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ALPHA-SPIRAL FUNCTIONS IN AN ELLIPTICAL DOMAIN

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Abstract

Let $E = \left\{ z = x + iy : \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 < 0 \right\}$, where $a > b > 0$. In this paper[1] we introduce the class of alpha-spiral functions in E. We obtain sufficient conditions for analytic functions in the elliptical domain E, to be alpha-spiral.

1 Introduction

Let g be a complex function in the unit disk $U = \{z \in \mathbf{C} : |\mathbf{z}| < \mathbf{1}\}$. For $z = x + iy \in U$ we put $u(x, y) = \text{Re}z$ and $v(x, y) = \text{Im}z$. The function g belongs to the class $C^1(U)$ if the functions $u = u(x, y)$ and $v = v(x, y)$ are continuous and have continuous first order partial derivatives in U. If $g \in C^1(U)$ we denote

$$Dg = z\frac{\partial g}{\partial z} - \bar{z}\frac{\partial g}{\partial \bar{z}} \qquad \text{and} \qquad Jg = \left|\frac{\partial g}{\partial z}\right|^2 - \left|\frac{\partial g}{\partial \bar{z}}\right|^2$$

where

$$\frac{\partial g}{\partial z} = \frac{1}{2}\left(\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial g}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y}\right).$$

Let $\alpha \in \mathbf{R}$ with $|\alpha| < \frac{\pi}{2}$ and let $z_0 \in \mathbf{C} \setminus \{\mathbf{0}\}$. The equality

$$z(t) = z_0 e^{-(\cos\alpha + i\sin\alpha)t} \quad , \quad t \in \mathbf{R}$$

defines an *α-spiral curve* in the complex plane.

Let D be a domain in $\mathbf{C}$ such that $0 \in D$. If for any $z_0 \in D \setminus \{0\}$, the arc of α-spiral curve which joins the points z_0 and 0, is contained in D, then D is an *α-spiral domain with* respect to 0.

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In 1981, H. S. Al-Amiri and P. T. Mocanu [1], introduced the class of nonanalytic α-spiral functions in U and obtained sufficient conditions for complex nonanalytic functions in U to be α-spiral.

Let $g \in C^1(U), g(0) = 0$ and let $\alpha \in \mathbf{R}, |\alpha| < \frac{\pi}{\mathbf{2}}$. The function g is an *α-spiral function* in U if g is injective and maps U into an α-spiral domain with respect to 0.

Theorem 1.1 [1] *Let $\alpha \in \mathbf{R}$, with $|\alpha| < \frac{\pi}{2}$. If the function g belongs to the class $\mathbf{C^1}(\mathbf{U})$ and satisfies the following conditions:*

(i) $g(0) = 0$ and $g(z) \neq 0$, for all $z \in U \setminus \{0\}$,

(ii) $Jg(z) > 0$, for all $z \in U$,

(iii) $\mathrm{Re}\left[e^{i\alpha}\frac{Dg(z)}{g(z)}\right] > 0$, for all $z \in U \setminus \{0\}$,

then g is an α-spiral function in U.

2 Alpha-spirallikeness conditions

Let $\alpha \in \mathbf{R}, |\alpha| < \frac{\pi}{\mathbf{2}}$. An analytic function $f : E \to \mathbf{C}, \mathbf{f}(\mathbf{0}) = \mathbf{0}$ is called *α-spiral in E* if it is univalent in E and $f(E)$ is an α-spiral domain with respect to the origin.

The following theorems provide sufficient conditions of α-spirallikeness.

Theorem 2.1 *Let f be an analytic function from E into $\mathbf{C}$ and $\alpha \in \mathbf{R}, |\alpha| < < \frac{\pi}{2}$. If f satisfies the conditions:*

(i) $f(0) = 0, f(z) \neq 0$, for all $z \in E \setminus \{0\}$ and $f'(z) \neq 0$, for all $z \in E$,

(ii) For each $z \in E$, the inequality

$$(1) \qquad \left(a^2+b^2\right)\mathrm{Re}\left[e^{i\alpha}\frac{zf'(z)}{f(z)}\right] - \left(a^2-b^2\right)\mathrm{Re}\left[e^{i\alpha}\frac{\bar{z}f'(z)}{f(z)}\right] > 0$$

holds, then f is an α-spiral function in E.

Proof. Let h be the function from U into E given by

$$(2) \qquad h(z) = \frac{a+b}{2}z + \frac{a-b}{2}\bar{z}.$$

Then $h \in C^1(U)$, h is injective in U and $h(U) = E$. We consider the function $g : U \to \mathbf{C}, \mathbf{g} = \mathbf{f} \circ \mathbf{h}$ and we shall prove that g satisfies the conditions of Theorem 1, when f satisfies the conditions $(i)-(ii)$ of Theorem 2. Hence g is an α-spiral function in U and since $f(E) = g(U)$ we obtain that f is α-spiral in E.

We have $g(z) = f\left(\frac{a+b}{2}z + \frac{a-b}{2}\bar{z}\right) \in C^1(U), g(0) = f(0)$ and $g(z) \neq 0$, for all $z \in U \setminus \{0\}$. We also have

$$Jg(z) = \left|\frac{\partial g}{\partial z}\right|^2 - \left|\frac{\partial g}{\partial \bar{z}}\right|^2 = ab\left|f'(u)\right|^2 > 0,$$

where $u = h(z) \in E$.

By using the definition of the operator D, we obtain

$$\frac{Dg(z)}{g(z)} = \frac{\left(\frac{a+b}{2}z - \frac{a-b}{2}\bar{z}\right) f'(u)}{f(u)}. \tag{3}$$

From $u = h(z) = \frac{a+b}{2}z + \frac{a-b}{2}\bar{z}$ and $\bar{u} = \frac{a-b}{2}z + \frac{a+b}{2}\bar{z}$ it results

$$z = \frac{1}{2ab}\left[(a+b)u - (a-b)\bar{u}\right] \tag{4}$$

By replacing (4) in (3), we obtain that the inequality

$$\operatorname{Re}\left[e^{i\alpha}\frac{Dg(z)}{g(z)}\right] > 0, \qquad z \in U \setminus \{0\}$$

holds, when the following inequality

$$\left(a^2 + b^2\right)\operatorname{Re}\left[e^{i\alpha}\frac{uf'(u)}{f(u)}\right] - \left(a^2 - b^2\right)\operatorname{Re}\left[e^{i\alpha}\frac{\bar{u}f'(u)}{f(u)}\right] > 0, \qquad u \in E$$

is true.

Remark. For $a = b$, we have $E = U$ and we obtain the well known condition of α-spiralikeness for analytic functions in U.

Theorem 2.2 *Let $\alpha \in \mathbf{R}$, with $|\alpha| < \frac{\pi}{2}$. If the function $f : E \to \mathbf{C}$ is analytic in E and satisfies the following conditions:*

(i) $f(0) = 0, f(z) \neq 0$, for all $z \in E \setminus \{0\}$ and $f'(z) \neq 0$, for all $z \in E$,

(ii) For each $z \in E$

$$\left|\alpha + \arg\frac{zf'(z)}{f(z)}\right| < \arccos\frac{a^2 - b^2}{a^2 + b^2}, \tag{5}$$

then f is an α-spiral function in E.

Proof. In order to prove that f is an α-spiral function in E, we shall show that the inequality is (1) true. Since

$$-\left(a^2-b^2\right)\operatorname{Re}\left[e^{i\alpha}\frac{\bar{z}f'(z)}{f(z)}\right] \geq -\left(a^2-b^2\right)\left|e^{i\alpha}\frac{zf'(z)}{f(z)}\right|, \qquad z \in E$$

we obtain

$$\left(a^2+b^2\right)\operatorname{Re}\left[e^{i\alpha}\frac{zf'(z)}{f(z)}\right]-\left(a^2-b^2\right)\operatorname{Re}\left[e^{i\alpha}\frac{\bar{z}f'(z)}{f(z)}\right] \geq$$

$$\geq \left(a^2+b^2\right)\operatorname{Re}\left[e^{i\alpha}\frac{zf'(z)}{f(z)}\right]-\left(a^2-b^2\right)\left|e^{i\alpha}\frac{zf'(z)}{f(z)}\right| =$$

$$= \left(a^2+b^2\right)\left|e^{i\alpha}\frac{zf'(z)}{f(z)}\right|\left\{\frac{\mathrm{xRe}\left[e^{i\alpha}\frac{zf'(z)}{f(z)}\right]}{\left|e^{i\alpha}\frac{zf'(z)}{f(z)}\right|}-\frac{a^2-b^2}{a^2+b^2}\right\} =$$

$$\left(a^2+b^2\right)\left|e^{i\alpha}\frac{zf'(z)}{f(z)}\right|\left\{\cos\left[\arg e^{i\alpha}\frac{zf'(z)}{f(z)}\right]-\frac{a^2-b^2}{a^2+b^2}\right\} > 0.$$

Hence, f is an α-spiral function in E.

Remark 2.1 *If $\alpha = 0$ the results concerning starlike functions in an elliptical domain are obtained* [2].

References

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