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ON UPPER AND LOWER M-CONTINUOUS MULTIFUNCTIONS

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Abstract

In this paper[1] we introduce upper/lower M-continuous multifunctions as a multifunction defined between sets satisfying certain minimal conditions. We obtain some characterizations and some properties of such multifunctions. Moreover, we define m-compactness and m-connectedness and investigate their properties.

1 Introduction

Semi-open sets, preopen sets, α-sets, β-open sets and δ-open sets play an important role in the researches of generalizations of continuity in topological spaces. By using these sets several authors introduced and studied various types of non-continuous functions. Certain of these non-continuous functions have properties similar to those of continuous functions and they hold, in many part, parallel to the theory of continuous functions. Further, the analogy in their definitions and results suggests the need of formulating a unified theory in the setting of multifunctions.

In this paper we introduce upper/lower M-continuous multifunctions as multifunctions defined between sets satisfying certain minimal conditions. We obtain some characterizations and some properties of such multifunctions. Moreover, we define m-compactness and m-connectedness and investigte their properties.

2 Preliminaries

Let X be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset

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A is said to be *regular closed* (resp. *regular open*) if $\mathrm{Cl}(\mathrm{Int}(A)) = A$ (resp. $\mathrm{Int}(\mathrm{Cl}(A)) = A$).

Definition 2.1 A subset A of a topological space (X, τ) is said to be *β-open* [1] (resp. *semi-open* [10], *preopen* [13], *α-open* [18]) if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A)))$, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))))$.

The family of all semi-open (resp. preopen, α-open, β-open) sets in X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$). The complement of a semi-open (resp. preopen, α-open, β-open) set is said to be *semi-closed* [5] (resp. *preclosed* [7], *α-closed* [15], *β-closed* [1]). If A is both semi-open and semi-closed, then it is said to be *semi-regular* [6]. The set of all semi-regular sets of X is denoted by $\mathrm{SR}(X)$. The intersection of all semi-closed (resp. preclosed, α-closed, β-closed, semi-regular) sets of X containing A is called the *semi-closure* [5] (resp. *preclosure* [7], *α-closure* [15], *β-closure* [2], *semi-θ-closure* [16]) of A and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_\beta\mathrm{Cl}(A)$, $s_\theta\mathrm{Cl}(A)$). The union of all semi-open (resp. preopen, α-open, β-open, semi-regular) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, *α-interior*, *β-interior*, *semi-θ-interior*) of A and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_\beta\mathrm{Int}(A)$, $s_\theta\mathrm{Int}(A)$).

Throughout the present paper, (X, τ) and (Y, σ) (or simply X and Y) always denote topological spaces and $F : (X, \tau) \to (Y, \sigma)$ presents a multivalued function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Definition 2.2 A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(1) *upper continuous* (resp. *upper semi-continuous* [8, 23], *upper precontinuous* [26], *upper α-continuous* [17], *upper β-continuous* [31], *upper s-θ-continuous* [9]) if for each $x \in X$ and each open set V of Y containing $F(x)$, there exists an open (resp. semi-open, preopen, α-open, β-open, semi-regular) set U of X containing x such that $F(U) \subset V$,

(2) *lower continuous*, (resp. *lower semi-continuous* [8, 23], *lower precontinuous* [26], *lower α-continuous* [17], *lower β-continuous* [31], *lower s-θ-continuous* [9]) if for each $x \in X$ and each open set V of Y such that $V \cap F(x) \neq \emptyset$, there exists an open (resp. semi-open, preopen, α-open, β-open, semi-regular) set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.

Definition 2.3 A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(1) *upper irresolute* [8] (resp. *upper preirresolute* [27], *upper α-irresolute* [17] , *upper β-irresolute* [28]) if for each $x \in X$ and each semi-open (resp. preopen, α-open, β-open) set V of Y containing $F(x)$,there exists a semi-open (resp. preopen, α-open, β-open) set U of X containing x such that $F(U) \subset V$,

(2) *lower irresolute* [8] (resp. *lower preirresolute* [27], *lower α-irresolute* [17], *lower β-irresolute* [28]) if for each $x \in X$ and each semi-open (resp. preopen, α-open, β-open) set V of Y such that $F(x) \cap V \neq \emptyset$,there exists a semi-open (resp. preopen, α-open, β-open) set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.

3 M-continuous multifunctions

Definition 3.1 A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) on X if $\emptyset \in m_X$ and $X \in m_X$. Each member of m_X is said to be *m_X-open* and the complement of a m_X-open set is said to be *m_X-closed*.

Remark 3.1 Let (X, τ) be a topological space. Then the families τ, SO(X), PO(X), $\alpha(X)$, $\beta(X)$, SR(X)) are all m-structures on X.

Definition 3.2 Let X be a nonempty set and m_X a m-structure on X. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [12] as follows:

(1) m_X-Cl$(A) = \cap\{F : A \subset F, X - F \in m_X\}$,

(2) m_X-Int$(A) = \cup\{U : U \subset A, U \in m_X\}$.

Remark 3.2 Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, $\beta(X)$, SR(X)), then we have

(1) m_X-Cl(A) = Cl(A) (resp. sCl(A), pCl(A), αCl(A), ${}_\beta$Cl(A), s_θCl(A)),

(2) m_X-Int(A) = Int(A) (resp. sInt(A), pInt(A), αInt(A), ${}_\beta$Int(A), s_θInt(A)).

Lemma 3.1 (Maki [12]). *Let X be a nonempty set and m_X a m-structure on X. For subsets A and B of X, the following hold:*

(1) m_X-Cl$(X - A) = X - (m_X$-Int$(A))$ *and* m_X-Int$(X - A) = X - (m_X$-Cl$(A))$,

(2) If $(X - A) \in m_X$, *then* m_X-Cl$(A) = A$ *and if* $A \in m_X$, *then* m_X-Int$(A) = A$,

(3) m_X-Cl$(\emptyset) = \emptyset$, m_X-Cl$(X) = X$, m_X-Int$(\emptyset) = \emptyset$ *and* m_X-Int$(X) = X$,

(4) If $A \subset B$, *then* m_X-Cl$(A) \subset m_X$-Cl(B) *and* m_X-Int$(A) \subset m_X$-Int(B),

(5) $A \subset m_X$-Cl(A) *and* m_X-Int$(A) \subset A$,

(6) m_X-Cl$(m_X$-Cl$(A)) = m_X$-Cl(A) *and* m_X-Int$(m_X$-Int$(A)) = m_X$-Int(A).

Lemma 3.2 *Let X be a nonempty set with a minimal structure m_X and A a subset of X. Then $x \in m_X$-Cl(A) if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

Proof. *Necessity.* Suppose that there exists $U \in m_X$ containing x such that $U \cap A = \emptyset$. Then $A \subset X - U$ and $X - (X - U) = U \in m_X$. Then m_X-Cl$(A) \subset X - U$. Since $x \in U$, we have $x \notin m_X$-Cl(A).

Sufficiency. Suppose that $x \notin m_X$-Cl(A). There exists a subset F of X such that $X - F \in m_X, A \subset F$ and $x \notin F$. Thus there exists $(X - F) \in m_X$ containing x such that $(X - F) \cap A = \emptyset$.

Definition 3.3 A minimal structure m_X on a nonempty set X is said to have *property* $(\mathcal{B})$ [12] if the union of any family of subsets belonging to m_X belongs to m_X.

Lemma 3.3 *For a minimal structure m_X on a nonempty set X, the following are equivalent:*

(1) m_X has property $(\mathcal{B})$;

(2) If m_X-Int$(V) = V$, then $V \in m_X$;

(3) If m_X-Cl$(F) = F$, then $X - F \in m_X$.

Proof. (1) $\Rightarrow$ (2): Let m_X-Int$(V) = V$. By Definition 3.2, we have m_X-Int$(V) \in m_X$ and hence $V \in m_X$.

(2) $\Rightarrow$ (3): Let m_X-Cl$(F) = F$. Then we have $X - F = X - m_X$-Cl$(F) = m_X$-Int$(X - F)$. Therefore, by (2) we obtain $X - F \in m_X$.

(3) $\Rightarrow$ (2): This proof is similar to that of the implication (2) $\Rightarrow$ (3).

(2) $\Rightarrow$ (1): Suppose that $A_\alpha \in m_X$ for each $\alpha \in \Delta$. Let $V = \cup_{\alpha\in\Delta} A_\alpha$. By Lemma 3.1, we have m_X-Int$(V) \subset V$. Let $x \in V$, then there exists $\alpha_0 \in \Delta$ such that $x \in A_{\alpha_0} \in m_X$. Therefore, $x \in A_{\alpha_0} = m_X$-Int$(A_{\alpha_0}) \subset m_X$-Int$(V)$. Thus we obtain $V \subset m_X$-Int(V) and hence m_X-Int$(V) = V$. By (2), we have $V = \cup_{\alpha\in\Delta} A_\alpha \in m_X$. This completes the proof.

Lemma 3.4 *Let X be a nonempty set and m_X a minimal structure on X satisfying $(\mathcal{B})$. For a subset A of X, the following properties hold:*

(1) $A \in m_X$ *if and only if* m_X-Int$(A) = A$,
(2) A *is* m_X*-closed if and only if* m_X-Cl$(A) = A$,
(3) m_X-Int$(A) \in m_X$ *and* m_X-Cl(A) *is* m_X*-closed.*

Proof. This follows immediatately from Lemmas 3.1 and 3.3.

Definition 3.4 Let (X, m_X) and (Y, m_Y) be nonempty sets X and Y with minimal structures m_X and m_Y, respectively. A multifunction $F : (X, m_X) \to (Y, m_Y)$ is said to be

(1) *upper M-continuous* if for each $x \in X$ and each $V \in m_Y$ containing $F(x)$, there exists $U \in m_X$ containing x such that $F(U) \subset V$,

(2) *lower M-continuous* if for each $x \in X$ and each $V \in m_Y$ such that $F(x) \cap V \neq \emptyset$, there exists $U \in m_X$ containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.

Theorem 3.1 *For a multifunction* $F : (X, m_X) \to (Y, m_Y)$*, where* (Y, m_Y) *satisfies property* $(\mathcal{B})$*, the following properties are equivalent:*
(1) F *is upper M-continuous;*
(2) $F^+(V) = m_X$-Int$(F^+(V))$ *for every* $V \in m_Y$*;*
(3) $F^-(K) = m_X$-Cl$(F^-(K))$ *for every* m_Y*-closed set* K*;*
(4) m_X-Cl$(F^-(B)) \subset F^-(m_Y$-Cl$(B))$ *for every subset* B *of* Y*;*
(5) $F^+(m_Y$-Int$(B)) \subset m_X$-Int$(F^+(B))$ *for every subset* B *of* Y*.*

Proof. (1) ⇒ (2): Let $V \in m_Y$ and $x \in F^+(V)$. Then $F(x) \subset V$. There exists $U \in m_X$ containing x such that $F(U) \subset V$. Thus $x \in U \subset F^+(V)$. This implies that $x \in m_X$-Int$(F^+(V))$. This shows that $F^+(V) \subset m_X$-Int$(F^+(V))$. By Lemma 3.1, we have m_X-Int$(F^+(V)) \subset F^{-1}(V)$. Therefore, $F^+(V) = m_X$-Int$(F^+(V))$.

(2) ⇒ (3): Let K be any m_Y-closed set. Since $Y - K \in m_X$, by Lemma 3.1 we have $X - F^-(K) = F^+(Y - K) = m_X$-Int$(F^+(Y - K)) = m_X$-Int$(X - F^-(K)) = X - m_X$-Cl$(F^-(K))$. Therefore, we obtain m_X-Cl$(F^-(K)) = F^-(K)$.

(3) ⇒ (4): Let B be any subset of Y. Then by Lemmas 3.1 and 3.4, m_Y-Cl(B) is m_Y-closed. By Lemma 3.1, we have $F^-(B) \subset F^-(m_Y$-Cl$(B))$ $= m_X$-Cl$(F^-(m_Y$-Cl$(B))$ and m_X-Cl$(F^-(B)) \subset F^-(m_Y$-Cl$(B))$.

(4) ⇒ (5): Let B be any subset of Y. Then by Lemma 3.1 we have

$$X - m_X\text{-Int}(F^+(B))) = m_X\text{-Cl}(X - F^+(B)) = m_X\text{-Cl}(F^-(Y - B))$$
$$\subset F^-(m_Y\text{-Cl}(Y - B)) = F^-(Y - m_Y\text{-Int}(B)) = X - F^+(m_Y\text{-Int}(B)).$$

Therefore, we obtain $F^+(m_Y$-Int$(B)) \subset m_X$-Int$(F^+(B))$.

(5) ⇒ (1): Let $x \in X$ and $V \in m_Y$ containing $F(x)$. Then

$$x \in F^+(V) = F^+(m_Y\text{-Int}(V)) \subset m_X\text{-Int}(F^+(V)).$$

There exists $U \in m_X$ containing x such that $U \subset F^+(V)$; hence $F(U) \subset V$. This implies that F is upper M-continuous.

Theorem 3.2 *For a multifunction $F : (X, m_X) \to (Y, m_Y)$, where (Y, m_Y) satisfies property ($\mathcal{B}$), the following properties are equivalent:*

(1) F is lower M-continuous;

(2) $F^-(V) = m_X\text{-Int}(F^-(V))$ for every $V \in m_Y$;

(3) $F^+(K) = m_X\text{-Cl}(F^+(K))$ for every m_Y-closed set K;

(4) $m_X\text{-Cl}(F^+(B)) \subset F^+(m_Y\text{-Cl}(B))$ for every subset B of Y;

(5) $F(m_X\text{-Cl}(A)) \subset m_Y\text{-Cl}(F(A))$ for every subset A of X;

(6) $F^-(m_Y\text{-Int}(B)) \subset m_X\text{-Int}(F^-(B))$ for every subset B of Y.

Proof. We prove only the implications (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (6) being the proofs of the other similar to those of Theorem 3.1.

(4) $\Rightarrow$ (5): Let A be any subset of X. By (4), we have $m_X\text{-Cl}(A) \subset m_X\text{-Cl}(F^+(F(A))) \subset F^+(m_Y\text{-Cl}(F(A)))$ and $F(m_X\text{-Cl}(A)) \subset m_Y\text{-Cl}(F(A))$.

(5) $\Rightarrow$ (6): Let B be any subset of Y. By(5), we have $F(m_X\text{-Cl}(F^+(Y - B))) \subset m_Y\text{-Cl}(F(F^+(Y - B))) \subset m_Y\text{-Cl}(Y - B)) = Y - m_Y\text{-Int}(B)$ and $F(m_X\text{-Cl}(F^+(Y - B))) = F(m_X\text{-Cl}(X - F^-(B))) = F(X - m_X\text{-Int}(F^-(B)))$.

This implies that $F^-(m_Y\text{-Int}(B)) \subset m_X\text{-Int}(F^-(B))$.

Remark 3.3 Let (X, τ) and (Y, σ) be two topological spaces.

(1) We put $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{SR}(X)$) and $m_Y = \sigma$. Then, an upper/lower M-continuous multifunction $F : (X, m_X) \to (Y, \sigma)$ is an upper/lower continuous (resp. semi-continuous, precontinuous, α-continuous, β-continuous, s-θ-continuous). Moreover, Theorems 3.1 and 3.2 establish their characterizations which are obtained in ([23], [26], [29], [31], [9]).

(2) We put $m_X = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$) and $m_Y = \mathrm{SO}(Y)$ (resp. $\mathrm{PO}(Y)$, $\alpha(Y)$, $\beta(Y)$), then an upper/lower M-continuous multifunction $F : (X, m_X) \to (Y, m_Y)$ is upper/lower irresolute (resp. preirresolute or M-precontinuous, α-irresolute or strongly feebly continuous, β-irresolute). Moreover, Theorems 3.1 and 3.2 establish their characterizations which are obtained in [25] (resp. [27], [20], [28]).

4 m-compact sets

Definition 4.1 A nonempty set X with a minimal structure m_X is said to be *m-compact* if every cover of X by m_X-open sets has a finite subcover.

A subset K of a nonempty set X with a minimal structure m_X is said to be *m-compact* if every cover of K by m_X-open sets has a finite subcover.

Theorem 4.1 *Let (X, m_X) and (Y, m_Y) be nonempty sets with m-structures and let m_Y satisfy property $\mathcal{B}$. If $F : (X, m_X) \to (Y, m_Y)$ is an upper M-continuous multifunction such that $F(x)$ is m-compact for each $x \in X$ and K is an m-compact set of (X, m_X), then $F(K)$ is m-compact.*

Proof. Let $\{V_i : i \in I\}$ be any cover of $F(K)$ by m_Y-open sets. For each $x \in K$, $F(x)$ is m-compact and there exists a finite subset $I(x)$ of I such that $F(x) \subset \cup\{V_i : i \in I(x)\}$. Now, set $V(x) = \cup\{V_i : i \in I(x)\}$. Then we have $F(x) \subset V(x)$ and $V(x) \in m_Y$. Since F is upper M-continuous, there exists $U(x) \in m_X$ containing x such that $F(U(x)) \subset V(x)$. The family $\{U(x) : x \in K\}$ is a cover of K by m_X-open sets. Since K is m-compact, there exists a finite number of points, say, $x_1, x_2, ..., x_n$, in K such that $K \subset \cup\{U(x_k) : x_k \in K, 1 \leq k \leq n\}$. Therefore, we obtain

$$\begin{aligned} F(K) &\subset \cup\{F(U(x_k)) : x_k \in K, 1 \leq k \leq n\} \\ &\subset \cup\{V_i : i \in I(x_k), x_k \in K, 1 \leq k \leq n\}. \end{aligned}$$

This shows that $F(K)$ is m-compact.

Corollary 4.1 *Let (X, m_X) and (Y, m_Y) be nonempty sets with m-structures and let m_Y satisfy property $\mathcal{B}$. If $F : (X, m_X) \to (Y, m_Y)$ is an upper M-continuous surjective multifunction such that $F(x)$ is m-compact for each $x \in X$ and (X, m_X) is m-compact, then (Y, m_Y) is m-compact.*

Definition 4.2 A subset A of a topological space X is said to be *α-compact* [19] (resp. *semi-compact, strongly compact* [14]) relative to X if every cover of A by α-open (resp. semi-open, preopen) sets has a finite subcover.

A topological space X is said to be *α-compact* [11] (resp. *semi-compact* [4], *strongly compact* [14]) if the set X is α-compact (resp. semi-compact, strongly compact) relative to X.

Corollary 4.2 *Let (X, τ) and (Y, σ) be topological spaces and $F : (X, \tau) \to (Y, \sigma)$ be a surjective multifunction such that $F(x)$ is compact for each $x \in X$. If (X, τ) is compact (resp. α-compact, semi-compact, strongly compact) and F is upper continuous (resp. upper α-continuous, upper semi-continuous, upper precontinuous), then (Y, σ) is compact.*

Proof. Let $m_X = \tau$ (resp. $\alpha(X)$, SO(X), PO(X)) and $F : (X, m_X) \to (Y, \sigma)$ be a surjective upper M-continuous multifunction such that $F(x)$ is m-compact for each $x \in X$. Then by Corollary 4.1 we obtain the result.

Remark 4.1 In case of α-compact spaces, the result of Corollary 4.2 is established in [3].

5 m-connected sets

Definition 5.1 A nonempty set X with a minimal structure m_X satisfying $\mathcal{B}$ is said to be *m-connected* if X can not be written as the union of two nonempty disjoint m_X-open sets.

Definition 5.2 A topological space (X, τ) is said to be *semi-connected* [22] (resp. *preconnected* [24], *β-connected* [30]) if X can not be written as the union of two nonempty disjoint semi-open (resp. preopen, β-open) sets.

Remark 5.1 If (X, τ) is a topological space and $m_X = \tau$ (resp. SO(X), PO(X), $\beta(X)$), then the definition of connectedness (resp. semi-connectedness [22], preconnectedness [24], β-connectedness [30]) are obtained from Definition 5.1.

Theorem 5.1 *Let (X, m_X) be a nonempty set with a minimal structure m_X satisfying property ($\mathcal{B}$) and (Y, σ) a topological space. If $F : (X, m_X) \to (Y, \sigma)$ is an upper or lower M-continuous surjective multifunction such that $F(x)$ is connected for each $x \in X$ and (X, m_X) is m-connected, then (Y, σ) is connected.*

Proof. Suppose that (Y, σ) is not connected. There exist nonempty open sets $U, V \in \sigma$ such that $U \cap V = \emptyset$ and $U \cup V = Y$. Since $F(x)$ is connected for each $x \in X$, either $F(x) \subset U$ or $F(x) \subset V$. If $x \in F^+(U \cup V)$, then $F(x) \subset U \cup V$ and hence $x \in F^+(U) \cup F^+(V)$. Moreover, since F is surjective, there exist x and y in X such that $F(x) \subset U$ and $F(y) \subset V$; hence $x \in F^+(U)$ and $y \in F^+(V)$. Therefore, we obtain the following:
(1) $F^+(U) \cup F^+(V) = F^+(U \cup V) = X$,
(2) $F^+(U) \cap F^+(V) = F^+(U \cap V) = \emptyset$, and
(3) $F^+(U) \neq \emptyset$ and $F^+(V) \neq \emptyset$.
Next, we show that $F^+(U)$ and $F^+(V)$ are m_X-open in X. (i) Let F be upper M-continuous. By Theorem 3.1 and Lemma 3.4, $F^+(U)$ and $F^+(V)$ are m_X-open in X. (ii) Let F be lower M-continuous. By Theorem 3.2 and Lemma 3.4, $F^+(U)$ is m_X-closed in X since U is clopen in (Y, σ). Therefore, $F^+(V)$ is m_X-open in X. Similarly, $F^+(U)$ is m_X-open in X. Consequently, X is not m-connected. This completes the proof.

Corollary 5.1 *Let (X, τ) and (Y, σ) be topological spaces and $F : (X, \tau) \to (Y, \sigma)$ be a surjective multifunction such that $F(x)$ is connected for each $x \in X$. If (X, τ) is connected (resp. semi-connected, preconnected, β-connected) and F is upper or lower continuous (resp. semi-continuous, precontinuous, β-continuous), then (Y, σ) is connected.*

Proof. Let $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\beta(X)$) and $F : (X, m_X) \to (Y, \sigma)$ be an upper M-continuous surjective multifunction such that $F(x)$ is connected for each $x \in X$. Then by Theorem 5.1 we obtain the result.

6 Applications

A subset A is said to be *δ-open* [33] if for each $x \in A$ there exists a regular open set G such that $x \in G \subset A$. A point $x \in X$ is called a *δ-cluster* point of A if $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$ for every open set V containing x. The set of all δ-cluster points of A is called the *δ-closure* of A and is denoted by ${}_\delta\mathrm{Cl}(A)$. The set $\{x \in X : x \in U \subset A$ for some regular open set U of $X\}$ is called the *δ-interior* of A and is denoted by ${}_\delta\mathrm{Int}(A)$. A subset A of X is said to be *δ-preopen* [32] (resp. *δ-semi-open* [21]) if $A \subset \mathrm{Int}({}_\delta\mathrm{Cl}(A))$ (resp. $A \subset \mathrm{Cl}({}_\delta\mathrm{Int}(A))$). The family of all δ-preopen (resp. δ-semi-open) sets in X is denoted by $\delta\mathrm{PO}(X)$ (resp. $\delta\mathrm{SO}(X)$). The complement of a δ-preopen (resp. δ-semi-open) set is said to be *δ-preclosed* [32] (resp. *δ-semi-closed* [21]). The intersection of all δ-preclosed (resp. δ-semi-closed) sets of X containing A is called the *δ-preclosure* [32] (resp. *δ-semi-closure* [21]) of A and is denoted by ${}_\delta\mathrm{pCl}(A)$ (resp. ${}_\delta\mathrm{sCl}(A)$). The union of all δ-preopen (resp. δ-semi-open) sets of X contained in A is called the *δ-preinterior* (resp. *δ-semi-interior*) of A and is denoted by ${}_\delta\mathrm{pInt}(A)$ (resp. ${}_\delta\mathrm{sInt}(A)$).

Let (X, τ) and (Y, σ) be topological spaces in the sequel. We can define some new types of generalized continuity for multifunctions.

Definition 6.1 A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(1) *upper δ-almost-continuous* (resp. *upper δ-semi-continuous*) if for each $x \in X$ and each open set V of Y containing $F(x)$, there exists a δ-preopen (resp. δ-semi-open) set U of X containing x such that $F(U) \subset V$,

(2) *lower δ-almost-continuous* (resp. *lower δ-semi-continuous*) if for each $x \in X$ and each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a δ-preopen (resp. δ-semi-open) set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.

Definition 6.2 A multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$ is said to be

(1) *upper δ-almost-irresolute* (resp. *upper δ-semi-irresolute* , *upper s-θ-irresolute*) if for each $x \in X$ and each δ-preopen (resp. δ-semi-open, semi-regular) set V of Y containing $F(x)$, there exists a δ-preopen (resp. δ-semi-open, semi-regular) set U of X containing x such that $F(U) \subset V$,

(2) *lower δ-almost-irresolute* (resp. *lower δ-semi-irresolute* , *lower s-θ-irresolute*) if for each $x \in X$ and each δ-preopen (resp. δ-semi-open, semi-regular) set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a δ-preopen (resp. δ-semi-open, semi-regular) set U of X containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.

Remark 6.1 Let (X, τ) be a topological space. Then the families $\delta\mathrm{PO}(X)$ and $\delta\mathrm{SO}(X)$ are all m-structures on X satisying property ($\mathcal{B}$). If $m_X = \delta\mathrm{PO}(X)$ (resp. $\delta\mathrm{SO}(X)$), then for a subset A of X we have

(1) m_X-$\mathrm{Cl}(A) = {}_\delta\mathrm{pCl}(A)$ (resp. ${}_\delta\mathrm{sCl}(A)$),

(2) m_X-$\mathrm{Int}(A) = {}_\delta\mathrm{pInt}(A)$ (resp. ${}_\delta\mathrm{sInt}(A)$) .

We obtain the following characterizations, for example, for lower/lower δ-almost-continuous multifunctions.

Theorem 6.1 *For a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper δ-almost-continuous;

(2) $F^+(V) \in \delta\mathrm{PO}(X)$ for every open set $V \in \sigma$;

(3) $F^-(K)$ δ-preclosed in (X, τ) for every closed set K of (Y, σ);

(4) ${}_\delta\mathrm{pCl}(F^-(B)) \subset F^-(\mathrm{Cl}(B))$ for every subset B of Y;

(5) $F^+(\mathrm{Int}(B)) \subset {}_\delta\mathrm{pInt}(F^+(B))$ for every subset B of Y.

Proof. In Theorem 3.1 we set $m_X = \delta\mathrm{PO}(X)$ and $m_Y = \sigma$, then this theorem follows from Theorem 3.1.

Theorem 6.2 *For a multifunction $F : (X, \tau) \rightarrow (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower δ-almost-continuous;

(2) $F^-(V) \in \delta\mathrm{PO}(X)$ for every open set $V \in \sigma$;

(3) $F^+(K)$ δ-preclosed in (X, τ) for every closed set K of (Y, σ);

(4) ${}_\delta\mathrm{pCl}(F^+(B)) \subset F^+(\mathrm{Cl}(B))$ for every subset B of Y;

(5) $F({}_\delta\mathrm{pCl}(A)) \subset \mathrm{Cl}(F(A))$ for every subset A of X;

(6) $F^-(\mathrm{Int}(B)) \subset {}_\delta\mathrm{pInt}(F^-(B))$ for every subset B of Y.

Proof. In Theorem 3.2 we set $m_X = \delta\mathrm{PO}(X)$ and $m_Y = \sigma$, then this theorem follows from Theorem 3.2.

Moreover, we define the following new multifunctions. Then we can obtain their characterizations by Theorems 3.1 and 3.2.

Definition 6.3 A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(1) *upper strongly β-irresolute* (resp. *upper strongly-semi-irresolute* , *upper strongly M-precontinuous*, *upper strongly α-irresolute*) if for each point $x \in X$ and each β-open (resp. semi-open, preopen, α-open) set V of (Y, σ) containing $F(x)$ there exists an open set U containing x such that $F(U) \subset V$,

(2) *lower strongly β-irresolute* (resp. *lower strongly-semi-irresolute* , *lower strongly M-precontinuous*, *lower strongly α-irresolute*) if for each point $x \in X$ and each β-open (resp. semi-open, preopen, α-open) set V of (Y, σ) such that $F(x) \cap V \neq \emptyset$ there exists an open set U containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$.

Remark 6.2 Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction, $m_X = \tau$ and $m_Y = \beta(Y)$ (resp. $\mathrm{SO}(Y)$, $\mathrm{PO}(Y)$, $\alpha(Y)$). Then,

(1) By Theorem 3.1, we obtain characterizations of upper strongly β-irresolute (resp. upper strongly-semi-irresolute, upper strongly M-precontinuous, upper strongly α-irresolute),

(2) By Theorem 3.2, we obtain characterizations of lower strongly β-irresolute (resp. lower strongly-semi-irresolute, lower strongly M-precontinuous, lower strongly α-irresolute).

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