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A UNIFIED FIXED POINT RESULT IN METRIC SPACES INVOLVING A TWO VARIABLE FUNCTION

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Abstract

In this paper[1] a unique fixed point theorem in complete metric spaces for a class of self mappings has been derived which satisfy certain inequality constraints involving a function of two variables. For particular choices of the function several fixed point theorems may be obtained.

1 Introduction

In existing literatures there have been a very large number of fixed point results for self-mappings satisfying various types of contractive inequalities. A detailed survey of these may be obtained in [1], [2] and [4]

In particular, fixed point results involving altering distances have been introduced in [3]. An altering distance is a mapping $\Phi : [0, \infty) \to [0, \infty)$ which satisfies

a) Φ is increasing and continuous, and

b) $\Phi(t) = 0$ if and only if $t = 0$.

Fixed points involving altering distances have also been studied in works like [5] and [6].

In this paper, we obtain a new fixed point result for self-mappings defined on complete metric spaces satisfying a contractive inequality which involves a function of two variables and acts on distances of two pair of points in a metric space. This function of two variables is an extension of the idea of altering distances [3].

We begin with the following definition.

Definition 1.1 [Condition – A] A function $\Psi : R^+ \times R^+ \to R^+$ is said to satisfy *Condition – A* if

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(i) Ψ is continuous,

(ii) Ψ is monotone increasing in both the arguments,

(iii) $\Psi(0,0)=0$ and $\Psi(\epsilon,0)=0$ implies $\epsilon=0$. (1.1)

Let $\Psi(\epsilon,\epsilon)=0$. Then $\Psi(\epsilon,0)\leq\Psi(\epsilon,\epsilon)=0$, or $\Psi(\epsilon,0)=0$, which implies $\epsilon=0$ (by (1.1)).

Therefore, $\Psi(\epsilon,\epsilon)=0$ implies $\epsilon=0$. (1.2)

Here R^+ is the set of all non-negative real numbers.

Examples of Ψ are:

(i) $\Psi(a,b)=(a^p+b^q)^k$,

(ii) $\Psi(a,b)=a^p\cdot b^q+a^k$,

where p, q and k are positive real numbers.

2 Fixed point results

Theorem 1 *Let $T:X\to X$ be a self-mapping from a complete metric space X to itself which satisfies the following inequality:*

$$\begin{aligned}&\Psi(d(Tx,Ty),d(x,Tx))+\Psi(d(y,Ty),d(y,T^2x))\\&\quad\leq c\Psi(d(x,y),d(x,Tx))+c'\Psi(d(y,Ty),d(y,Tx)),\end{aligned}\tag{2.1}$$

where $0<c<1$, $0<c'\leq 1$, $x,y\in X$ and Ψ satisfies condition-A (Definition 1.1). Then T has a unique fixed point.

Proof. For any $x_0\in X$, we construct the sequence $\{x_n\}$ by

$$x_n=Tx_{n-1}=T^nx_0,\quad n=1,2,... \tag{2.2}$$

Substituting $y=Tx$ in (2.1) we have

$$\begin{aligned}&\Psi(d(Tx,T^2x),d(x,Tx))+\Psi(d(Tx,T^2x),d(Tx,T^2x))\\&\quad\leq c\Psi(d(x,Tx),d(x,Tx))+c'\Psi((d(Tx,T^2x),d(Tx,Tx)).\end{aligned}\tag{2.3}$$

As $0<c'<1$ and Ψ satisfies condition (ii) of Definition 1.1,

$$\begin{aligned}c'\Psi(d(Tx,T^2x),0)&\leq c'\Psi(d(Tx,T^2x),d(Tx,T^2x))\\&<\Psi(d(Tx,T^2x),d(Tx,T^2x))\end{aligned}$$

and consequently, from (2.3),

$$\Psi(d(Tx,T^2x),d(x,Tx))\leq c\Psi(d(x,Tx),d(x,Tx)\leq\Psi(d(x,Tx),d(x,Tx)) \tag{2.4}$$

which implies

$$d(Tx,T^2x)\leq d(x,Tx). \tag{2.5}$$

Setting $x=x_{n-1}$, we have

$$0\leq d(x_{n+1},x_n)\leq d(x_n,x_{n-1}),\quad n=1,2,... \tag{2.6}$$

This shows that $\{d(x_n, x_{n+1})\}$ converges.

Let $d(x_n, x_{n+1}) = a$ (say). From (2.4), again setting $x = x_{n-1}$, we obtain

$$\Psi(d(x_n, x_{n+1}), d(x_{n-1}, x_n) \leq c\Psi(d(x_{n-1}, x_n), d(x_{n-1}, x_n). \tag{2.7}$$

Making $n \to \infty$ and by virtue of the fact that Ψ is continuous, we have $\Psi(a, a) \leq c\Psi(a, a)$, or $\Psi((a, a) = 0$ (as $0 < c < 1$), which implies that $a = 0$ (using (1.2)). Therefore

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.8}$$

We next show that $\{x_n\}$ is a Cauchy sequence. Otherwise, there exist $\epsilon > 0$ and corresponding subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that for $m(k) < n(k)$

$$d(x_{n(k)}, x_{m(k)}) \geq \epsilon \text{ and } d(x_{n(k)-1}, x_{m(k)}) < \epsilon. \tag{2.9}$$

Then we have

$$\epsilon \leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) < d(x_{n(k)}, x_{n(k)-1}) + \epsilon.$$

Making $k \to \infty$ and using (2.8)

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \epsilon. \tag{2.10}$$

Again,

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)})$$

and

$$d(x_{n(k)-1}, x_{m(k)-1}) \leq d(x_{n(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1}).$$

Making $k \to \infty$ and using (2.8) and (2.10), we obtain

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) = \epsilon. \tag{2.11}$$

Also

$$d(x_{m(k)-1}, x_{n(k)+1}) \leq d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1})$$

and

$$d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)}).$$

Making $k \to \infty$ and using (2.8) and (2.10), we obtain

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \epsilon. \tag{2.12}$$

Lastly,

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)})$$

and

$$d(x_{n(k)}, x_{m(k)-1}) \leq d(x_{n(k)}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1}).$$

Making $k \to \infty$ and using (2.8) and (2.10), we obtain

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)-1}) = \epsilon. \tag{2.13}$$

Now substituting $x = x_{n(k)-1}$ and $y = x_{m(k)-1}$ in (2.1) one has

$$\begin{aligned}\Psi(d(x_{n(k)}, x_{m(k)}), d(x_{n(k)-1}, x_{n(k)})) + \Psi(d(x_{m(k)-1}, x_{m(k)}), d(x_{m(k)-1}, x_{n(k)+1})\\ \leq c\Psi(d(x_{n(k)-1}, x_{m(k)-1}), d(x_{n(k)-1}, x_{n(k)}))\\ +c'\Psi(d(x_{m(k)-1}, x_{m(k)}), d(x_{m(k)-1}, x_{n(k)}).\end{aligned}$$

Making $k \to \infty$ in the above inequality, using (2.8) and (2.10)–(2.13) and using the fact that Ψ is continuous, we obtain,

$$\Psi(\epsilon, 0) + \Psi(0, \epsilon) \leq c\Psi(\epsilon, 0) + c'\Psi(0, \epsilon),$$

which implies $\Psi(\epsilon, 0) \leq c\Psi(\epsilon, 0)$ (as $0 < c' \leq 1$), and consequently $\Psi(\epsilon, 0) = 0$ (as $0 < c < 1$). So, using condition (iii) of Definition 1.1, we obtain $\epsilon = 0$, which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence and hence is convergent in the complete metric space X.

Let $x_n \to z$ (say) as $n \to \infty$. Again, putting $y = z$, $x = x_n$ in (2.1) we obtain

$$\begin{aligned}\Psi(d(x_{n+1}, Tz), d(x_n, x_{n+1})) + \Psi(d(z, Tz), d(z, x_{n+2}))\\ \leq c\Psi(d(x_n, z), d(x_n, x_{n+1})) + c'\Psi(d(z, Tz), d(z, x_{n+1})).\end{aligned}$$

Making $n \to \infty$, considering (2.8), $x_n \to z$, and using the continuity of Ψ we obtain

$$\Psi(d(z, Tz), 0) + \Psi(d(z, Tz), 0) \leq c\Psi(0, 0) + c'\Psi(d(z, Tz), 0),$$

which implies

$$\begin{aligned}\Psi(d(z, Tz), 0) &\leq c\Psi(0, 0) \quad (\text{ as } 0 < c' \leq 1)\\ &\leq c\Psi(d(z, Tz), 0) \quad (\text{ using condition (ii) of Definition 1.1 }).\end{aligned}$$

Consequently, $\Psi(d(z, Tz), 0) = 0$ (as $0 < c < 1$), so that $d(z, Tz) = 0$ (by (1.1)), which implies that $z = Tz$.

Next, we prove the uniqueness of the fixed point. If possible, let z_1 and z_2 be two fixed points of T. Then from (2.1) we obtain

$$\Psi(d(z_1,z_2),d(z_1,z_1)) + \Psi(d(z_2,z_2),d(z_2,z_1))$$

$$\leq c\Psi(d(z_1,z_2),d(z_1,z_1)) + c'\Psi(d(z_2,z_2),d(z_2,z_1)),$$

or

$$\Psi(d(z_1,z_2),0) + \Psi(0,d(z_2,z_1)) \leq c\Psi(d(z_1,z_2),0) + c'\Psi(0,d(z_2,z_1)),$$

or

$$\Psi(d(z_1,z_2),0) \leq c\Psi(d(z_1,z_2),0) \text{ (as } 0 < c' \leq 1),$$

which implies $\Psi(d(z_1,z_2),0) = 0$ and consequently $d(z_1,z_2) = 0$ (using condition (iii) of Definition 1.1), or $z_1 = z_2$.

This completes the proof of the theorem.

With different choices of Ψ it is possible to obtain different fixed point theorems. In particular, we have the following corollary.

Corollary 1 *Let $T : X \to X$ be a self-mapping from a complete metric space to itself and satisfy*

$$\begin{aligned}&[(d(Tx,Ty)^p + r(d(x,Tx))^q]^k + [(d(y,Ty))^p + r(d(y,T^2x))^q]^k\\ &\qquad \leq c[(d(x,y))^p + r(d(x,Tx))^q]^k + c'[(d(y,Ty))^p + r(d(y,Tx))^q]^k,\end{aligned}$$

where $x,y \in X$, $p,k > 0$, $r,q \geq 0$ and $0 < c < 1$, $0 < c' \leq 1$. Then T has a unique fixed point.

The proof of the corollary follows by the specific choice of the function Ψ as

$$\Psi(a,b) = (a^p + rb^q)^k, \quad p,k > 0, \quad r,q \geq 0.$$

It may be noted that for particular choice of $p = k = 1$, $r = q = 0$ and $c' = 1$, we obtain the Banach fixed point theorem in complete metric spaces [2].

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