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AN EXTENSION OF KELLEY'S CLOSED RELATION THEOREM TO RELATOR SPACES

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Abstract

In this paper[1] we prove a straightforward extension of the dual of Kelley's famous closed relation theorem to a pair of relations on one relator (generalized uniform) space to another.

In particular, we show that an almost uniformly lower semicontinuous closed relation on a topologically semisymmetric relator space to a complete metric type relator space is uniformly lower semicontinuous.

Introduction

The following theorem, proved in [13 , p. 202] , is usually called Kelley's closed relation theorem since it easily yields some natural extensions of Banach's closed graph theorem [44] .

Theorem 1. *Let* R *be a closed subset of the product of a complete pseudometric space* (X, d) *with the uniform space* $(Y, \mathcal{V})$ *and suppose that for each positive* r *there is* V *in* $\mathcal{V}$ *such that* $R\big[U_r[x]\big]^-$ *contains* $V[y]$ *for each* (x, y) *in* R*. Then for each* r *and each positive* e *it is true that*

$$R\big[U_{r+e}[x]\big] \supset R\big[U_r[x]\big]^- \supset V[y]\,.$$

This theorem has subsequently been generalized by several authors in various directions. See, for instance, Mah–Naimpally [18] and Wilhelm [45] .

However, the $F = G$ particular case of our following extension of the dual of Theorem 1 is certainly not included in the existing generalizations.

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Theorem 2. *If (F, G) is an almost uniformly lower semicontinuous closed pair of relations on a topologically semisymmetric relator space $X(\mathcal{R})$ to a sequentially convergence-adherence complete metric type relator space $Y(\mathcal{S})$ such that $G \subset F$, then*

$$\mathrm{cl}_{\mathcal{R}}\Big(G^{-1}\big(V(y)\big)\Big) \subset F^{-1}\Big(W\big(V(y)\big)\Big)$$

for all $V, W \in \mathcal{S}$ and $y \in Y$. Thus, in particular, the relation F and the pair (G, F) are uniformly lower semicontinuous.

The necessary prerequisites concerning relations and relators (relational systems), which are possibly unfamiliar to the reader, will be briefly laid out in the subsequent preparatory sections. Unfortunately, our present terminology and notation may differ from those of the earlier papers.

1. A few basic facts on relations and relators

A subset F of a product set $X \times Y$ is called a relation on X to Y. In particular, the relations $\Delta_X = \{(x, x): \ x \in X\}$ and $X^2 = X \times X$ are called the identity and the universal relations on X, respectively.

Namely, if in particular $X = Y$, then we may simply say that F is a relation on X. Note that if F is a relation on X to Y, then F is also a relation on $X \cup Y$. Therefore, it is frequently not a severe restriction to assume that $X = Y$.

If F is a relation on X to Y, and moreover $x \in X$ and $A \subset X$, then the sets $F(x) = \{y \in Y: \ (x, y) \in F\}$ and $F[A] = \bigcup_{x \in A} F(x)$ are called the images of x and A under F, respectively. Whenever $A \in X$ seems unlikely, we may write $F(A)$ in place of $F[A]$.

If F is a relation on X to Y, then the sets $D_F = \{x \in X: \ F(x) \neq \emptyset\}$ and $R_F = F(D_F)$ are called the domain and the range of F, respectively. Whenever, $X = D_F$ (and $Y = R_F$), we say that F is a relation of X into (onto) Y.

A relation F on X to Y is said to be a function if for each $x \in D_F$ there exists a unique $y \in Y$ such that $y \in F(x)$. In this case, by identifying singletons with their elements, we usually write $F(x) = y$ in place of $F(x) = \{y\}$.

If F is a relation on X to Y, then the values $F(x)$, where $x \in X$, uniquely determine F since we have $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the inverse F^{-1} can be defined such that $F^{-1}(y) = \{x \in X: \ y \in F(x)\}$ for all $y \in Y$.

Moreover, if F is a relation on X to Y, and G is a relation on Z to W, then the composition $G \circ F$ and the box product $F \boxtimes G$ can be defined such that $(G \circ F)(x) = G\big(F(x)\big)$ and $\big(F \boxtimes G\big)(x, z) = F(x) \times G(z)$ for all $x \in X$ and $z \in Z$.

A relation R on X is called reflexive, symmetric, transitive, and directive if $\Delta_X \subset R$, $R \subset R^{-1}$, $R \circ R \subset R$, and $X^2 \subset R^{-1} \circ R$, respectively. Moreover, a reflexive relation is called a preorder (tolerance) if it is transitive (symmetric), and a directive preorder is called a direction.

If R is a relation on X, then we write $R^n = R \circ R^{n-1}$ for all $n \in \mathbb{N}$ by agreeing that $R^0 = \Delta_X$. Moreover, we also write $R^\infty = \bigcup_{n=0}^{\infty} R^n$. Note that thus R^∞ is the smallest preorder on X such that $R \subset R^\infty$.

A nonvoid family $\mathcal{R}$ of relations on a nonvoid set X is called a *relator* on X, and the ordered pair $X(\mathcal{R}) = (X, \mathcal{R})$ is called a *relator space*. Relator spaces are straightforward generalizations of ordered sets and uniform spaces [30]. They are mainly motivated by the following two facts.

If $\mathcal{D}$ is a nonvoid family of certain distance functions on X, then the relator $\mathcal{R}_\mathcal{D}$ consisting of all surroundings $B_\varepsilon^d = \{(x, y) \in X^2 : \ d(x, y) < \varepsilon\}$, where $d \in \mathcal{D}$ and $\varepsilon > 0$, is a more convenient mean of defining the basic notions of analysis in the space $X(\mathcal{D})$, than the family of all open subsets of $X(\mathcal{D})$, or even the the family $\mathcal{D}$ itself.

Moreover, all reasonable generalizations of the usual topological structures (such as proximities, closures, topologies, filters and convergences, for instance) can be easily derived from relators (according to the results of [36] and [29]), and thus they need not be studied separately.

For instance, if $\mathcal{A}$ is a certain generalized topology or a stack (ascending system) in X, then $\mathcal{A}$ can easily be derived (according to the forthcoming definitions of the families $\tau_\mathcal{R}$, $\mathcal{T}_\mathcal{R}$ and $\mathcal{E}_\mathcal{R}$) from the Davis–Pervin relator $\mathcal{R}_\mathcal{A}$ consisting of all preorders $V_A = A^2 \cup (X \setminus A) \times X$, where $A \in \mathcal{A}$. Note that, in contrast to these preorders, the surroundings B_ε^d are usually tolerances on X.

2. Structures derived from relators and operations on relators

If $\mathcal{R}$ is a relator on X, then for any $A, B \subset X$ and $x, y \in X$ we write:

(1) $B \in \operatorname{Int}_\mathcal{R}(A)\ \big(B \in \operatorname{Cl}_\mathcal{R}(A)\big)$ if $R(B) \subset A\ \big(R(B) \cap A \neq \emptyset\big)$ for some (all) $R \in \mathcal{R}$;

(2) $x \in \operatorname{int}_\mathcal{R}(A)\ \big(x \in \operatorname{cl}_\mathcal{R}(A)\big)$ if $\{x\} \in \operatorname{Int}_\mathcal{R}(A)\ \big(\{x\} \in \operatorname{Cl}_\mathcal{R}(A)\big)$;

(3) $y \in \sigma_\mathcal{R}(x)\ \ \big(y \in \rho_\mathcal{R}(x)\big)$ if $y \in \operatorname{int}_\mathcal{R}\big(\{x\}\big)\ \ \big(y \in \operatorname{cl}_\mathcal{R}\big(\{x\}\big)$;

and moreover

(4) $A \in \tau_\mathcal{R}\ \ \big(A \in \tau\!\!\!-_\mathcal{R}\big)$ if $A \in \operatorname{Int}_\mathcal{R}(A)\ \ \big(X \setminus A \notin \operatorname{Cl}_\mathcal{R}(A)\big)$;

(5) $A \in \mathcal{T}_\mathcal{R}\ \ \big(A \in \mathcal{F}_\mathcal{R}\big)$ if $A \subset \operatorname{int}_\mathcal{R}(A)\ \ \big(\operatorname{cl}_\mathcal{R}(A) \subset A\big)$;

(6) $A \in \mathcal{E}_\mathcal{R}\ \ \big(A \in \mathcal{D}_\mathcal{R}\big)$ if $\operatorname{int}_\mathcal{R}(A) \neq \emptyset\ \ \big(\operatorname{cl}_\mathcal{R}(A) = X\big)$.

The relations $\mathrm{Int}_{\mathcal{R}}$, $\mathrm{int}_{\mathcal{R}}$ and $\sigma_{\mathcal{R}}$ are called the proximal, the topological and the infinitesimal interiors induced by $\mathcal{R}$ on X, respectively. While, the members of the families $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$ are called the proximally open, the topologically open and the fat subsets of $X(\mathcal{R})$, respectively.

The interiors and the fat sets are frequently more important tools, then the open sets. For instance, if $\prec$ is a preorder on X, then $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{E}_{\prec}$ are precisely the families of all ascending and residual subsets of the preordered set $X(\prec)$, respectively.

A function x of a preordered set Γ into a set X is called a Γ-net in X. The Γ-net x is said to be residually (cofinally) in a subset A of X if $x^{-1}(A)$ is a residual (cofinal) subset of Γ. Note that these definitions would actually allow Γ to be an arbitrary relator space.

Moreover, if $\mathcal{R}$ is a relator on X, then for any Γ-nets x and y in X and $a \in X$ we write:

(7) $y \in \mathrm{Lim}_{\mathcal{R}}(x)$ $\big(y \in \mathrm{Adh}_{\mathcal{R}}(x) \big)$ if the net (y, x) is residually (cofinally) in each $R \in \mathcal{R}$;

(8) $a \in \mathrm{lim}_{\mathcal{R}}(x)$ $\big(a \in \mathrm{adh}_{\mathcal{R}}(x) \big)$ if $a_{\Gamma} \in \mathrm{Lim}_{\mathcal{R}}(x)$ $\big(a_{\Gamma} \in \mathrm{Adh}_{\mathcal{R}}(x) \big)$, where $a_{\Gamma} = \Gamma \times \{a\}$.

If $\mathcal{R}$ is a relator on X, then the relators

$$
\begin{aligned}
\mathcal{R}^{*} &= \big\{ S \subset X^2 : \quad \exists\ R \in \mathcal{R} : \quad R \subset S \big\}, \\
\mathcal{R}^{\#} &= \big\{ S \subset X^2 : \quad \forall\ A \subset X : \quad A \in \mathrm{Int}_{\mathcal{R}}\big(S(A) \big) \big\}, \\
\mathcal{R}^{\wedge} &= \big\{ S \subset X^2 : \quad \forall\ x \in X : \quad x \in \mathrm{int}_{\mathcal{R}}\big(S(x) \big) \big\}, \\
\mathcal{R}^{\triangle} &= \big\{ S \subset X^2 : \quad \forall\ x \in X : \quad S(x) \in \mathcal{E}_{\mathcal{R}} \big\}
\end{aligned}
$$

are called the uniform, the proximal, the topological and the paratopological refinements of $\mathcal{R}$, respectively.

Moreover, if $\mathcal{R}$ is a relator on X, then the relators

$$
\mathcal{R}^{-1} = \big\{ R^{-1} : \quad R \in \mathcal{R} \big\} \qquad \text{and} \qquad \mathcal{R}^{\infty} = \big\{ R^{\infty} : \quad R \in \mathcal{R} \big\}
$$

are called the inverse and the preorder modification of $\mathcal{R}$, respectively. While, if $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then the relators

$$
R \wedge \mathcal{S} = \big\{ R \cap S : \quad R \in \mathcal{R},\, S \in \mathcal{S} \big\} \ \text{ and } \ \mathcal{R} \square \mathcal{S} = \big\{ R \square S : \quad R \in \mathcal{R},\, S \in \mathcal{S} \big\},
$$

where $\square = \circ$ $(\square = \boxtimes)$, are called the meet and the composition (the box product) of $\mathcal{R}$ and $\mathcal{S}$, respectively.

The importance of the topological refinement of relators lies mainly in the next two theorems proved in [30] and [37].

Theorem 2.1. *If $\mathcal{R}$ is a relator on X, then $\mathcal{R}^{\wedge}$ is the largest relator on X such that* $\operatorname{int}_{\mathcal{R}^{\wedge}} = \operatorname{int}_{\mathcal{R}}$ $\big(\operatorname{cl}_{\mathcal{R}^{\wedge}} = \operatorname{cl}_{\mathcal{R}} \big)$, *resp.* $\lim_{\mathcal{R}^{\wedge}} = \lim_{\mathcal{R}}$ $\big(\operatorname{adh}_{\mathcal{R}^{\wedge}} = \operatorname{adh}_{\mathcal{R}} \big)$.

Therefore, two relators $\mathcal{R}$ and $\mathcal{S}$ on X are said to be *topologically equivalent* if $\mathcal{R}^{\wedge} = \mathcal{S}^{\wedge}$. In particular, a relator is called *topologically simple* if it is topologically equivalent to a singleton relator. Moreover, the relator $\mathcal{R}$ is called *topologically fine* if $\mathcal{R}^{\wedge} = \mathcal{R}$.

Theorem 2.2. *If $\mathcal{R}$ is a relator on X, then $\mathcal{R}^{\wedge}$ is the largest relator on X such that for each $A, B \subset X$ we have $B \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(A)$ $\big(B \in \operatorname{Cl}_{\mathcal{R}^{\wedge}}(A) \big)$ if and only if $B \subset \operatorname{int}_{\mathcal{R}}(A)$ $\big(B \cap \operatorname{cl}_{\mathcal{R}}(A) \neq \emptyset \big)$.*

By using this theorem, we can at once see that $\tau_{\mathcal{R}^{\wedge}} = \mathcal{T}_{\mathcal{R}}$ $\big(\digamma_{\mathcal{R}^{\wedge}} = \mathcal{F}_{\mathcal{R}} \big)$. Moreover, we can easily prove that $\mathcal{R}^{\wedge\,-1\,\wedge} = \{ \rho_{\mathcal{R}} \}^{\wedge}$, and thus the relator $\mathcal{R}^{\vee} = \mathcal{R}^{\wedge\,-1}$ is always topologically simple.

The importance of the preorder modification of relators lies mainly in the following theorem proved in [20].

Theorem 2.3. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then the assertions $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{S}}$, $\mathcal{F}_{\mathcal{R}} = \mathcal{F}_{\mathcal{S}}$, $\mathcal{R}^{\wedge\infty} = \mathcal{S}^{\wedge\infty}$ and $\mathcal{R}^{\wedge\infty\wedge} = \mathcal{S}^{\wedge\infty\wedge}$ are equivalent.*

Namely, in contrast to Theorem 2.1, we can only state that $\mathcal{R}^{\wedge\infty}$ is the largest preorder relator on X such that $\mathcal{T}_{\mathcal{R}^{\wedge\infty}} = \mathcal{T}_{\mathcal{R}}$ $\big(\mathcal{F}_{\mathcal{R}^{\wedge\infty}} = \mathcal{F}_{\mathcal{R}} \big)$.

3. Metric type relators

A relator space $X(\mathcal{R})$ is usually said to have a property P if the relator $\mathcal{R}$ has the property P. Among the various useful properties of relators, we shall only need here some uniform and topological type ones.

Definition 3.1. A relator $\mathcal{R}$ on X will be called *uniformly countable* if there exists a sequence $(R_n)_{n=1}^{\infty}$ of relations on X such that

$$\mathcal{R}^{*} = \Big(\{ R_n \}_{n=1}^{\infty} \Big)^{*}.$$

Moreover, a uniformly countable relator $\mathcal{R}$ on X will be said to be of *metric type* if it is uniformly symmetric, transitive and filtered in the sense that

$$\mathcal{R}^{-1} \subset \mathcal{R}^{*}, \qquad \mathcal{R} \subset (\mathcal{R} \circ \mathcal{R})^{*}, \qquad \mathcal{R} \subset (\mathcal{R} \wedge \mathcal{R})^{*}.$$

Remark 3.2. Quite similarly, a uniformly countable, transitive and filtered relator may be called quasi-metric type.

The above terminology is mainly motivated by the following obvious

Example 3.3. If d is a function of X^2 to $[0, +\infty]$, then the relator $\mathcal{R}_d$ consisting of all surroundings B_ε^d, where $\varepsilon > 0$, is uniformly countable and filtered.

Moreover, if in addition the distance function d is symmetric and satisfies the triangle inequality, then the relator $\mathcal{R}_d$ is of metric type.

Simple applications of the corresponding definitions immediately yield the following

Proposition 3.4. *If $\mathcal{R}$ is a relator on X, then the following assertions hold:*

(1) $\mathcal{R}$ is uniformly symmetric if and only if $\mathcal{R}^*$ is properly symmetric in the sense that $\left(\mathcal{R}^*\right)^{-1} \subset \mathcal{R}^*$;

(2) $\mathcal{R}$ is uniformly filtered if and only if $\mathcal{R}^*$ is properly filtered in the sense that $\mathcal{R}^* \wedge \mathcal{R}^* \subset \mathcal{R}^*$;

(3) if $\mathcal{R}$ is a uniformly transitive and filtered, then for each $R \in \mathcal{R}^*$ there exists an $S \in \mathcal{R}$ such that $S \subset R$ and $S \circ S \subset R$.

Hint. To prove the assertion (3), note that if $R \in \mathcal{R}^*$, then there exists an $R_1 \in \mathcal{R}$ such that $R_1 \subset R$. Moreover, if $\mathcal{R}$ is uniformly transitive, then there exist $R_2, R_3 \in \mathcal{R}$ such that $R_2 \circ R_3 \subset R_1$. Furthermore, if $\mathcal{R}$ is uniformly filtered, then there exists an $S \in \mathcal{R}$ such that $S \subset R_1 \cap R_2 \cap R_3$. Hence, it is clear that S has the required properties.

Remark 3.5. Note that in the assertions (1) and (2) actually the equalities are also true.

Now, by using Proposition 3.4, we can also easily prove the following

Theorem 3.6. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is of metric type;

(2) there exists a decreasing sequence $\left(R_n\right)_{n=1}^\infty$ of symmetric relations on X such that $R_{n+1}^2 \subset R_n$ for all $n \in \mathbb{N}$ and $\mathcal{R}^* = \left(\left\{ R_n \right\}_{n=1}^\infty \right)^*$.

Proof. If the assertion (1) holds, then there exists a sequence $\left(V_n\right)_{n=1}^\infty$ of relations on X such that

$$\mathcal{R}^* = \left(\left\{ V_n \right\}_{n=1}^\infty \right)^*.$$

Define $R_1 = V_1 \cap V_1^{-1}$. Then, it is clear that R_1 is a symmetric relation on X such that $R_1 \subset V_1$. Moreover, from the assertions (1) and (2) of Proposition 3.4, we can see that $R_1 \in \mathcal{R}^*$. Therefore, by the assertion

(3) of Proposition 3.4, there exists an $S_1 \in \mathcal{R}$ such that $S_1 \subset R_1$ and $S_1 \circ S_1 \subset R_1$. Define

$$R_2 = (S_1 \cap V_2) \cap (S_1 \cap V_2)^{-1}.$$

Then, it is clear that R_2 is a symmetric relation on X such that $R_2 \subset R_1$, $R_2 \circ R_2 \subset R_1$ and $R_2 \subset V_2$. Moreover, from the assertions (1) and (2) of Proposition 3.4, we can see that $R_2 \in \mathcal{R}^*$.

Hence, by induction, it is clear that there exists a decreasing sequence $\big(R_n\big)_{n=1}^{\infty}$ of symmetric relations on X such that $R_{n+1}^2 \subset R_n$,

$$R_n \subset V_n \qquad \text{and} \qquad R_n \in \mathcal{R}^*$$

for all $n \in \mathbb{N}$. Therefore, we have

$$\mathcal{R}^* = \Big(\big\{ V_n \big\}_{n=1}^{\infty} \Big)^* \subset \Big(\big\{ R_n \big\}_{n=1}^{\infty} \Big)^* \subset \mathcal{R}^{**} = \mathcal{R}^*,$$

and thus the assertion (2) also holds.

The proof of the converse implication $(2) \Longrightarrow (1)$ is even more obvious.

Remark 3.7. Note that if $\big(R_n\big)_{n=1}^{\infty}$ is as in the assertion (2) of Theorem 3.6, then by defining $S_n = R_{2n-1}$ for all $n \in \mathbb{N}$ we can get a decreasing sequence $\big(S_n\big)_{n=1}^{\infty}$ of symmetric relations on X such that $S_{n+1}^3 \subset S_n$ for all $n \in \mathbb{N}$ and $\mathcal{R}^* = \Big(\big\{ S_n \big\}_{n=1}^{\infty} \Big)^*$.

Therefore, by using a standard construction of pseudo-metrics [13, p. 185], we could prove a certain converse of the second assertion of Example 3.3. However, this converse would be of no particular importance for us now since the metric type relators are usually more convenient means than the corresponding generalized metrics.

4. Topological relators

Definition 4.1. If $\mathcal{R}$ is a relator on X , then we say that:

(1) $\mathcal{R}$ is reflexive if $\Delta_X \subset \bigcap \mathcal{R}$;
(2) $\mathcal{R}$ is topologically symmetric if $\mathcal{R}^{\wedge -1} \subset \mathcal{R}^{\wedge}$;
(3) $\mathcal{R}$ is topologically semisymmetric if $\mathcal{R}^{-1} \subset \mathcal{R}^{\wedge}$;
(4) $\mathcal{R}$ is topologically transitive if $\mathcal{R} \subset \big(\mathcal{R}^{\wedge} \circ \mathcal{R}\big)^{\wedge}$;
(5) $\mathcal{R}$ is uniformly topologically transitive if $\mathcal{R} \subset \big(\mathcal{R} \circ \mathcal{R}\big)^{\wedge}$;
(6) $\mathcal{R}$ is topologically regular if $\mathcal{R} \subset \big(\mathcal{R}^{\wedge -1} \circ \mathcal{R}\big)^{\wedge}$;
(7) $\mathcal{R}$ is topologically filtered if $\mathcal{R} \subset \big(\mathcal{R} \wedge \mathcal{R}\big)^{\wedge}$.

Moreover, we say that a relator is *topological (uniformly topological)* if it is reflexive and topologically (uniformly topologically) transitive.

To let the reader feel the appropriateness of the above definitions, we shall only mention here the following theorems which were proved in [34].

Theorem 4.2. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is reflexive;*
(2) $B \in \operatorname{Int}_{\mathcal{R}}(A)$ *implies* $B \subset A$ *for all* $A, B \subset X$;
(3) $B \cap A \neq \emptyset$ *implies* $B \in \operatorname{Cl}_{\mathcal{R}}(A)$ *for all* $A, B \subset X$;
(4) $\operatorname{int}_{\mathcal{R}}(A) \subset A$ $\big(A \subset \operatorname{cl}_{\mathcal{R}}(A) \big)$ *for all* $A \subset X$;
(5) $\rho_{\mathcal{R}}$ *is reflexive.*

Remark 4.3. To prove the implication $(5) \Longrightarrow (1)$, it is convenient to note that

$$\operatorname{cl}_{\mathcal{R}}(A) = \bigcap_{R \in \mathcal{R}} R^{-1}(A)$$

for all $A \subset X$, and thus in particular $\rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1} = \big(\bigcap \mathcal{R} \big)^{-1}$.

Theorem 4.4. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is topologically transitive;*
(2) $x \in \operatorname{int}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}\big(R(x)\big)\big)$ *for all* $R \in \mathcal{R}$ *and* $x \in X$;
(3) $\operatorname{int}_{\mathcal{R}}\big(R(x)\big) \in \mathcal{T}_{\mathcal{R}}$ *for all* $R \in \mathcal{R}$ *and* $x \in X$;
(4) $\operatorname{int}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}$ $\big(\operatorname{cl}_{\mathcal{R}}(A) \in \mathcal{F}_{\mathcal{R}} \big)$ *for all* $A \subset X$;
(5) $\mathcal{R}^{\wedge}$ *is uniformly topologically transitive.*

Remark 4.5. If $\mathcal{R}$ is topologically transitive, then we can prove that the relator $\mathcal{S} = \mathcal{R}^{\wedge}$ is actually strictly proximally transitive in the sense that $\mathcal{S} \subset (\mathcal{S} \bullet \mathcal{S})^{\#}$, where $\mathcal{S} \bullet \mathcal{S} = \{ S \circ S : \ S \in \mathcal{S} \}$.

Theorem 4.6. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is topological;*
(2) $\operatorname{int}_{\mathcal{R}}(A) = \bigcup \big\{ V \in \mathcal{T}_{\mathcal{R}} : \ V \subset A \big\}$ *for all* $A \subset X$;
(3) $\operatorname{cl}_{\mathcal{R}}(A) = \bigcap \big\{ W \in \mathcal{F}_{\mathcal{R}} : \ A \subset W \big\}$ *for all* $A \subset X$;
(4) $\mathcal{R}$ *is topologicaly equivalent to* $\mathcal{R}_{\mathcal{T}_{\mathcal{R}}}$ $\big(\mathcal{R}^{\wedge\infty} \big)$;
(5) $\mathcal{R}$ *is topologicaly equivalent to a preorder relator on* X.

Because of Theorem 4.4, in addition to Definition 4.1, we may also have

Definition 4.7. If $\mathcal{R}$ is a relator on X, then we say that:

(1) $\mathcal{R}$ is weakly topologically transitive if $\rho_{\mathcal{R}}(x) \in \mathcal{F}_{\mathcal{R}}$ for all $x \in X$;
(2) $\mathcal{R}$ is strongly topologically transitive if $R(x) \in \mathcal{T}_{\mathcal{R}}$ for all $R \in \mathcal{R}$ and $x \in X$.

Moreover, we say that a relator is *weakly (strongly) topological* if it is reflexive and weakly (strongly) topologically transitive.

Namely, in addition to Theorem 4.6, we can also prove

Theorem 4.8. *If $\mathcal{R}$ is a relator on X and R° is a relation on X for each $R \in \mathcal{R}$ such that*

$$R^\circ(x) = \operatorname{int}_{\mathcal{R}}\big(R(x)\big) \qquad \big(x \in X\big),$$

then $\mathcal{R}^\circ = \big\{R^\circ : \quad R \in \mathcal{R}\big\}$ is a strongly topological relator on X such that $\mathcal{R}$ and $\mathcal{R}^\circ$ are topologically equivalent if and only if $\mathcal{R}$ is topological.

Unfortunately, concerning the pointwise closures of relators we can only prove

Theorem 4.9. *If $\mathcal{R}$ is a reflexive, topologically regular relator on X and R^- is a relation on X for each $R \in \mathcal{R}$ such that*

$$R^-(x) = \operatorname{cl}_{\mathcal{R}}\big(R(x)\big) \qquad \big(x \in X\big),$$

then $\mathcal{R}^- = \big\{R^- : \quad R \in \mathcal{R}\big\}$ is a reflexive relator on X such that $\mathcal{R}$ and $\mathcal{R}^-$ are topologically equivalent.

Proof. Since $\mathcal{R}$ is reflexive, by Theorem 4.2, it is clear that $R \subset R^-$ for all $R \in \mathcal{R}$. Thus, in particular $\mathcal{R} \subset \mathcal{R}^* \subset \mathcal{R}^\wedge$, and hence $\mathcal{R}^\wedge \subset \mathcal{R}^{\wedge\wedge} = \mathcal{R}^\wedge$.

Moreover, since $\mathcal{R}$ is topologically regular for each $R \in \mathcal{R}$ and $x \in X$ there exist $U \in \mathcal{R}$ and $V \in \mathcal{R}^\wedge$ such that $V^{-1}\big(U(x)\big) \subset R(x)$. Hence, by Theorem 2.1 and Remark 4.3, it is clear that

$$U^-(x) = \operatorname{cl}_{\mathcal{R}}\big(U(x)\big) = \operatorname{cl}_{\mathcal{R}^\wedge}\big(U(x)\big) = \bigcap_{V \in \mathcal{R}^\wedge} V^{-1}\big(U(x)\big) \subset R(x).$$

Therefore, $\mathcal{R} \subset \mathcal{R}^{-\wedge}$, and hence $\mathcal{R}^\wedge \subset \mathcal{R}^{-\wedge\wedge} = \mathcal{R}^{-\wedge}$ is also true.

Remark 4.10. If in particular $\mathcal{R}$ is topologically semisymmetric and uniformly topological, then we can prove that $\mathcal{R}^-$ is also uniformly topological.

5. Complete relators

The following natural definition of Cauchy nets has been established in [38].

Definition 5.1. A net x in a relator space $X(\mathcal{R})$ is called *convergent (adherent)* if $\lim_{\mathcal{R}}(x) \neq \emptyset$ $\big(\operatorname{adh}_{\mathcal{R}}(x) \neq \emptyset\big)$.

In particular, a net x in a relator space $X(\mathcal{R})$ is called *convergence (adherence) Cauchy* if it is convergent (adherent) in each of the spaces $X(R)$, where $R \in \mathcal{R}$.

Remark 5.2. By the corresponding definitions, it is clear that

$$\lim_{\mathcal{R}}(x) = \bigcap_{R \in \mathcal{R}} \lim_R(x) \qquad \Big(\operatorname{adh}_{\mathcal{R}}(x) = \bigcap_{R \in \mathcal{R}} \operatorname{adh}_R(x) \Big).$$

Therefore, a convergent (adherent) net is, in particular, convergence (adherence) Cauchy.

Moreover, in [38] we have shown that a net x in a relator a space $X(\mathcal{R})$ is convergent (adherent) if and only if it is convergence (adherence) Cauchy in the space $X(\mathcal{R}^{\wedge})$. In contrast to this, note that the net x is convergence (adherence) Cauchy in the space $X(\mathcal{R}^{*})$ if and only if it is convergence (adherence) Cauchy in $X(\mathcal{R})$.

Now, by applying Definition 5.1, we may also have the following important

Definition 5.3. A relator $\mathcal{R}$ on X is called *convergence-adherence complete* if each convergence Cauchy net in $X(\mathcal{R})$ is adherent.

Moreover, the relator $\mathcal{R}$ is called *directedly (sequentially) convergence-adherence complete* if each directed convergence Cauchy net (convergence Cauchy sequence) in $X(\mathcal{R})$ is adherent.

Remark 5.4. In [38] we have shown that an adherent convergence Cauchy net in an uniformly topologically transitive, proximally symmetric and uniformly filtered relator space is convergent.

Therefore, a metric type relator $\mathcal{R}$ on X is, in particular, directedly (sequentially) convergence-adherence complete if and only if it is directedly (sequentially) convergence-convergence complete in the sense that each directed convergence Cauchy net (convergence Cauchy sequence) in $X(\mathcal{R})$ is convergent.

Concerning metric type relators, we can also prove the next useful

Theorem 5.5. *If $\mathcal{R}$ is a metric type relator on X, then the following assertions are equivalent;*

(1) $\mathcal{R}$ is sequentially convergence-adherence complete;

(2) $\mathcal{R}$ is directedly convergence-adherence complete.

Proof. Since $\mathcal{R}$ is of metric type, by Theorem 3.6, there exists a decreasing sequence $(R_n)_{n=1}^{\infty}$ of symmetric relations on X such that

$$R_{n+1}^2 \subset R_n \qquad (n \in \mathbb{N}) \qquad \text{and} \qquad \mathcal{R}^{*} = \Big(\{ R_n \}_{n=1}^{\infty} \Big)^{*}.$$

Therefore, if $(x_\alpha)_{\alpha \in \Gamma}$ is a convergence Cauchy net in $X(\mathcal{R})$, then for each $n \in \mathbb{N}$ there exist $u_n \in X$ and $\alpha_n \in \Gamma$ such that

$$x_\alpha \in R_n(u_n), \qquad \text{i.e.} \qquad u_n \in R_n(x_\alpha)$$

for all $\alpha \geq \alpha_n$. Hence, it is clear that for each $k \in \mathbb{N}$, $n \geq k+1$ and $\alpha \in \Gamma$, with $\alpha \geq \alpha_n$ and $\alpha \geq \alpha_{k+1}$, we have

$$u_n \in R_n(x_\alpha) \subset R_n\big(R_{k+1}(u_{k+1})\big) \subset R_{k+1}^2(u_{k+1}) \subset R_k(u_{k+1}).$$

Therefore, if Γ is directed, then the sequence $(u_n)_{n=1}^{\infty}$ is also convergence Cauchy in the space $X(\mathcal{R})$. Thus, if the assertion (1) holds, then there exists a $u \in X$ such that

$$u \in \operatorname{adh}_{\mathcal{R}} (u_n)_{n=1}^{\infty}.$$

This implies that for each $k \in \mathbb{N}$ there exists an $n \geq k+1$ such that

$$u_n \in R_{k+1}(u).$$

Therefore, for each $\alpha \geq \alpha_n$, we have

$$x_\alpha \in R_n(u_n) \subset R_n\big(R_{k+1}(u)\big) \subset R_{k+1}^2(u) \subset R_k(u).$$

Hence, since Γ is directed, it is clear that we also have

$$u \in \operatorname{adh}_{\mathcal{R}} (x_\alpha)_{\alpha \in \Gamma}.$$

Therefore, the assertion (2) also holds. Now, since the converse implication $(2) \Longrightarrow (1)$ is quite obvious, the proof is complete.

Remark 5.6. The most important directedly convergence-adherence complete relators are the topologically compact ones.

A relator $\mathcal{R}$ on X has been called *topologically compact* in [43] if its topological refinement $\mathcal{R}^{\wedge}$ is properly compact in the sense that for each $R \in \mathcal{R}^{\wedge}$ there exists a finite subset A of X such that $R(A) = X$.

6. Upper and lower semicontinuity of relations

The following definitions and theorems have been established in [27].

Definition 6.1. A pair (F, G) of relations on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$ is called *uniformly (topologically) upper semicontinuous* if

$$\mathcal{S}^* \circ F \subset \big(G \circ \mathcal{R}\big)^* \qquad \Big(\mathcal{S}^{\wedge} \circ F \subset \big(G \circ \mathcal{R}\big)^{\wedge} \Big).$$

Theorem 6.2. *If F and G are relations on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$, then the following assertions are equivalent:*

(1) (F, G) *is uniformly upper semicontinuous;*

(2) for each $S \in \mathcal{S}$ *there exists an* $R \in \mathcal{R}$ *such that for each* $x \in X$ *we have* $G\big(R(x)\big) \subset S\big(F(x)\big)$*;*

(3) for each $S \in \mathcal{S}$ *there exists an* $R \in \mathcal{R}$ *such that for each* $x \in X$ *and* $u \in R(x)$ *we have* $G(u) \subset S\big(F(x)\big)$.

Theorem 6.3. *If F and G are relations on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$, then the following assertions are equivalent:*

(1) (F, G) *is topologically upper semicontinuous;*

(2) for each $S \in \mathcal{S}^{\wedge}$ *and* $x \in X$ *there exists an* $R \in \mathcal{R}$ *such that* $G\big(R(x)\big) \subset S\big(F(x)\big)$*;*

(3) for each $S \in \mathcal{S}^{\wedge}$ *and* $x \in X$ *there exists an* $R \in \mathcal{R}$ *such that for each* $u \in R(x)$ *we have* $G(u) \subset S\big(F(x)\big)$.

Theorem 6.4. *If F and G are relations on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$, then the following assertions are equivalent:*

(1) (F, G) *is topologically upper semicontinuous;*

(2) $\operatorname{cl}_{\mathcal{R}}\big(G^{-1}(A)\big) \subset F^{-1}\big(\operatorname{cl}_{\mathcal{S}}(A)\big)$ *for all* $A \subset Y$.

Corollary 6.5. *If F and G are relations on an arbitrary relator space $X(\mathcal{R})$ to a topological relator space $Y(\mathcal{S})$, then the following assertions are equivalent:*

(1) (F, G) *is topologically upper semicontinuous;*

(2) $\operatorname{cl}_{\mathcal{S}}(B) \subset A$ *implies* $\operatorname{cl}_{\mathcal{R}}\big(G^{-1}(B)\big) \subset F^{-1}(A)$ *for all* $A, B \subset Y$*;*

(3) $A \in \mathcal{F}_{\mathcal{S}}$ *implies* $\operatorname{cl}_{\mathcal{R}}\big(G^{-1}(A)\big) \subset F^{-1}(A)$.

Remark 6.6. The implication $(1) \Longrightarrow (2)$ does not require the relator $\mathcal{S}$ to be topological.

Moreover, a pair (A, B) of subsets of a relator space $X(\mathcal{R})$ may be called closed if $\operatorname{cl}_{\mathcal{R}}(B) \subset A$.

Definition 6.7. A pair (F, G) of relations on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$ is called *uniformly (topologicaly) lower semicontinuous* if

$$G^{-1} \circ \mathcal{S} \subset \big(\mathcal{R}^{*} \circ F^{-1}\big)^{*} \qquad \Big(\, G^{-1} \circ \mathcal{S} \subset \big(\mathcal{R}^{\wedge} \circ F^{-1}\big)^{\wedge} \,\Big).$$

Theorem 6.8. *If F and G are relations on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$, then the following assertions are equivalent:*

(1) (F, G) *is uniformly lower semicontinuous;*

(2) for each $S \in \mathcal{S}$ *there exists an* $R \in \mathcal{R}$ *such that for each* $y \in Y$ *we have* $R\big(F^{-1}(y)\big) \subset G^{-1}\big(S(y)\big)$ *;*

(3) for each $S \in \mathcal{S}$ *there exists an* $R \in \mathcal{R}$ *such that for each* $x \in X$ *,* $y \in F(x)$ *and* $u \in R(x)$ *we have* $G(u) \cap S(y) \neq \emptyset$ *;*

Theorem 6.9. *If* F *and* G *are relations on one relator space* $X(\mathcal{R})$ *to another* $Y(\mathcal{S})$*, then the following assertions are equivalent:*

(1) (F, G) *is topologically lower semicontinuous;*

(2) for each $S \in \mathcal{S}$ *there exists an* $R \in \mathcal{R}^{\wedge}$ *such that for each* $y \in Y$ *we have* $R\big(F^{-1}(y)\big) \subset G^{-1}\big(S(y)\big)$ *;*

(3) for each $S \in \mathcal{S}$*,* $x \in X$ *and* $y \in F(x)$ *there exists an* $R \in \mathcal{R}$ *such that for each* $u \in R(x)$ *we have* $G(u) \cap S(y) \neq \emptyset$*.*

Theorem 6.10. *If* F *and* G *are relations on one relator space* $X(\mathcal{R})$ *to another* $Y(\mathcal{S})$*, then the following assertions are equivalent:*

(1) (F, G) *is topologically lower semicontinuous;*

(2) $F^{-1}\big(\operatorname{int}_{\mathcal{S}}(A)\big) \subset \operatorname{int}_{\mathcal{R}}\big(G^{-1}(A)\big)$ *for all* $A \subset Y$*.*

Corollary 6.11. *If* F *and* G *are relations on an arbitrary relator space* $X(\mathcal{R})$ *to a topological relator space* $Y(\mathcal{S})$*, then the following assertions are equivalent:*

(1) (F, G) *is topologically lower semicontinuous;*

(2) $A \subset \operatorname{int}_{\mathcal{S}}(B)$ *implies* $F^{-1}(A) \subset \operatorname{int}_{\mathcal{R}}\big(G^{-1}(B)\big)$ *for all* $A, B \subset Y$*;*

(3) $A \in \mathcal{T}_{\mathcal{S}}$ *implies* $F^{-1}(A) \subset \operatorname{int}_{\mathcal{R}}\big(G^{-1}(A)\big)$*.*

Remark 6.12. The implication $(1) \Longrightarrow (2)$ does not require the relator $\mathcal{S}$ to be topological.

Moreover, a pair (A, B) of subsets of a relator space $X(\mathcal{R})$ may be called open if $A \subset \operatorname{int}_{\mathcal{R}}(B)$.

7. Almost uniform and topological lower semicontinuity of relations

Analogously to Definition 6.7 and Theorems 6.8, 6.9 and 6.10, we may now also have the following definition and theorems.

Definition 7.1. A pair (F, G) of relations on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$ will be called *almost uniformly (topologically) lower semicontinuous* if

$$\big(G^{-1} \circ \mathcal{S}\big)^{-} \subset \big(\mathcal{R}^{*} \circ F^{-1}\big)^{*} \qquad \Big(\big(G^{-1} \circ \mathcal{S}\big)^{-} \subset \big(\mathcal{R}^{\wedge} \circ F^{-1}\big)^{\wedge}\Big).$$

Theorem 7.2. *If* F *and* G *are relations on one relator space* $X(\mathcal{R})$ *to another* $Y(\mathcal{S})$, *then the following assertions are equivalent:*

(1) (F, G) *is almost uniformly lower semicontinuous;*

(2) for each $S \in \mathcal{S}$ *there exists an* $R \in \mathcal{R}$ *such that for each* $y \in Y$ *we have* $R\big(F^{-1}(y)\big) \subset \mathrm{cl}_{\mathcal{R}}\big(G^{-1}\big(S(y)\big)\big)$;

(3) for each $S \in \mathcal{S}$ *there exists an* $R \in \mathcal{R}$ *such that for each* $x \in X$, $y \in F(x)$, $u \in R(x)$ *and* $U \in \mathcal{R}$ *there exists a* $v \in U(u)$ *such that* $G(v) \cap S(y) \neq \emptyset$;

Hint. If the assertion (1) holds, then for each $S \in \mathcal{S}$ there exists an $R \in \mathcal{R}$ such that $R \circ F^{-1} \subset \big(G^{-1} \circ V\big)^{-}$. Hence, it follows that

$$R\big(F^{-1}(y)\big) \subset \mathrm{cl}_{\mathcal{R}}\big(G^{-1}\big(S(y)\big)\big)$$

for all $y \in Y$. Therefore, if $x \in X$ and $y \in F(x)$, i.e., $x \in F^{-1}(y)$, then

$$R(x) \subset \mathrm{cl}_{\mathcal{R}}\big(G^{-1}\big(S(y)\big)\big) .$$

Thus, for each $u \in R(x)$, we have $u \in \mathrm{cl}_{\mathcal{R}}\big(G^{-1}\big(S(y)\big)\big)$. Hence, it follows that for each $U \in \mathcal{R}$ there exist $v \in U(u)$ such that $v \in G^{-1}\big(S(y)\big)$, i.e., $G(v) \cap S(y) \neq \emptyset$. Therefore, the assertion (3) also holds.

Theorem 7.3. *If* F *and* G *are relations on one relator space* $X(\mathcal{R})$ *to another* $Y(\mathcal{S})$, *then the following assertions are equivalent:*

(1) (F, G) *is almost topologically lower semicontinuous;*

(2) for each $S \in \mathcal{S}$ *there exists an* $R \in \mathcal{R}^{\wedge}$ *such that for each* $y \in Y$ *we have* $R\big(F^{-1}(y)\big) \subset \mathrm{cl}_{\mathcal{R}}\big(G^{-1}\big(S(y)\big)\big)$;

(3) for each $S \in \mathcal{S}$, $x \in X$ *and* $y \in F(x)$ *there exists an* $R \in \mathcal{R}$ *such that for each* $u \in R(x)$ *and* $U \in \mathcal{R}$ *there exists a* $v \in U(u)$ *such that* $G(v) \cap S(y) \neq \emptyset$.

Hint. If the assertion (3) holds, then for each $S \in \mathcal{S}$, $y \in Y$ and $x \in F^{-1}(y)$ there exists an $R_x \in \mathcal{R}$ such that for each $u \in R(x)$ and $U \in \mathcal{R}$ there exists a $v \in U(u)$ such that $G(v) \cap S(y) \neq \emptyset$. Hence, similarly as in the proof of Theorem 7.2, we can infer that $R_x(x) \subset \mathrm{cl}_{\mathcal{R}}\big(G^{-1}\big(S(y)\big)\big)$. Therefore,

$$F^{-1}(y) \subset \mathrm{int}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{R}}\big(G^{-1}\big(S(y)\big)\big)\big) .$$

Hence, by Theorem 2.2, it follows that

$$F^{-1}(y) \in \mathrm{Int}_{\mathcal{R}^{\wedge}}\big(\mathrm{cl}_{\mathcal{R}}\big(G^{-1}\big(S(y)\big)\big)\big) .$$

Therefore, there exists an $R \in \mathcal{R}^{\wedge}$ such that

$$\big(R \circ F^{-1} \big) (y) = R \big(F^{-1}(y) \big) \subset \operatorname{cl}_{\mathcal{R}} \Big(G^{-1} \big(S(y) \big) \Big) = \big(G^{-1} \circ S \big)^{-} (y) .$$

Therefore, $\big(G^{-1} \circ S \big)^{-} \in \big(\mathcal{R}^{\wedge} \circ F^{-1} \big)^{\wedge}$, and thus the assertion (1) also holds.

Theorem 7.4. *If F and G are relations on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$, then the following assertions are equivalent:*

(1) (F, G) is almost topologically lower semicontinuous;

(2) $F^{-1}\big(\operatorname{int}_{\mathcal{S}}(A) \big) \subset \operatorname{int}_{\mathcal{R}} \Big(\operatorname{cl}_{\mathcal{R}} \big(G^{-1}(A) \big) \Big)$ for all $A \subset Y$.

Hint. If $A \subset Y$ and $x \in F^{-1}\big(\operatorname{int}_{\mathcal{S}}(A) \big)$, then there exists a $y \in \operatorname{int}_{\mathcal{S}}(A)$ such that $x \in F^{-1}(y)$. Therefore, there exists an $S \in \mathcal{S}$ such that $S(y) \subset A$. Moreover, if the assertion (1) holds, then by Theorem 7.3 there exists an $R \in \mathcal{R}^{\wedge}$ such that $R\big(F^{-1}(y) \big) \subset \operatorname{cl}_{\mathcal{R}} \Big(G^{-1}\big(S(y) \big) \Big)$. Hence, it is clear that $R(x) \subset \operatorname{cl}_{\mathcal{R}}\big(G^{-1}(A) \big)$. Therefore, $x \in \operatorname{int}_{\mathcal{R}} \Big(\operatorname{cl}_{\mathcal{R}}\big(G^{-1}(A) \big) \Big)$, and thus the assertion (2) also holds.

Corollary 7.5. *If F and G are relations on an arbitrary relator space $X(\mathcal{R})$ to a topological relator space $Y(\mathcal{S})$, then the following assertions are equivalent:*

(1) (F, G) is almost topologically lower semicontinuous;

(2) $A \subset \operatorname{int}_{\mathcal{S}}(B)$ implies $F^{-1}(A) \subset \operatorname{int}_{\mathcal{R}} \Big(\operatorname{cl}_{\mathcal{R}}\big(G^{-1}(B) \big) \Big)$ for all $A, B \subset Y$;

(3) $A \in \mathcal{T}_{\mathcal{S}}$ implies $F^{-1}(A) \subset \operatorname{int}_{\mathcal{R}} \Big(\operatorname{cl}_{\mathcal{R}}\big(G^{-1}(A) \big) \Big)$.

Hint. If $A \subset Y$, then by Theorem 4.4 we have $\operatorname{int}_{\mathcal{S}}(A) \in \mathcal{T}_{\mathcal{S}}$. Therefore, if the assertion (2) holds, then

$$F^{-1}\big(\operatorname{int}_{\mathcal{S}}(A) \big) \subset \operatorname{int}_{\mathcal{R}} \Big(\operatorname{cl}_{\mathcal{R}} \Big(G^{-1}\big(\operatorname{int}_{\mathcal{S}}(A) \big) \Big) \Big) .$$

Moreover, by Theorem 4.2, we have $\operatorname{int}_{\mathcal{S}}(A) \subset A$. Therefore,

$$F^{-1}\big(\operatorname{int}_{\mathcal{S}}(A) \big) \subset \operatorname{int}_{\mathcal{R}} \Big(\operatorname{cl}_{\mathcal{R}}\big(G^{-1}(A) \big) \Big) .$$

Thus, by Theorem 7.4, the assertion (1) also holds.

Remark 7.6. Note that the implication (1) $\Longrightarrow$ (2) does not require the relator $\mathcal{S}$ to be topological.

Moreover, a pair (A, B) of subsets of a relator space $X(\mathcal{R})$ may be called almost open if $A \subset \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(B) \big)$.

8. Relationships between the closedness and semicontinuity properties of relations

The following definition and theorems have also been established in [27].

Definition 8.1. A pair (F, G) of relations on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$ is called *closed-valued* if $\big(F(x), G(x)\big)$ is a closed pair of subsets of $Y(S)$ for all $x \in X$.

Moreover, the pair (F, G) is called *closed* if (F, G) is a closed pair of subsets of the product space $X \times Y(\mathcal{R} \boxtimes \mathcal{S})$.

Remark 8.2. Recall that a pair (A, B) of subsets of a relator space $X(\mathcal{R})$ has been called closed if $\operatorname{cl}_{\mathcal{R}}(B) \subset A$.

Theorem 8.3. *If (F, G) is a closed pair of relations on a reflexive relator space $X(\mathcal{R})$ to an arbitrary relator space $Y(\mathcal{S})$, then (F, G) is closed-valued.*

Theorem 8.4. *If (F, G) is a closed pair of relations on a topologically filtered relator space $X(\mathcal{R})$ to a topologically compact relator space $Y(\mathcal{S})$, then (F, G) is topologically upper semicontinuous.*

Corollary 8.5. *If (F, G) is a closed pair of relations on a topologically compact relator space $X(\mathcal{R})$ to a topologically filtered relator space $Y(\mathcal{S})$, then $\big(F(A), G(B)\big)$ is a closed pair of subsets of $Y(\mathcal{S})$ whenever (A, B) is a closed pair of subsets of $X(\mathcal{R})$.*

Theorem 8.6. *If (F, G) is a closed-valued and topologically upper semicontinuous pair of relations on an arbitrary relator space $X(\mathcal{R})$ to a topologically regular relator space $Y(\mathcal{S})$ such that $F \subset G$, then (F, G) is closed.*

Theorem 8.7. *If (F, G) is a closed-valued and uniformly lower semicontinuous pair of relations on a topologically semisymmetric relator space $X(\mathcal{R})$ to a uniformly topologically transitive relator space $Y(\mathcal{S})$ such that $G \subset F$, then (F, G) is closed.*

Corollary 8.8. *If (F, G) is a closed-valued and uniformly lower semicontinuous pair of relations on a topologically semisymmetric and filtered relator space $X(\mathcal{R})$ to a uniformly topologically transitive and topologically compact relator space $Y(\mathcal{S})$ such that $G \subset F$, then (F, G) is topologically upper semicontinuous.*

Remark 8.9. Note that if (F, G) is a closed-valued pair of relations on an arbitrary relator space $X(\mathcal{R})$ to a reflexive relator space $Y(\mathcal{S})$, then $G(x) \subset \operatorname{cl}_{\mathcal{S}}\big(G(x)\big) \subset F(x)$ for all $x \in X$, and thus $G \subset F$.

In view of Corollary 8.8, it is worth mentioning that we also have

Theorem 8.10. *If $X(\mathcal{R})$ and $Y(\mathcal{S})$ are relator spaces and F and G are relations on X to Y, then following assertions are equivalent:*

(1) (F, G) *is a uniformly lower semicontinuous pair of relations on* $X(\mathcal{R})$ *to* $Y(\mathcal{S})$*;*

(2) (G, F) *is an uniformly upper semicontinuous pair of relations on* $X\big(\mathcal{R}^{-1}\big)$ *to* $Y\big(\mathcal{S}^{-1}\big)$*.*

Hint. Make use of the facts that $\mathcal{U}^{*\,-1} = \mathcal{U}^{-1\,*}$ and $\big(\mathcal{U} \circ \mathcal{V}\big)^{-1} = \mathcal{V}^{-1} \circ \mathcal{U}^{-1}$ hold for any relators $\mathcal{U}$ and $\mathcal{V}$.

Corollary 8.11. *If* F *and* G *are relations on one uniformly symmetric relator space* $X(\mathcal{R})$ *to another* $Y(\mathcal{S})$*, then the following assertions are equivalent:*

(1) (F, G) *is uniformly lower semicontinuous;*

(2) (G, F) *is uniformly upper semicontinuous.*

Hint. By the assumptions, we now have $\mathcal{R}^* = \mathcal{R}^{*\,-1}$ and $\mathcal{S}^* = \mathcal{S}^{*\,-1}$.

Remark 8.12. Unfortunately, the analogues of Theorem 8.10 and Corollary 8.11 fail to hold for the topological lower and upper semicontinuities.

Namely, according to [19, Theorem 6.5], the equality $\mathcal{R}^{\wedge -1} = \mathcal{R}^{-1\,\wedge}$ can hold true if and only if both $\mathcal{R}$ and $\mathcal{R}^{-1}$ are topologically simple.

9. An extension of Kelley's closed relation theorem

Now, having the necessary preparations, we are ready to prove the following generalization of the dual of Kelley's closed relation theorem.

Theorem 9.1. *If* (F, G) *is an almost uniformly lower semicontinuous closed pair of relations on a topologically semisymmetric relator space* $X(\mathcal{R})$ *to a sequentially convergence-adherence complete metric type relator space* $Y(\mathcal{S})$ *such that* $G \subset F$*, then*

$$\mathrm{cl}_{\mathcal{R}}\Big(G^{-1}\big(V(y)\big)\Big) \subset F^{-1}\Big(W\big(V(y)\big)\Big)$$

for all $V, W \in \mathcal{S}$ *and* $y \in Y$*.*

Proof. Let $V, W \in \mathcal{S}$ and $y_0 \in Y$, and assume that

$$x \in \mathrm{cl}_{\mathcal{R}}\Big(G^{-1}\big(V(y_0)\big)\Big).$$

Since $\mathcal{S}$ is of metric type, we can find a decreasing sequence $\big(S_n\big)_{n=1}^{\infty}$ of symmetric relations on Y such that

$$S_1^3 \subset W, \qquad S_{n+1}^2 \subset S_n \qquad (n \in \mathbb{N}), \qquad \mathcal{S}^* = \Big(\big\{S_n\big\}_{n=1}^{\infty}\Big)^*.$$

Moreover, from the almost uniform lower semicontinuity of F, it follows that there exists a sequence $\left(R_n\right)_{n=1}^{\infty}$ in $\mathcal{R}$ such that

$$R_n\left(F^{-1}(y)\right) \subset cl_{\mathcal{R}}\Big(G^{-1}\left(S_n(y)\right)\Big)$$

for all $n \in N$ and $y \in Y$.

Now, since

$$x \in \mathrm{cl}_{\mathcal{R}}\Big(G^{-1}\left(V(y_0)\right)\Big) \subset R_1\Big(G^{-1}\left(V(y_0)\right)\Big) \subset R_1\Big(F^{-1}\left(V(y_0)\right)\Big),$$

we can see that there exists an $y_1 \in V(y_0)$ such that

$$x \in R_1\left(F^{-1}(y_1)\right) \subset \mathrm{cl}_{\mathcal{R}}\Big(G^{-1}\left(S_1(y_1)\right)\Big).$$

Quite similarly, since

$$x \in \mathrm{cl}_{\mathcal{R}}\Big(G^{-1}\left(S_1(y_1)\right)\Big) \subset R_2\Big(G^{-1}\left(S_1(y_1)\right)\Big) \subset R_2\Big(F^{-1}\left(S_1(y_1)\right)\Big),$$

we can also see that there exists an $y_2 \in S_1(y_1)$ such that

$$x \in R_2\left(F^{-1}(y_2)\right) \subset \mathrm{cl}_{\mathcal{R}}\Big(G^{-1}\left(S_2(y_2)\right)\Big).$$

Hence, by induction, it is clear that there exists a sequence $(y_n)_{n=1}^{\infty}$ in Y such that

$$y_1 \in V(y_0), \qquad y_{n+1} \in S_n(y_n), \qquad x \in \mathrm{cl}_{\mathcal{R}}\Big(G^{-1}\left(S_n(y_n)\right)\Big)$$

for all $n \in N$.

Now, by noticing that for each $S \in \mathcal{S}$ there exists a $k > 1$ such that $S_{k-1} \subset S$, we can easily see that

$$\begin{aligned}
&y_{k+1} \in S_k(y_k) \subset S_{k-1}(y_k) \subset S(y_k), \\
&y_{k+2} \in S_{k+1}(y_{k+1}) \subset S_{k+1}\left(S_k(y_k)\right) \subset S_k^2(y_k) \subset S_{k-1}(y_k) \subset S(y_k), \\
&y_{k+3} \in S_{k+2}(y_{k+2}) \subset S_{k+2}\left(S_{k+1}\left(S_k(y_k)\right)\right) \subset S_{k+1}^2\left(S_k(y_k)\right) \subset \\
&\qquad\qquad \subset S_k^2(y_k) \subset S_{k-1}(y_k) \subset S(y_k).
\end{aligned}$$

Hence, it is clear that $y_n \in S(y_k)$ for all $n > k$, and thus

$$y_k \in \lim_{\{S\}}(y_n)_{n=1}^{\infty}.$$

Therefore, $(y_n)_{n=1}^{\infty}$ is a convergence Cauchy sequence in $Y(\mathcal{S})$. And thus, by the assumed completeness of $Y(\mathcal{S})$, there exists an $y \in Y$ such that

$$y \in \operatorname{adh}_{\mathcal{S}}(y_n)_{n=1}^{\infty}.$$

Now, by noticing that for each $S \in \mathcal{S}$ there exist $k > 1$ and $n > k$ such that

$$S_{k-1} \subset S \qquad \text{and} \qquad y_n \in S_k(y),$$

we can easily see that

$$S_n(y_n) \subset S_n\big(S_k(y)\big) \subset S_k^2(y) \subset S_{k-1}(y) \subset S(y),$$

and thus

$$x \in \operatorname{cl}_{\mathcal{R}}\Big(G^{-1}\big(S_n(y_n)\big)\Big) \subset \operatorname{cl}_{\mathcal{R}}\Big(G^{-1}\big(S(y)\big)\Big).$$

This implies that for each $R \in \mathcal{R}$ we have

$$R(x) \cap G^{-1}\big(S(y)\big) \cap \neq \emptyset,$$

and hence

$$\big(R(x) \times S(y)\big) \cap G \neq \emptyset.$$

Therefore, we also have

$$(x, y) \in \operatorname{cl}_{\mathcal{R} \boxtimes \mathcal{S}}(G) \subset F,$$

and hence $y \in F(x)$.

Now, by noticing that there exists a $n > 2$ such that $y_n \in S_1(y)$, we can easily see that

$$y \in S_1(y_n) \subset S_1\Big(S_{n-1}\big(\cdots S_1(y_1)\big)\Big) \subset S_1^3(y_1) \subset W(y_1).$$

Therefore, $y \in W\big(V(y_0)\big)$, and thus we also have

$$x \in F^{-1}(y) \subset F^{-1}\Big(W\big(V(y_0)\big)\Big).$$

Remark 9.2. Note that if (F, G) is a closed pair of relations on one reflexive relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$, then $G \subset \operatorname{cl}_{\mathcal{R} \boxtimes \mathcal{S}}(G) \subset F$.

10. Some useful consequences of the closed relation theorem

A simple application of Theorem 9.1 immediately yields the following

Theorem 10.1. *If* (F, G) *is an almost uniformly lower semicontinuous closed pair of relations on a topologically semisymmetric relator space* $X(\mathcal{R})$ *to a sequentially convergence-adherence complete metric type relator space* $Y(\mathcal{S})$ *such that* $G \subset F$, *then the relation* F *is uniformly lower semicontinuous.*

Proof. If $S \in \mathcal{S}$, then because of the uniform transitivity of S, there exist $V, W \in \mathcal{S}$ such that
$$W\big(V(y)\big) \subset S(y)$$
for all $y \in Y$. Moreover, because of the almost uniform lower semicontinuity of (F, G) there exists an $R \in \mathcal{R}$ such that
$$R\big(F^{-1}(y)\big) \subset cl_{\mathcal{R}}\Big(G^{-1}\big(V(y)\big)\Big)$$
for all $y \in Y$.

On the other hand, from Theorem 6.1 we know that
$$\mathrm{cl}_{\mathcal{R}}\Big(G^{-1}\big(V(y)\big)\Big) \subset F^{-1}\Big(W\big(V(y)\big)\Big)$$
for all $y \in Y$. Therefore, we also have
$$R\big(F^{-1}(y)\big) \subset F^{-1}\big(S(y)\big)$$
for all $y \in Y$. And thus the relation F is uniformly lower semicontinuous.

Remark 10.2. Note that if (F, G) is an almost uniformly lower semicontinuous pair of relations on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$ such that $G \subset F$, then both F and G are almost uniformly lower semicontinous.

Moreover, if (F, G) is a pair of relations on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$ such that $G \subset F$ and the relation F is uniformly lower semicontinuous, then the pair (G, F) is also uniformly lower semicontinuous.

The $F = G$ particular case of Theorem 10.1 at once gives the next practically important

Corollary 10.3. *If* F *is an almost uniformly lower semicontinuous closed relation on a topologically semisymmetric relator space* $X(\mathcal{R})$ *to a sequentially convergence-adherence complete metric type relator space* $Y(\mathcal{S})$, *then* F *is uniformly lower semicontinuous.*

Now, combining Corollary 10.3 with the $F = G$ particular cases of Theorems 8.3 and 8.7, we can also easily establish the following important

Theorem 10.4. *If F is a relation on a reflexive, topologically semisymmetric relator space $X(\mathcal{R})$ to a sequentially convergence-adherence complete metric type relator space $Y(\mathcal{S})$, then the following assertions are equivalent:*

(1) F is closed and almost uniformly lower semicontinuous;

(2) F is closed-valued and uniformly lower semicontinuous.

Proof. If the assertion (1) holds, then Theorem 8.3 shows that F is closed-valued. Moreover, Corollary 10.3 shows that F is uniformly lower semicontinuous. Therefore, the assertion (2) also holds.

On the other hand, if the assertion (2) holds, then Theorem 8.7 shows that F is closed. Moreover, because of the reflexivity of $\mathcal{R}$, it is clear that F is also almost uniformly lower semicontinuous. Therefore, the assertion (1) also holds.

Combining Theorem 10.1 with the $F = G$ particular case of Corollary 8.11, we can also at once state

Theorem 10.5. *If (F, G) is an almost uniformly lower semicontinuous closed pair of relations on an uniformly symmetric relator space $X(\mathcal{R})$ to a sequentially convergence-adherence complete metric type relator space $Y(\mathcal{S})$ such that $G \subset F$, then the relation F is uniformly upper semicontinuous.*

Remark 10.6. Note that if (F, G) is a pair of relations on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$ such that $G \subset F$ and the relation F is uniformly upper semicontinuous, then the pair (F, G) is also uniformly upper semicontinuous.

In this respect it is also worth mentioning that by using Theorem 9.1 we can also easily get the following

Theorem 10.7. *If (f, g) is an almost uniformly lower semicontinuous closed pair of functions on a topologically semisymmetric relator space $X(\mathcal{R})$ to a sequentially convergence-adherence complete metric type relator space $Y(\mathcal{S})$ such that $g \subset f$, then*

$$\mathrm{cl}_{\mathcal{R}}\Big(g^{-1}\big(V(y) \big) \Big) \subset f^{-1}\Big(\mathrm{cl}_{\mathcal{S}}\big(V(y) \big) \Big)$$

for all $V \in \mathcal{S}$ and $y \in Y$.

Proof. Since f is a function, by Theorem 9.1 and the uniform symmetry of $\mathcal{S}$, it is clear that

$$\mathrm{cl}_{\mathcal{R}}\Big(g^{-1}\big(V(y) \big) \Big) \subset \bigcap_{S \in \mathcal{S}} f^{-1}\Big(S\big(V(y) \big) \Big) =$$

$$= f^{-1}\Big(\bigcap_{S \in \mathcal{S}} \Big(S\big(V(y) \big) \Big) \Big) = f^{-1}\Big(\mathrm{cl}_{\mathcal{S}}\big(V(y) \big) \Big)$$

for all $V \in \mathcal{S}$ and $y \in Y$.

Remark 10.8. Therefore, if in addition to the conditions of Theorem 10.7 for each $A \subset Y$ there exist $V \in \mathcal{S}$ and $y \in Y$ such that $A = V(y)$, then by Theorem 6.4, the pair (f, g) is topologically upper semicontinuous.

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