixteen [5] are shown in Table 1. From this table it's clear that our results are more accurate. In Figure 1 we plot,

(a) The approximate solution $u_5(x)$, the corrected approximate solution $u_5(x) + e^*_{14}(x)$ and the exact solution $u(x)$ in the interval $x \in [0, 4]$.

(b) The absolute error $\mid u(x) - u_5(x) \mid$ and the estimate absolute error $\mid e^*_{14}(x) \mid$.

(c) The corrected absolute error $\mid u(x) - (u_5(x) + e^*_{14}(x)) \mid$ by applying the residual correction procedure.

(d) Estimate the absolute error $\mid u_6(x) - u_5(x) \mid$.

From this Figure the corrected approximate solution and the corrected absolute error are more better than the approximate solution and the absolute error respectively. Moreover the upper bound can be estimated if the exact solution is unknown. Figure 2 shown the numerical results of γ_m. It can be easily seen that γ_m are less than one.

Table 1: The maximum absolute errors and run time in seconds of problem 1.

n	Non-Polynomial spline method [5] $\Vert E \Vert_\infty$	t	m	Present method $\Vert E \Vert_\infty$	t
16	4.41×10^{-16}	0.45	16	9.58×10^{-16}	0.890
32	3.23×10^{-21}	0.51	20	2.01×10^{-21}	1.061
64	1.74×10^{-26}	0.55	24	2.03×10^{-27}	1.311
128	7.89×10^{-32}	0.82	28	1.10×10^{-33}	1.545
256	3.28×10^{-37}	1.48	32	3.49×10^{-40}	1.841
512	1.31×10^{-42}	6.19	36	6.87×10^{-47}	2.168
1024	5.10×10^{-48}	36.44	40	8.85×10^{-54}	2.543

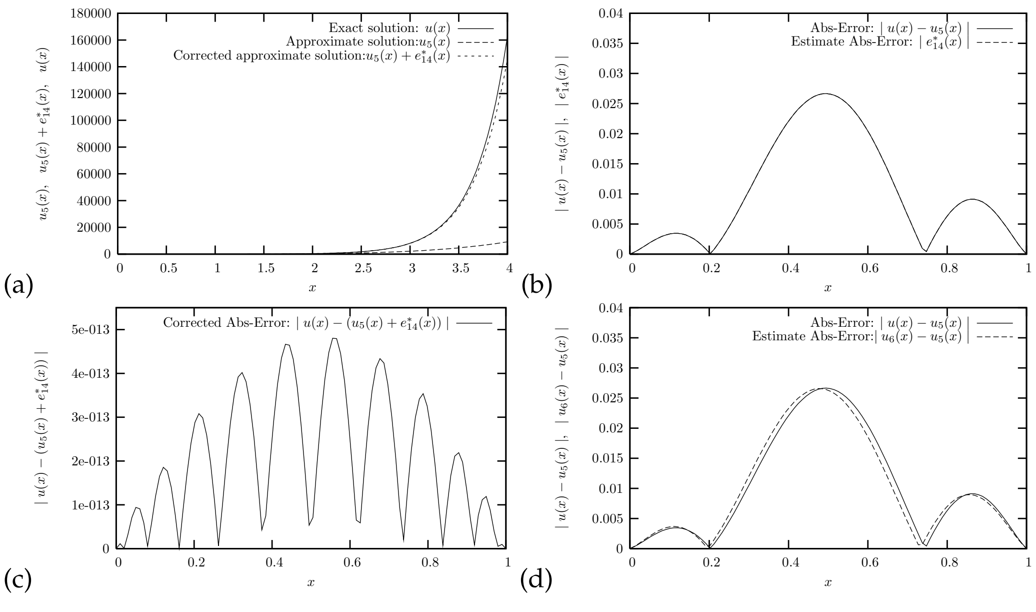

Figure 1: Numerical results of BPM for problem (1).

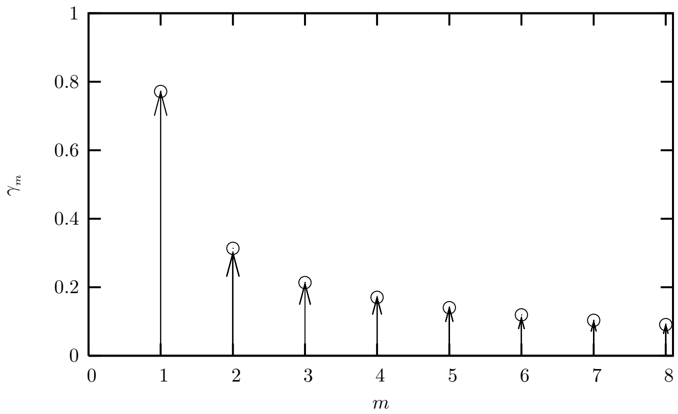

Figure 2: Numerical results of γ_m for problem (1). Here γ_m are less than one.

7.2. Problem 2

We consider the following variational problem:

$$\min v = \int_0^1 \frac{1+u^2(x)}{u'^2(x)}\,\mathrm{d}x, \tag{24}$$

subject to the boundary conditions

$$u(0) = 0, \quad u(1) = 0.5. \tag{25}$$

The corresponding Euler-Lagrange equation is given by

$$u'' + u''u^2 - uu'^2 = 0, \tag{26}$$

or equivalently

$$u'' = \frac{uu'^2}{1+u^2}, \tag{27}$$

with boundary conditions (25). The exact solution of the variational problem (24) is

$$u(t) = \sinh(0.481211825\,x). \tag{28}$$

By the same manipulations as in the previous example and assuming $m = 4$, we have

$$u_4(x) = 0.4812265234773x + 0.0000701645796x^2 + 0.0181546818671x^3 + 0.0005486300758x^4,$$

and also we have for $n = 5$

$$\begin{aligned} e_5^*(x) \quad = \quad & 0.1289687526 \times 10^{-11}x + 0.324759071 \times 10^{-12}x^2 - 0.183769252 \times 10^{-11}x^3 \\ & +0.1261456470 \times 10^{-10}x^4 - 0.1239131878 \times 10^{-10}x^5. \end{aligned}$$

The maximum absolute errors for the present method and the B–Spline Collocation Method [4] are shown in Table 2. Further reconfirms the present results are more accurate. In Figure 5 we plot,

(a) The approximate solution $u_5(x)$, the corrected approximate solution $u_5(x) + e_{14}^*(x)$ and the exact solution $u(x)$ in the interval $x \in [0, 4]$.

(b) The absolute error $\mid u(x) - u_5(x) \mid$ and the estimate absolute error $\mid e_{14}^*(x) \mid$.

(c) The corrected absolute error $|\, u(x) - (u_5(x) + e^*_{14}(x)) \,|$ by applying the residual correction procedure.

(d) Estimate the absolute error $|\, u_6(x) - u_5(x) \,|$.

From this Figure the corrected approximate solution and the corrected absolute error are more better than the approximate solution and the absolute error respectively. Moreover the upper bound can be estimated if the exact solution is unknown. Figure 4 shown the numerical results of γ_m. It can be easily seen that γ_m are less than one.

Table 2: The maximum absolute errors and run time in seconds of problem 2.

N	B–Spline Collocation Method [4] $\|\| E \|\|_\infty$	t	m	Present Method $\|\| E \|\|_\infty$	t
8	2.18×10^{-6}	–	5	2.57×10^{-8}	0.890
16	5.43×10^{-7}	–	6	1.11×10^{-9}	0.968
32	1.35×10^{-7}	–	7	6.66×10^{-11}	1.124
64	3.35×10^{-8}	–	8	6.66×10^{-11}	1.280
128	8.01×10^{-9}	–	9	6.66×10^{-11}	1.669
256	1.64×10^{-9}	–	10	6.66×10^{-11}	2.028

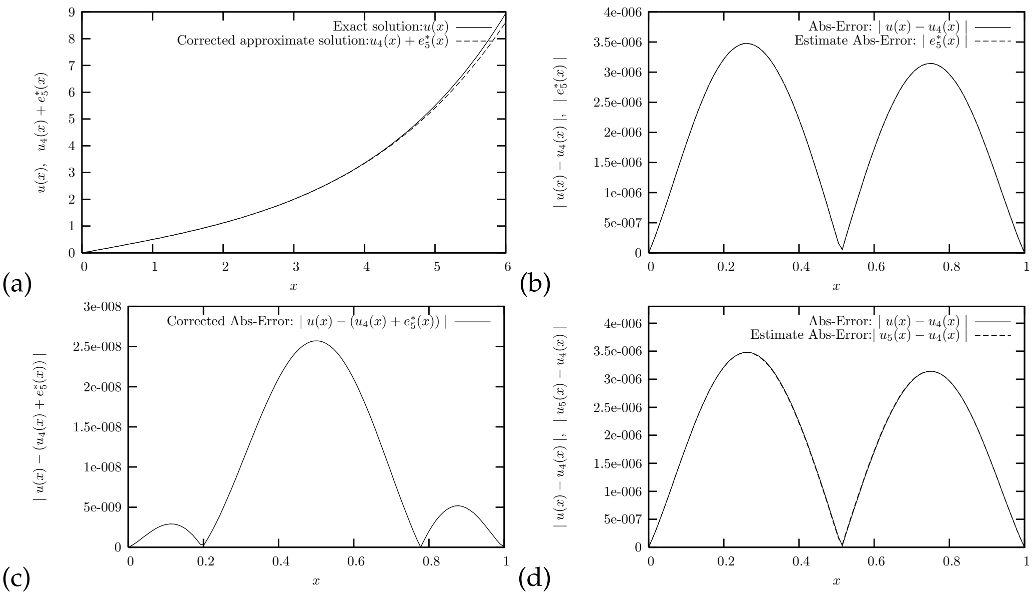

Figure 3: Numerical results of BPM for problem (2).

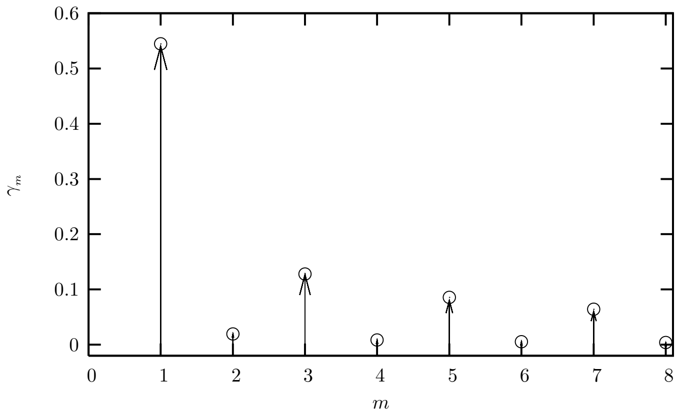

Figure 4: Numerical results of γ_m for problem (2). Here γ_m are less than one.

7.3. Problem 3

We consider the following brachistochrone problem:

$$\min v = \int_0^1 \left[\frac{1 + u'^2(x)}{1 - u(x)}\right]^2 \mathrm{d}x, \tag{29}$$

subject to the boundary conditions

$$u(0) = 0, \quad u(1) = -0.5. \tag{30}$$

The corresponding Euler-Lagrange equation is given by

$$u'' = \frac{1}{2}\left(\frac{1 + u'^2}{1 - u}\right), \tag{31}$$

or equivalently

$$2u'' - 2uu'' - u'^2 - 1 = 0, \tag{32}$$

with boundary conditions (30). The exact solution of the brachistochrone problem (29) in implicit form is

$$\begin{aligned} F(x, u) \quad = \quad & 0.5938731505 - x - \sqrt{-u^2 + 0.381510869u + 0.618489131} \\ & -0.8092445655 \tan^{-1}\left(\frac{u - 0.1907554345}{\sqrt{-u^2 + 0.381510869u + 0.618489131}}\right). \end{aligned}$$

The 4−term HPM approximate solution is given by [42]

$$\begin{aligned} \phi_4 \quad = \quad & -\frac{209379982158073}{262813488537600}t + \frac{71856289}{180633600}t^2 - \frac{630915349}{4843238400}t^3 - \frac{11645089}{722534400}t^4 \\ & + \frac{152564089}{2690688000}t^5 + \frac{202189357}{10838016000}t^6 - \frac{468036601}{11300889600}t^7 + \frac{3691}{322560}t^8 \\ & - \frac{1400257}{270950400}t^9 + \frac{1189}{241920}t^{10} + \frac{149843}{124185600}t^{11} - \frac{3743}{3801600}t^{12} - \frac{5167}{15654912}t^{13} \\ & + \frac{19031}{82790400}t^{14} + \frac{394403}{19372953600}t^{16} - \frac{537553}{158083301376}t^{18} + \frac{1069}{1982955520}t^{20}. \end{aligned} \tag{33}$$

It is clearly seen that the 4−term HPM approximate solution is a polynomial of degree t^{20}. Based on that for $m = 20$ and by the same manipulations as in the previous example we have

$$\begin{aligned} u_{20}(x) \;\; = \;\; & -0.786441t + 0.404622t^2 - 0.212141t^3 + 0.179700t^4 - 0.173144t^5 + 0.182541t^6 \\ & -0.203607t^7 + 0.236176t^8 - 0.280738t^9 + 0.335585t^{10} - 0.392033t^{11} \\ & +0.430653t^{12} - 0.425975t^{13} + 0.363484t^{14} - 0.256885t^{15} + 0.144266t^{16} \\ & -0.061324t^{17} + 0.018422t^{18} - 0.003471t^{19} + 0.000308t^{20}. \end{aligned} \tag{34}$$

Table 3 shows the maximum absolute errors function for different values of m to display the convergence of the solutions. It is clear that the efficiency of this approach can be dramatically enhanced by increasing m. In Figure 5 we plot,

(a) The absolute error function $|F(x, \phi_4)|$ for the solution (33) obtained by HPM [42] in the interval $[0, 1]$.

(b) The absolute error function $|F(x, u_{20})|$ for the solution (34) obtained by the present method in the interval $[0, 1]$.

(c) The approximate solution $F(x, u_5)$ obtained by the present method in the interval $[0, 1]$.

It is clear that the present solution is more accurate than the solution of HPM. Figure 6 shown the numerical results of γ_m. It can be easily seen that γ_m are less than one.

Table 3: The maximum absolute errors and run time in seconds of problem 2.

m	$\| F(x, u_m) \|_\infty$	t
5	9.94×10^{-3}	0.312
6	2.34×10^{-3}	0.375
7	5.45×10^{-4}	0.453
8	1.26×10^{-4}	0.530
9	2.92×10^{-5}	0.608
10	6.47×10^{-6}	0.780

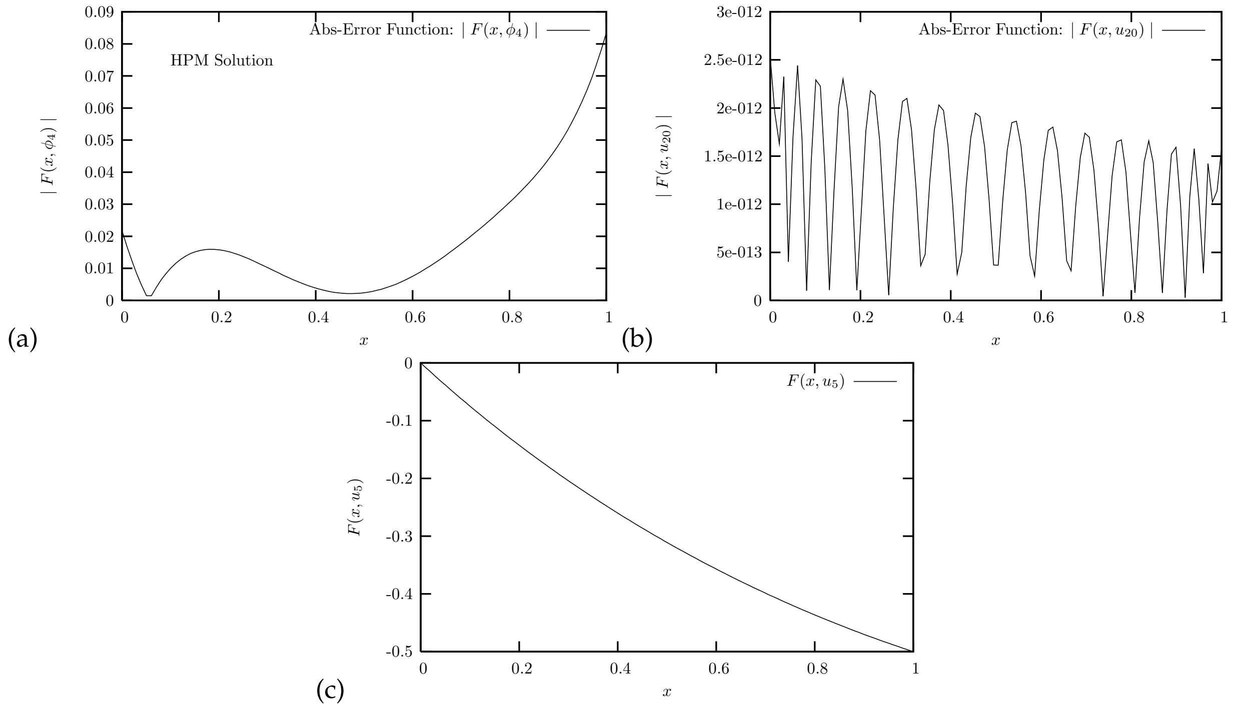

Figure 5: Numerical results of BPM and HPM for problem (3).

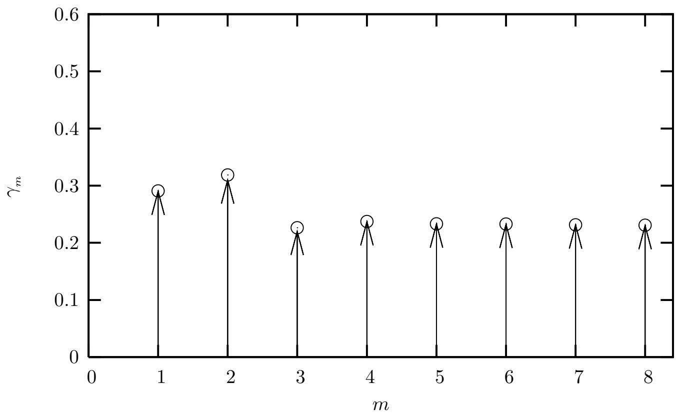

Figure 6: Numerical results of γ_m for problem (3). Here γ_m are less than one.

7.4. Problem 4

We consider the following variational problem:

$$v[u(x), w(x)] = \int_0^{\pi/2} [u'^2(x) + w'^2(x) + 2u(x)w(x)]\, dx, \tag{35}$$

subject to the following boundary conditions:

$$u(0) = 0, \quad u(\pi/2) = 1, \tag{36}$$
$$w(0) = 0, \quad w(\pi/2) = -1. \tag{37}$$

Table 4: The maximum absolute errors and run time in seconds of problem 4.

N	B–Spline Collocation Method [4] $\Vert E_1 \Vert_\infty = \Vert E_2 \Vert_\infty$	t	m	Present Method $\Vert E_1 \Vert_\infty = \Vert E_2 \Vert_\infty$	t
8	8.9855×10^{-4}	–	5	0.5767×10^{-4}	1.014
16	2.2507×10^{-4}	–	6	1.1169×10^{-4}	1.029
32	5.6321×10^{-5}	–	7	1.0528×10^{-5}	1.045
64	1.4082×10^{-5}	–	8	1.6762×10^{-6}	1.092
128	3.5207×10^{-6}	–	9	1.1803×10^{-7}	1.124
256	8.8018×10^{-7}	–	10	1.6021×10^{-8}	1.202

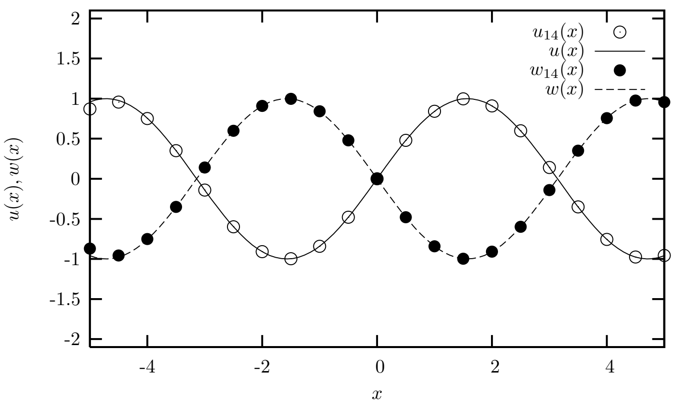

Figure 7: Comparison between the BPM solutions of $u_{14}(x), w_{14}(x)$ and the exact solution (40) on the interval $[-5, 5]$ to Problem (4).

The corresponding system of Euler's differential equations is given by

$$u'' - w = 0, \qquad (38)$$
$$w'' - u = 0, \qquad (39)$$

with boundary conditions (36)–(37). The exact solution of the variational problem (35) is as follows [15]:

$$u(x) = \sin(x), \qquad w(x) = -\sin(x). \qquad (40)$$

In Table 4 the maximum absolute errors to the present method for different values of m is compare with maximum absolute errors of the B–Spline collocation method [4] for different values of N. We see that the present method achieves higher accuracy than the B–Spline collocation method. In Figure 7 we compar the exact solutions $u(x)$ and $w(x)$ defined in (40) with the present approximate solutions of $u_{14}(x)$ and $w_{14}(x)$ on the interval $[-5, 5]$. Figure 8 shown the numerical results of γ_m. It can be easily seen that γ_m are less than one.

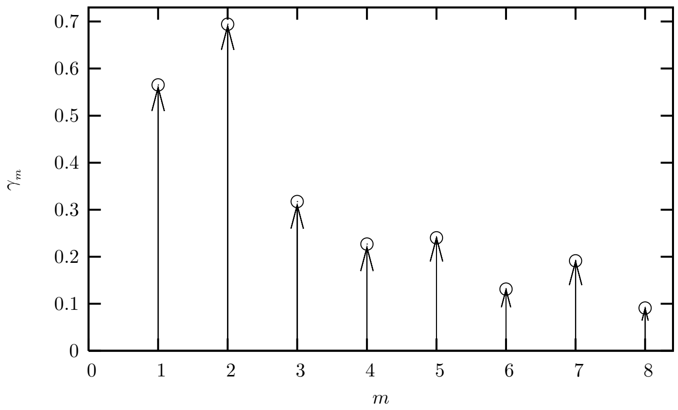

Figure 8: Numerical results of γ_m for problem (4). Here γ_m are less than one.

8. Conclusions

In this paper, the BPM technique was applied to obtain an approximate analytical solutions of some problems in calculus of variations. In most cases, BPM provides more realistic series solutions that converge very rapidly, usually only small size operational matrix is required to provide the solution at high accuracy. Some examples of variational problems were solved using BPM to illustrate the efficiency and accuracy of the method.

References

[1] O. P. Agrawal, Formulation of Euler-Lagrange equations for fractional variational problems, Journal of Mathematical Analysis and Applications 272 (2002) 368-379.
[2] I. M. Gelfand, S. V. Fomin, Calculus of Variations, Prentice-Hall, Englewood Cliffs, N. J., 1963.
[3] L. Elsgolts, Differential Equations and the Calculus of Variations, Mir Publisher, Moscow, 1977.
[4] M. Zarebnia, M. Birjandi, The Numerical Solution of Problems in Calculus of Variation Using B-Spline Collocation Method, Journal of Applied Mathematics 2012 (2012) doi:10.1155/2012/605741.
[5] K. Farajeyan, J. Rashidinia, R. Jalilian Classes of high-order numerical methods for solution of certain problem in calculus of variations, Cogent Mathematics 4 (2017) http://dx.doi.org/10.1080/23311835.2017.1288307.
[6] R. Jalilian, J. Rashidinia, K. Farajyan, H. Jalilian Non-Polynomial Spline for the Numerical Solution of Problems in Calculus of Variations, International Journal of Mathematical Modelling and Computations 05 (2015) 1–14.
[7] C. F. Chen, C. H. Hsiao, A Walsh series direct method for solving variational problems, Journal of the Franklin Institute 300 (1975) 265-280.
[8] C. Hwang, Y. P. Shih, Laguerre series direct method for variational problems, Journal of Optimization Theory and Applications 39 (1983) 143-149.
[9] R. Y. Chang, M. L. Wang, Shifted Legendre direct method for variational problems series, Journal of Optimization Theory and Applications 39 (1983) 299-307.
[10] I. R. Horng, J. H. Chou, Shifted Chebyshev direct method for solving variational problems, International Journal of Systems Science 16 (1985) 855–861.
[11] M. Razzaghi, M. Razzaghi, Fourier series direct method for variational problems, International Journal of Control 48 (1988) 887–895.
[12] M. Razzaghi, S. Yousefi, Legendre wavelets direct method for variational problems, Mathematical and Computer Modelling 53 (2000) 185-192.
[13] W. Glabisz, Direct Walsh-wavelet packet method for variational problems, Applied Mathematics and Computation 159 (2004) 769–781.
[14] M. Dehghan, M. Tatari, The use of Adomian decomposition method for solving problems in calculus of variations, Mathematical Problems in Engineering, 2006 (2006) http://dx.doi.org/10.1155/MPE/2006/65379.
[15] M. Tatari, M. Dehghan, Solution of problems in calculus of variations via He's variational iteration method, Physics Letters A 362 (2007) 401-406.
[16] A. Saadatmandi, Bernstein operational matrix of fractional derivatives and its applications, Applied Mathematical Modelling 38 (2014) 1365–1372.
[17] J. A. Rad, S. Kazem, M. Shaban, K. Parand, A. Yildirim, Numerical solution of fractional differential equations with a Tau method based on Legendre and Bernstein polynomials, Mathematical Methods in the Applied Sciences 37 (2014) 329–342.

[18] D. Rostamy, K. Karimi, ”A new operational matrix method based on the Bernstein polynomials for solving the backward inverse heat conduction problems”, International Journal of Numerical Methods for Heat and Fluid Flow 24 (2014) 669–678.
[19] A. Jafarian, S. Measoomy Nia, A. K. Golmankhaneh, D. Baleanu, On Bernstein Polynomials Method to the System of Abel Integral Equations, Abstract and Applied Analysis 2014 (2014), Article ID 796286.
[20] R.K. Pandey, N. Kumar, Solution of Lane-Emden type equations using Bernstein operational matrix of differentiation, New Astronomy 17 (2012) 303–308.
[21] O. R. Isik, M. Sezer, Bernstein series solution of a class of Lane-Emden type equations, Mathematical Problems in Engineering 3013 (2013). doi.org/10.1155/2013/423797.
[22] S. A. Yousefi, M. Behroozifar, Operational matrices of Bernstein polynomials and their applications, International Journal of Systems Science 41 (2010) 709–716.
[23] M. I. Bhatti, P. Bracken, Solutions of differential equations in a Bernstein polynomial basis, Journal of Computational and Applied Mathematics 205 (2007) 272-280.
[24] A. A. Dascioglu, N. Isler, Bernstein collocation method for solving nonlinear differential equations, Mathematical and Computational Applications 18 (2013) 293-300.
[25] I. elik, Collacation method and residual correction using Chebyshev series, Applied Mathematics and Computation 174 (2006) 910–920.
[26] F. A. Oliveira, Collacation and residual correction, Numerische Mathematik 36 (1980) 27–31.
[27] A. S. Bataineh, O. Isik, N. Aloushoush, N. Shawagfeh Bernstein Operational Matrix with Error Analysis for Solving High Order Delay Differential Equations International Journal of Applied and Computational Mathematics 3 (3), 1749-1762.
[28] M. H. T. Alshbool, A. S. Bataineh, I. Hashim, O.R. Isik, Solution of fractional-order differential equations based on the operational matrices of new fractional Bernstein functions, Journal of King Saud University-Science 29 (2017), 1–18.
[29] M. H. T. Alshbool, A. S. Bataineh, I. Hashim, O. R. Isik, Approximate solutions of singular differential equations with estimation error by using Bernstein polynomials, International Journal of Pure and Applied Mathematics 100 (2015) 109–125.
[30] A. S. Bataineh, O. R. Isik, I. Hashim, Bernstein method for the MHD flow and heat transfer of a second grade fluid in a channel with porous wall, Alexandria Engineering Journal 55 (2016) 2149–2156.
[31] F. Bani-Ahmad, A. K. Alomari, A. S. Bataineh, J. Sulaiman, I. Hashim, On the approximate solutions of systems of odes by legendre operational matrix of differentiation, Italian Journal of Pure and Applied Mathematics 36 (2016) 483-494.
[32] A. S. Bataineh, Novel method for solving singular two-point BVPs, Journal of Physics: Conference Series 435 (2013) 012044 doi:10.1088/1742-6596/435/1/012044.
[33] A. S. Bataineh, A. K. Alomari, I. Hashim Approximate solutions of singular two-point BVPs using Legendre operational matrix of differentiation, Journal of Applied Mathematics 2013 (3013) http://dx.doi.org/10.1155/2013/547502.
[34] S. Shahmorad, Numerical solution of general form linear FredholmVolterra integro differantial equations by the tau method with an error estimation, Applied Mathematics and Computation 167 (2005) 1418–1429.
[35] S. Yzbasi, An efficient algorithm for solving multi-pantograph equation systems, Computers and Mathematcs with Applications 64 (2012) 589–603.
[36] S. Yzbasi, E. Gk, M. Sezer, Laguerre matrix method with the residual error estimation for solutions of a class of delay differential equations, Mathematical Methods in the Applied Sciences 37 (2014) 453–463.
[37] A. Quarteroni, R. Sacco, F. Saleri, Numerical Mathematics, Springer, Berlin Heidelberg 2007.
[38] O. R. Isik, Z. Guney, M. Sezer, Bernstein series solutions of pantograph equations using polynomial interpolation, Journal of Difference Equations and Applications 16 (2010) 1–18.
[39] O. R. Isik, M. Sezer, Z. Guney, Bernstein series solution of a class of linear integro-differential equations with weakly singular kernel, Applied Mathematics and Computation 217 (2011) 7009–7020.
[40] O. R. Isik, M. Sezer, Z. Guney, A rational approximation based on bernstein polynomials for high order initial and boundary values problems, Applied Mathematics and Computation 217 (2011) 9438–9450.
[41] O. K. Kurkcu, E. Aslan, M. Sezera. A numerical method for solving some model problems arising in science and convergence analysis based on residual function, Applied Numerical Mathematics http://dx.doi.org/10.1016/j.apnum.2017.06.015.
[42] O. Abdulaziz, I. Hashim, M. S. H. Chowdhury, Solving variational problems by homotopy-perturbation method International Journal for Numerical Methods in Engineering 75 (2008) 09–721.