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A Note on Successive Coefficients of Spirallike Functions

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Abstract. Even there were several facts to show that $||a_{n+1}(f)| - |a_n(f)|| \leq 1$ is not true for the whole class of normalised univalent functions in the unit disk with with the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. In 1978, Leung[7] proved $||a_{n+1}(f)| - |a_n(f)||$ is actually bounded by 1 for starlike functions and by this result it is easy to get the conclusion $|a_n| \leq n$ for starlike functions. Since $||a_{n+1}(f)| - |a_n(f)|| \leq 1$ implies the Bieberbach conjecture (now the de Brange theorem), so it is still interesting to investigate the bound of $||a_{n+1}(f)| - |a_n(f)||$ for the class of spirallike functions as this class of functions is closely related to starlike functions. In this article we prove that this functional is bounded by 1 and equality occurs only for the starlike case. We are also able to give a precise form of extremal functions. Furthermore we also try to find the sharp bound of $||a_{n+1}(f)| - |a_n(f)||$ for non-starlike spirallike functions. By using the Carathéodory-Toeplitz theorem, we obtain the sharp lower and upper bounds of $|a_{n+1}(f)| - |a_n(f)|$ for $n = 1$ and $n = 2$. These results disprove the expected inequality $||a_{n+1}(f)| - |a_n(f)|| \leq \cos\alpha$ for α-spirallike functions.

1. Introduction

Let $\mathcal{A}$ denote the set of analytic functions f on the unit disk $\mathbb{D}$ normalized so that $f(0) = f'(0) - 1 = 0$ and $\mathcal{S}$ denote the subclass of functions $f \in \mathcal{A}$ which are univalent on $\mathbb{D}$. A function $f \in \mathcal{S}$ is called *starlike* if the image $f(\mathbb{D})$ is starlike with respect to the origin. It is well-known that the function $f \in \mathcal{A}$ is starlike if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, z \in \mathbb{D}. \tag{1}$$

The class of starlike functions is denoted by $\mathcal{S}^*$.

The class of spirallike functions was introduced by Špaček in 1932. A logarithmic spiral is a curve in the complex plane of the form

$$w = w_0 e^{-\lambda t}, \quad -\infty < t < \infty,$$

where w_0 and λ are complex constants with $w_0 \neq 0$ and $\operatorname{Re}\lambda \neq 0$. Without loss of generality we assume $\lambda = e^{i\alpha}$ with $-\pi/2 < \alpha < \pi/2$. The curve is then called an *α-spiral*. When $\alpha = 0$, it deduce to a line passing through w_0.

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A domain D containing the origin is said to be α-spirallike if for each point $w_0 \neq 0$ in D the arc of the α-spiral from w_0 to the origin lies entirely in D. A function $f \in \mathcal{A}$ with $f(0) = 0$, is said to be α-spirallike if its range is α-spirallike. We denote this class of functions by $\mathcal{SP}(\alpha)$. We remark that $\mathcal{SP}(0) = \mathcal{S}^*$. Špaček also proved that a function $f \in \mathcal{A}$ is *α-spirallike* if and only if for any $z \in \mathbb{D}$, $\operatorname{Re} e^{-i\alpha} zf'(z)/f(z) > 0$.

In 1963, Hayman [6] proved that the difference of successive coefficients is bounded for all $f \in \mathcal{S}$:

$$\mathcal{D}_n(f) := ||a_{n+1}(f)| - |a_n(f)|| \leq A, \quad n = 2, 3, \cdots, \tag{2}$$

where A is an absolute constant. It is an interesting topic to find out the value of A. Up to now, the best estimate was given by Grinspan [4] with $A < 3.61$. There is also some results to show that A can not be 1. In this paper we consider this functional $\mathcal{D}_n(f)$ for spirallike functions. For simplicity we write $a_n = a_n(f)$.

In 1978 Leung [7] considered the functional $\mathcal{D}_n(f)$ for the class of starlike functions and obtained the following result.

Theorem 1.1 (Leung). *For every* $f \in \mathcal{S}^*$,

$$||a_{n+1}| - |a_n|| \leq 1, \quad n = 1, 2, 3, \cdots.$$

For fixed n equality occurs only for the functions

$$f(z) = \frac{z}{(1 - \gamma z)(1 - \xi z)} \tag{3}$$

for some γ *and* ξ *with* $|\gamma| = |\xi| = 1$.

Based on the proof of Theorem 1.1 by Leung, for spirallike functions we get the following result:

Theorem 1.2. *For every function* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{SP}(\alpha)$,

$$||a_{n+1}| - |a_n|| \leq 1, \ n = 2, 3, \cdots. \tag{4}$$

Equality occurs only when $\alpha = 0$, *that is* $f \in \mathcal{S}^*$, *and* f *is in the form of*

$$K_\phi(z) = \frac{z}{1 - 2z\cos\phi + z^2} = \sum_{n=1}^{\infty} \frac{\sin n\phi}{\sin\phi} z^n. \tag{5}$$

or its rotation with $\phi = k\pi/n$ *or* $k\pi/(n+1)$ *for an integer* $0 \leq k < (n+1)/2$.

Here we remark that the inequality (4) can also be obtained by using a result of Hamilton in [5]. In our proof we find this inequality is sharp only for the special case ($\alpha = 0$) and this does not follow from Hamilton's result immediately. The extremal function in (5) is the precise presentation of (3). This statement already appeared in the author's paper [8] without proof.

By this theorem we know that the value 1 is not a sharp bound for α-spirallike functions for a fixed $\alpha \neq 0$. It is therefore of interest to find the sharp bound for $SP(\alpha)$. For the further analysis we recall a result by Basgöze and Keogh (see [1]), which gives a useful correspondence between α-spirallike functions and starlike functions.

Theorem 1.3 (Basgöze and Keogh). *For* $\alpha \in (-\pi/2, \pi/2)$, *a function* $f \in \mathcal{SP}(\alpha)$ *if and only if there corresponds a unique starlike function* $g \in \mathcal{S}^*$ *such that*

$$\frac{f(z)}{z} = \left(\frac{g(z)}{z}\right)^{\mu}, \quad z \in \mathbb{D}, \tag{6}$$

with $\mu = e^{i\alpha}\cos\alpha$. *Here the branch of complex power is chosen so that each side of the equation has the value 1 when* $z = 0$.

For $\log(g(z)/z) = 2\sum_{n=1}^{\infty}\gamma_n z^n$, it is well-known that if $g(z) \in \mathcal{S}^*$, $|\gamma_n| \leq 1/n$. So for $f(z) \in \mathcal{SP}(\alpha)$ and $\log(f(z)/z) = 2\sum_{n=1}^{\infty}\delta_n z^n$, by using Theorem 1.3 it is easy to see that $|\delta_n| \leq (\cos\alpha)/n$. In 1962 Zamorski (see [15]) proved the following sharp inequality for a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{SP}(\alpha)$:

$$|a_n| \leq \prod_{j=0}^{n-2} \frac{|2e^{-i\alpha}\cos\alpha + j|}{j+1} =: A_n, \qquad n = 2, 3, ...$$

It is easy to find that $A_{n+1} \leq (1 + 1/n)A_n$ and $A_2 = 2\cos\alpha$, so $|a_n| \leq A_n \leq n\cos\alpha$. Thus we may guess that $||a_{n+1}| - |a_n|| \leq \cos\alpha$ for general α-spirallike functions. Unfortunately, it is not true and we have the following result.

We define $T(\alpha) = \sqrt{5 + 4\cos(2\alpha)} + 1$. Then we know $2 < T(\alpha) \leq 4$ for $\alpha \in (-\pi/2, \pi/2)$.

Theorem 1.4. *For any real number $\alpha \in (-\pi/2, \pi/2)$, let $\mu = e^{i\alpha}\cos\alpha$. For every function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $\mathcal{SP}(\alpha)$, we have*

$$-1 \leq |a_2| - 1 \leq \cos\alpha. \tag{7}$$

Equality holds on the right hand side if and only if f is of the form in (18) given in the following section and on the left hand side equality holds if and only if f is given in (19) below. We also have

$$-\frac{2\cos\alpha}{\sqrt{T(\alpha)}} \leq |a_3| - |a_2| \leq \cos\alpha, \tag{8}$$

Equality holds on the right hand side when f is of the form in (22) given below, and equality holds on the left hand side when f is given by (23) below.

By noticing $2 < T(\alpha) < 4$ for $\alpha \in (-\pi/2, \pi/2)$ and $\alpha \neq 0$, then we can see that $1 < 2/\sqrt{T(\alpha)} < \sqrt{2}$, which means the absolute value of the quantity on the left hand side of (8) is greater than $\cos\alpha$.

2. Preliminaries

In 1960s, Milin systematically developed the idea of exponentiating inequalities to obtain information about the coefficients of univalent function itself. The inequality in the following lemma is known as the Third Lebedev-Milin Inequality, which estimates the new coefficients in terms of the previously given coefficients.

Lemma 2.1. (see [2, p.143]) *If $\phi(z) = \sum_{k=1}^{\infty}\alpha_k z^k$, $\varphi(z) = e^{\phi(z)} = \sum_{k=0}^{\infty}\beta_k z^k$ with $\sum_{k=1}^{\infty} k|\alpha_k|^2 < \infty$, then*

$$|\beta_n|^2 \leq \exp\left\{\sum_{k=1}^{n}\left(k|\alpha_k|^2 - \frac{1}{k}\right)\right\} \tag{9}$$

with equality if and only if $\alpha_k = \gamma^k/k, k = 1, 2, \cdots, n$ for some complex constant γ with $|\gamma| = 1$.

Let $\mathcal{P}$ denote the class of analytic functions P with positive real part on $\mathbb{D}$ which have the form

$$P(z) = 1 + 2\sum_{n=1}^{\infty} p_n z^n.$$

A member of $\mathcal{P}$ is called a Carathéodory function. We prove the following lemma for Carathéodory functions. This is an extension of Leung's result (see [7]), where is is discussed for the case $\mu = 1$.

Lemma 2.2. *Let $P(z) = 1 + 2\sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$, $\mu = \sigma + i\tau \in \mathbb{C}$. If $\alpha_n \in \mathbb{R}$ and $q(z) = 2\mu\sum_{n=1}^{\infty}\alpha_n p_n z^n$ is analytic in $\mathbb{D}$ and $\operatorname{Re} q(z) \leq M$ for some real number M on $|z| < 1$, then $2\sigma\sum_{n=1}^{\infty}\alpha_n|p_n|^2 \leq M$.*

Proof. Let $p_n = c_n + id_n$, $u(r,\theta) = \operatorname{Re} P(re^{i\theta})$ and $v(r,\theta) = \operatorname{Re} q(re^{i\theta})$. By simple calculations we know

$$u(r,\theta) = 1 + 2\sum_{n=1}^{\infty}(c_n \cos n\theta - d_n \sin n\theta)r^n,$$

$$v(r,\theta) = 2\sigma\sum_{n=1}^{\infty}\alpha_n(c_n \cos n\theta - d_n \sin n\theta)r^n - 2\tau\sum_{n=1}^{\infty}\alpha_n(c_n \sin n\theta + d_n \cos n\theta)r^n.$$

When $m \neq n$, then $\int_0^{2\pi}\cos m\theta \sin n\theta d\theta = 0$, $\int_0^{2\pi}\cos m\theta \cos n\theta d\theta = 0$ and $\int_0^{2\pi}\sin m\theta \sin n\theta d\theta = 0$, so

$$\int_0^{2\pi} u(r,\theta)v(r,\theta)d\theta = 4\sigma\pi\sum_{n=1}^{\infty}\alpha_n(c_n^2 + d_n^2)r^{2n} = 4\sigma\pi\sum_{n=1}^{\infty}\alpha_n|p_n|^2 r^{2n}.$$

By noticing $u \geq 0$ and $v \leq M$, we obtain

$$\int_0^{2\pi} u(r,\theta)v(r,\theta)d\theta \leq M\int_0^{2\pi} u(r,\theta)d\theta = 2\pi M.$$

Thus $2\sigma\sum_{n=1}^{\infty}\alpha_n|p_n|^2 r^{2n} \leq M$. Letting $r \to 1$, we have the statement. □

For any real number a, let $\mu = 1/(1 - ai)$. It is easy to check that $\sigma = \operatorname{Re}\mu = |\mu|^2$. Then by using Lemma 2.2, we get the following corollary.

Corollary 2.3. *For every $P(z) = 1 + 2\sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$ and every positive integer n, $\mu = 1/(1 - ai)(a \in \mathbb{R})$, there exists a complex number ξ with $|\xi| = 1$ such that*

$$\sum_{k=1}^{n}\frac{1}{k}|2\mu p_k - \xi^k|^2 \leq \sum_{k=1}^{n}\frac{1}{k}.$$

Proof. Applying Lemma 2.2 with $q(z) = 2\mu\sum_{k=1}^{n} p_k z^k/k$, and choosing proper ξ with $|\xi| = 1$ so that $\operatorname{Re}\{q(\overline{\xi})\} = M = \max_{z\in\overline{\mathbb{D}}}\operatorname{Re}\{q(z)\}$, we get

$$\sum_{k=1}^{n}\frac{1}{k}|2\mu p_k - \xi^k|^2 = 4\sum_{k=1}^{n}\frac{1}{k}|\mu p_k|^2 - 2\operatorname{Re}\{q(\overline{\xi})\} + \sum_{k=1}^{n}\frac{1}{k} \leq 2M - 2\operatorname{Re}\{q(\overline{\xi})\} + \sum_{k=1}^{n}\frac{1}{k} = \sum_{k=1}^{n}\frac{1}{k}.$$

□

To prove Theorem 1.4, the following well-known results for Carathéodory function will be used. The first lemma is known as Carathéodory's lemma (see e.g. [2, p. 41]).

Lemma 2.4 (Carathéodory's lemma). *For a function $P(z) = 1 + 2\sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$, the sharp inequality $|p_n| \leq 1$ holds for each n.*

The sharpness can be observed through the example $P_0(z) = (1 + z)/(1 - z) = 1 + 2z + 2z^2 + \ldots$. We will use also the following result due to Carathéodory and Toeplitz (see [3] or [14]).

Lemma 2.5 (Carathéodory-Toeplitz Theorem). *Let $P(z) = 1 + 2\sum_{n=1}^{\infty} p_n z^n$, then P represents a Carathéodory function if and only if the determinant*

$$D_n = \begin{vmatrix} 1 & p_1 & p_2 & \cdots & p_n \\ p_{-1} & 1 & p_1 & \cdots & p_{n-1} \\ p_{-2} & p_{-1} & 1 & \cdots & p_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{-n} & p_{-n+1} & p_{-n+2} & \cdots & 1 \end{vmatrix} \tag{10}$$

is non-negative for each $n \geq 1$, *where* $p_{-j} = \overline{p_j}$ *for* $j \geq 1$.
Moreover, if $D_1 > 0, \dots, D_{k-1} > 0$ *and if* $D_k = 0$, *then* $P(z)$ *is of the following form:*

$$P(z) = \sum_{j=1}^{k} t_j \frac{1+\varepsilon_j z}{1-\varepsilon_j z}, \quad t_j > 0, \quad |\varepsilon_j| = 1, \quad \varepsilon_j \neq \varepsilon_l \ (j \neq l). \tag{11}$$

Since $P(0) = 1$, the numbers t_j must satisfy $t_1 + \cdots + t_k = 1$.

We may assume without loss of generality that $0 \leq p_1 \leq 1$. On using (10) for $n = 2$, we get

$$D_2 = \begin{vmatrix} 1 & p_1 & p_2 \\ p_{-1} & 1 & p_1 \\ p_{-2} & p_{-1} & 1 \end{vmatrix} = 1 + 2\mathrm{Re}\,(p_1^2 p_2) - |p_2|^2 - 2p_1^2 = (1-p_1^2)^2 - |p_1^2 - p_2|^2 \geq 0,$$

which is equivalent to

$$p_2 = p_1^2 + x(1-p_1^2). \tag{12}$$

for some $|x| \leq 1$. A similar assertion is firstly made by Libera and Złotkiewicz [9].

As a special case of Lemma 2.5, we have the following useful assertion.

Lemma 2.6. *Let* $P(z) = 1 + 2p_1 z + 2p_2 z^2 + \cdots$ *be a Carathéodory function with* $p_1 \in [0,1]$. *If* $p_2 = p_1^2 + x(1-p_1^2)$ *with* $|x| = 1$, *then* P *must be of the form*

$$P(z) = \begin{cases} \dfrac{1+xz^2}{1-xz^2} & \text{when } p_1 = 0, \\ \dfrac{1+z}{1-z} & \text{when } p_1 = 1, \\ \dfrac{1}{1+t^2}\dfrac{1+\varepsilon_1 z}{1-\varepsilon_1 z} + \dfrac{t^2}{1+t^2}\dfrac{1+\varepsilon_2 z}{1-\varepsilon_2 z} & \text{when } p_1 \neq 0, 1 \end{cases}$$

with $\varepsilon_1 = p_1 - te^{i\theta}\sqrt{1-p_1^2}$, $\varepsilon_2 = p_1 + t^{-1}e^{i\theta}\sqrt{1-p_1^2}$ *and* $t = (p_1 \cos\theta + \sqrt{1-p_1^2 \sin^2\theta})/\sqrt{1-p_1^2}$, *for* $\theta = \frac{1}{2}\arg x \in (-\pi/2, \pi/2]$.

Proof. We observe that $D_2 = 0$ by assumption, where D_2 is given in (10) with $n = 2$. When $D_1 = 0$, which is equivalent to the condition $p_1 = 1$, the assertion holds clearly. Thus, we may assume that $D_1 = 1 - p_1^2 > 0$. Lemma 2.5 now implies that P has the form

$$P(z) = t_1 \frac{1+\varepsilon_1 z}{1-\varepsilon_1 z} + t_2 \frac{1+\varepsilon_2 z}{1-\varepsilon_2 z} = 1 + 2(t_1\varepsilon_1 + t_2\varepsilon_2)z + 2(t_1\varepsilon_1^2 + t_2\varepsilon_2^2)z^2 + \cdots$$

for $t_j > 0$ and $\varepsilon_j \in \partial\mathbb{D}$ $(j = 1, 2)$ with $t_1 + t_2 = 1$ and $\varepsilon_1 \neq \varepsilon_2$. By comparing the coefficients of z and z^2 in the above equation, we obtain the relations

$$t_1\varepsilon_1 + t_2\varepsilon_2 = p_1 \quad \text{and} \quad t_1\varepsilon_1^2 + t_2\varepsilon_2^2 = p_2 = p_1^2 + x(1-p_1^2).$$

Take $t = \sqrt{t_2/t_1}$ and $\theta = \frac{1}{2}\arctan x \in (-\pi/2, \pi/2)$, therefore by solving the equations we obtain

$$\varepsilon_1 = p_1 - \sqrt{1-p_1^2}\, te^{i\theta}, \qquad \varepsilon_2 = p_1 + \sqrt{1-p_1^2}\, t^{-1}e^{i\theta}. \tag{13}$$

$$\text{or} \quad \varepsilon_1 = p_1 + \sqrt{1-p_1^2}\, t^{-1}e^{i\theta} \qquad \varepsilon_2 = p_1 - \sqrt{1-p_1^2}\, te^{i\theta}. \tag{14}$$

Without loss of generality we just consider the first case.

If $p_1 = 0$, then $\varepsilon_2 = -\varepsilon_1 = e^{i\theta}$ and $t_1 = t_2 = 1/2$, so $P(z) = (1+xz^2)/(1-xz^2)$.

When $p_1 \neq 0, 1$, since $t = \sqrt{t_2/t_1}$ and $t_1 + t_2 = 1$, we have $t_1 = 1/(1+t^2)$ and $t_2 = t^2/(1+t^2)$. By using (13), $|\varepsilon_1|^2 = 1$ is equivalent to

$$t^2(1-p_1^2) - 2tp_1\cos\theta\sqrt{1-p_1^2} + p_1^2 = 1.$$

By solving the equation we obtain a positive solution

$$t = \frac{p_1\cos\theta + \sqrt{1-p_1^2\sin^2\theta}}{\sqrt{1-p_1^2}}. \tag{15}$$

Thus we conclude that in this case $P(z)$ is of the following form

$$P(z) = \frac{1}{1+t^2}\frac{1+\varepsilon_1 z}{1-\varepsilon_1 z} + \frac{t^2}{1+t^2}\frac{1+\varepsilon_2 z}{1-\varepsilon_2 z}$$

with t in the form of (15) and ε_1, ε_2 with form of (13). This completes the proof of the Lemma. □

3. Proof of the main results

Proof. [Proof of Theorem 1.2] Simple integration of (17) gives

$$\log\frac{f(z)}{z} = \mu\int_0^z \frac{P(t)-1}{t}dt = 2\mu\sum_{k=1}^{\infty}\frac{1}{k}p_k z^k.$$

For $|\xi| = 1$, let

$$\log\left\{(1-\xi z)\frac{f(z)}{z}\right\} = \sum_{k=1}^{\infty}\alpha_k z^k,$$

where $\alpha_k = (2\mu p_k - \xi^k)/k$.

On the other hand,

$$(1-\xi z)\frac{f(z)}{z} = \sum_{k=0}^{\infty}\beta_k z^k,$$

where $\beta_k = a_{k+1} - \xi a_k$. Then by applying Lemma 2.1 we get

$$|a_{n+1} - \xi a_n|^2 \leq \exp\left\{\sum_{k=1}^{n}\frac{1}{k}|2\mu p_k - \xi^k|^2 - \sum_{k=1}^{n}\frac{1}{k}\right\}.$$

By Corollary 2.3, we can pick some ξ with $|\xi| = 1$ to make the exponent nonpositive. Hence $|a_{n+1} - \xi a_n| \leq 1$. Because $||a_{n+1}| - |a_n|| \leq |a_{n+1} - \xi a_n|$ for all $|\xi| = 1$, this completes the proof of the inequality.

When $||a_{n+1}| - |a_n|| = 1$, by Lemma 2.1, we get $\alpha_k = (2\mu p_k - \xi^k)/k = \gamma^k/k$ with $|\gamma| = 1$, and thus $2\mu p_k = \xi^k + \gamma^k$. Since $\mu = 1/(1-ai)$ we have $2p_1 = (1-ai)(\xi+\gamma)$, $2p_2 = (1-ai)(\xi^2+\gamma^2)$, and therefore the Toeplitz determinant

$$D_2 = \begin{vmatrix} 1 & (1-ai)(\xi+\gamma)/2 & (1-ai)(\xi^2+\gamma^2)/2 \\ (1+ai)(\overline{\xi}+\overline{\gamma})/2 & 1 & (1-ai)(\xi+\gamma)/2 \\ (1+ai)(\overline{\xi}^2+\overline{\gamma}^2)/2 & (1+ai)(\overline{\xi}+\overline{\gamma})/2 & 1 \end{vmatrix} = -a^2.$$

By Lemma 2.5 we know that $D_2 \geq 0$. Since a is real, the only possibility is $a = 0$, which means $f \in \mathcal{S}^*$. By Theorem 1.1 we know $f(z) = z/((1-\gamma z)(1-\xi z))$ for some γ and ξ with $|\gamma| = |\xi| = 1$.

Based on Leung's result, we give more details on this extremal function. Let $\zeta = \xi/\gamma$. Then we know $|\zeta| = 1$ and

$$a_k = \frac{\gamma^k - \xi^k}{\gamma - \xi} = \frac{1-\zeta^k}{1-\zeta}. \tag{16}$$

Then we can see that $||a_{n+1}| - |a_n|| = 1$ is equivalent to $|1-\zeta^{n+1}| - |1-\zeta^n| = |1-\zeta|$ or $|1-\zeta^{n+1}| - |1-\zeta^n| = -|1-\zeta|$. Since $|1-\zeta^{n+1}| - |1-\zeta^n| \le |\zeta^{n+1} - \zeta^n| = |1-\zeta|$, we know $1-\zeta^{n+1}$ and $1-\zeta^n$ are collinear. Hence we have three cases to consider:

Case 1: $\zeta^n = 1$ and $|1-\zeta^{n+1}| - |1-\zeta^n| = |1-\zeta|$. This means $a_n = 0$ and $|a_{n+1}| = 1$. In this case $f(z)$ is in the form of (5) with $\phi = k\pi/n$, for integer $1 \le k < n/2$ or its rotation.

Case 2: $\zeta^{n+1} = 1$ and $|1-\zeta^{n+1}| - |1-\zeta^n| = -|1-\zeta|$. This means $a_{n+1} = 0$ and $|a_n| = 1$. In this case $f(z)$ is K_ϕ with $\phi = k\pi/(n+1)$ for integer $1 \le k < (n+1)/2$.

Case 3: $\zeta = 1$, which means $\xi = \gamma$, that is, $f(z)$ is the Koebe function or its rotation. We assume this is the case $\phi = 0$ for K_ϕ.

This completes the proof of Theorem 1.2. □

Next we look closely to the difference of successive coefficients. By the definition of spirallike function we know for every function $f \in \mathcal{SP}(\alpha)$, there exists a function $P(z) = 1 + 2p_1 z + 2p_2 z^2 + \cdots \in \mathcal{P}$, such that

$$e^{-i\alpha}\frac{zf'(z)}{f(z)} = P(z)\cos\alpha - i\sin\alpha.$$

This is equivalent to

$$f(z) = z\exp\left\{\mu\int_0^z \frac{P(t)-1}{t}dt\right\}, \tag{17}$$

where $\mu = e^{i\alpha}\cos\alpha = \frac{1}{1-ai}$ and $a = \tan\alpha$.

Proof. [Proof of Theorem 1.2] We assume that $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*$ is the function in (6). Since $g \in \mathcal{S}^*$, by (1) we assume there is $P(z) = 1 + 2p_1 z + 2p_2 z^2 + \cdots \in \mathcal{P}$ such that $zg'(z)/g(z) = P(z)$. Comparing the coefficients on both sides we get

$$2b_2 = 2p_1 + b_2, \quad 3b_3 = 2p_2 + b_3 + 2b_2 p_1,$$

that is $b_2 = 2p_1,\ b_3 = 2p_1^2 + p_2$. Using (6) we get

$$1 + \sum_{n=2}^{\infty} a_n z^{n-1} = \left(1 + \sum_{n=2}^{\infty} b_n z^{n-1}\right)^{\mu}.$$

Comparing the coefficients on both sides, we obtain

$$a_2 = \mu b_2, \quad a_3 = \mu b_3 + \frac{\mu(\mu-1)}{2}b_2^2.$$

So $|a_2| - 1 = |b_2|\cos\alpha - 1 \le \cos\alpha(|b_2| - 1) \le \cos\alpha$ and equality holds if and only if $|b_2| = 2$. Therefore $|p_1| = 1$ and $D_1 = 1 - |p_1|^2 = 0$. By Lemma 2.5, we get $P(z) = (1+\varepsilon z)/(1-\varepsilon z)$, then by (17) we have

$$f(z) = z(1-\varepsilon z)^{-2\mu} \quad \text{with} \quad |\varepsilon| = 1. \tag{18}$$

On the other hand, $1 - |a_2| = 1 - |b_2|\cos\alpha \le 1$, with equality holds if and only if $|b_2| = 0$, thus $p_1 = 0$. By Lemma 2.6 and (17),

$$f(z) = z\left(1-\varepsilon z^2\right)^{-\mu} \quad \text{with} \quad |\varepsilon| = 1. \tag{19}$$

For $n = 2$, to simplify the calculation we assume $p_1 = p \in [0, 1]$, since $|a_3| - |a_2|$ is invariant under rotation. By using (12),

$$|a_3| - |a_2| = \left|\mu b_3 + \frac{\mu(\mu-1)}{2} b_2^2\right| - |\mu b_2| = \cos\alpha\left(\left|b_3 + \frac{\mu-1}{2} b_2^2\right| - |b_2|\right)$$
$$= \cos\alpha\left(\left|2p^2 + p_2 + 2(\mu-1)p^2\right| - 2p\right)$$
$$= \cos\alpha\left(\left|p^2 + x(1-p^2) + 2\mu p^2\right| - 2p\right).$$

Leting $x = re^{2i\theta}$, we have

$$|a_3| - |a_2| = \cos\alpha\left(\left|p^2 + r(1-p^2)e^{i\theta} + 2p^2e^{2i\alpha}\cos\alpha\right| - 2p\right)$$
$$= \cos\alpha\left(\sqrt{A\cos(2\theta) + B\sin(2\theta) + C} - 2p\right), \tag{20}$$

where A, B, and C are given as $A = 2r(1-p^2)p^2(\cos(2\alpha) + 2)$, $B = 2r(1-p^2)p^2\sin(2\alpha)$ and $C = (1-p^2)^2r^2 + p^4(5 + 4\cos(2\alpha))$.

By using (20) and the notation $T(\alpha) = \sqrt{5 + 4\cos(2\alpha)} + 1$,

$$|a_3| - |a_2| \le \cos\alpha\left(\sqrt{\sqrt{A^2 + B^2} + C} - 2p\right)$$
$$= \cos\alpha\left(\sqrt{2(1-p^2)p^2(T(\alpha)-1)r + (1-p^2)^2r^2 + p^4(T(\alpha)-1)^2} - 2p\right)$$
$$\le \cos\alpha\left(p^2(T(\alpha)-2) - 2p + 1\right). \tag{21}$$

First equality holds when $\sin(2\theta) = \sin(2\alpha)/(T(\alpha)-1)$, $\cos(2\theta) = (\cos(2\alpha)+2)/(T(\alpha)-1)$ and equality for (21) is attained if $r = 1$.

Since $\alpha \in (-\pi/2, \pi/2)$, $p \in [0, 1]$, it is easy to check $T(\alpha) \ge 2$ and $2/(T(\alpha)-2) \ge 1$, so $|a_3| - |a_2| \le \cos\alpha$. Equality holds when $p = 0$, $|x| = 1$. We can check that $D_2 = 0$. Then by Lemma 2.6,

$$f(z) = z\left(1 - \varepsilon z^2\right)^{-\mu} \quad \text{with} \quad \varepsilon = x^{1/2} = e^{i\theta/2}. \tag{22}$$

On the other hand, by (20)

$$|a_3| - |a_2| \ge \cos\alpha\left(\sqrt{-\sqrt{A^2 + B^2} + C} - 2p\right)$$
$$= \cos\alpha\left(\sqrt{-2(1-p^2)p^2(T(\alpha)-1)r + (1-p^2)^2r^2 + p^4(T(\alpha)-1)^2} - 2p\right)$$
$$\ge \cos\alpha\left(\left|p^2T(\alpha) - 1\right| - 2p\right).$$

We notice that the first equality holds for $\sin(2\theta) = -\sin(2\alpha)/(T(\alpha)-1)$, $\cos(2\theta) = -(\cos(2\alpha)+2)/(T(\alpha)-1)$ and the last one holds when $r = 1$.

When $p^2 \le 1/T(\alpha)$, $|a_3| - |a_2| \ge \cos\alpha(-p^2T(\alpha) - 2p + 1)$. Since $T(\alpha) > 0$, we know $-p^2T(\alpha) - 2p + 1$ is decreasing in the interval $[0, 1/\sqrt{T(\alpha)}]$.

When $p^2 \ge 1/T(\alpha)$, $|a_3| - |a_2| \ge \cos\alpha(p^2T(\alpha) - 2p - 1)$. Since $1/\sqrt{T(\alpha)} > 1/T(\alpha)$, we know $p^2T(\alpha) - 2p - 1$ is increasing in the interval $[1/\sqrt{T(\alpha)}, 1]$. So

$$|a_3| - |a_2| \ge \cos\alpha\left(-\left(1/\sqrt{T(\alpha)}\right)^2 T(\alpha) - 2/\sqrt{T(\alpha)} + 1\right) = -2\cos\alpha/\sqrt{T(\alpha)}$$

Equality is attained when $p = 1/\sqrt{T(\alpha)}$ and $x = -(\sin(2\alpha) + (\cos(2\alpha) + 2)i)/\sqrt{5 + 4\cos(2\alpha)}$. It is easy to check that $|x| = 1$ and $D_2 = 0$. Then by applying Lemma 2.6 and (17), we know

$$f(z) = z(1 - \varepsilon_1 z)^{-2\mu/(1+t^2)}(1 - \varepsilon_2 z)^{-2t^2\mu/(1+t^2)} \tag{23}$$

with $\varepsilon_1 = (1 - te^{i\theta}\sqrt{T(\alpha) - 1})/\sqrt{T(\alpha)}$ and $\varepsilon_2 = (1 + t^{-1}e^{i\theta}\sqrt{T(\alpha) - 1})/\sqrt{T(\alpha)}$. Here $t = (\cos\theta + \sqrt{T(\alpha) - \sin^2\theta})\ /\ \sqrt{T(\alpha) - 1}$ and $\theta = (\pi + \theta_0)/2$ with $\theta_0 = \arctan\frac{\sin(2\alpha)}{2+\cos(2\alpha)} \in (0, \pi/4)$. This completes the proof of Theorem 1.4. □

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References

[1] T. Başgöze and F. R. Keogh, *The hardy class of a spirallike function and its derivative*, Proc. Amer. Math. Soc. **26** (1970), 266-269.
[2] P. L. Duren, *Univalent Functions*, Springer-Verlag, New York, 1983.
[3] U. Grenander and G. Szegő, *Toeplitz forms and their applications*, University of California Press, Berkeley-Los Angeles, 1958.
[4] A. Z. Grinspan, *Improved bounds for the difference of the moduli of adjacent coefficients of univalent functions*, in Some Questions in the Modern Theory of Functions, Sib. Inst. Mat., Novoibirsk, 1976, pp.41-45 (in Russian).
[5] D. H. Hamilton, *On a conjecture of M. S. Robertson*, J. London Math Soc.(2) **21** (1980), 265-278.
[6] W. K. Hayman, *On successive coefficients of univalent functions*, J. London Math Soc **38** (1963), 228-243.
[7] Y. Leung, *Successive coefficients of starlike functions*, Bull. London Math. Soc. **10** (1978) 193–196.
[8] M. Li and T. Sugawa, *A note on successive coefficients of convex functions*, Comput. Methods Func. Theory **17** (2017) 179–193.
[9] R. J. Libera and E. J. Złotkiewicz, *Early coefficients of the inverse of a regular convex function*, Proc. Amer. Math. Soc. **85** (1982) 225–230.
[10] I. M. Milin, *Adjacent coefficients of uivalent functions*, Dokl. Akad. Nauk SSSR **180** (1968) 1294–1297 (in Russian).
[11] A. Pfluger, *Some coefficient problems for starlike functions*, Ann. Acad. Sci. Fenn. Ser. AI **2** (1976) 383–396.
[12] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.
[13] L. Špaček, *Přispěvek k teorii funkcí prostych*, Časopis Pěst. Mat. **62** (1933) 12–19.
[14] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo, 1959.
[15] J. Zamorski, *About the extremal spiral schlicht functions*, Ann. Polon. Math. **9** (1962) 265–273.