High order ε-uniform methods for singularly perturbed reaction-diffusion problems with discontinuous coefficients and singular sources

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Abstract

We consider the reaction-diffusion equation with discontinues coefficients and singular sources in one dimension. In this work[1], we construct ε-uniformly convergent High Order Compact (HOC) monotone finite difference schemes defined on a priori Shishkin meshes, which have order two, three and four except for a logarithmic factor. Numerical experiments are presented and discussed.

1 Introduction

In this paper we are interested in the construction and validation of high order difference approximations to problems of type

$$L_\varepsilon u \equiv -\varepsilon^2 (p(x)u')' + q(x)u = f(x) \text{ on } \Omega = \bigcup_{s=1}^{S+1} \Omega_s,\ \Omega_s = (\xi_{s-1}, \xi_s), \tag{1}$$

$$s = 1, \ldots, S+1,\ -1 = \xi_0 < \xi_1 < \ldots < \xi_S < \xi_{S+1} = 1,$$

where ε is a parameter in $(0,1]$ and the coefficients $p,\ q$ satisfy the inequalities

$$p_1 \geq p(x) \geq p_0 > 0,\ q_1 \geq q(x) \geq q_0 > 0 \text{ for } x \in \overline{\Omega}. \tag{2}$$

On the interfaces the jump (interface) relations hold

$$[u]_{\xi_s} = u(\xi_s + 0) - u(\xi_s - 0) = 0,\ \varepsilon\,[pu']_{\xi_s} = Q_s u(\xi_s) + R_s,\ s = 1, \ldots, S \tag{3}$$

and on the boundary $\{-1, 1\}$ the Dirichlet condition

$$u(-1) = u_{-1},\ u(1) = u_1. \tag{4}$$

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More in detail we shall study the problem

$$L_\varepsilon u = f\left(x\right),\ x \in \Omega,\ \left[u\right]_0 = 0,\ \left[pu'\right]_0 = Qu\left(0\right) + R,\ u\left(-1\right) = u_{-1}, u\left(1\right) = u_1. \tag{5}$$

i.e. the case of single interface $S = 1,\ \xi_1 = 0,\ Q_1 = Q,\ R_1 = R$.

Let define the function $\Phi_1\left(x\right)$ and the real number $Q\left(\varepsilon\right)$ as follows:

$$L_\varepsilon \Phi_1 = 0,\ x \in \Omega,\ \Phi_1(-1) = 0,\ \Phi_1(0) = 1,\ \Phi_1(1) = 0,$$

$$Q\left(\varepsilon\right) \equiv -\varepsilon\left(p\left(+0\right)\Phi_1'\left(+0\right) - p\left(-0\right)\Phi_1'\left(-0\right)\right).$$

In [4] is shown that $Q\left(\varepsilon\right) > 0$ and if $Q \neq -Q\left(\varepsilon\right)$ then

$$u\left(x\right) = v\left(x\right) - \frac{Qv\left(0\right) + R}{Q + Q\left(\varepsilon\right)}\Phi_1\left(x\right), \tag{6}$$

where $y(x)$ is the solution of the differential problem with only discontinuous coefficients, i.e.

$$L_\varepsilon v = f\left(x\right),\ x \in \Omega,\ \left[v\right]_0 = 0,\ \left[pv'\right]_0 = 0,\ v(-1) = u_{-1},\ v(1) = u_1. \tag{7}$$

Therefore, we can focus on problem (7). Our aim of this paper is to find uniformly convergent finite difference methods of high order on Shiskin meshes [8,9]. Previously finite difference schemes (FDS) of arbitrary order for (7) at $\varepsilon = 1$ was developed in [5]. Now, we follow the idea of Gartland [5], where classical methods are constructed on the base of an integral identity which is attributed to Marchuk. Below, we prove that methods constructed in this way,with $k = 2,\ 3,\ 4$ are uniformly convergent with $O\left(N^{-k}\ln^k N\right)$ if Shiskin meshes is used, cf [3]. ε–uniform methods for singularly perturbed problems with discontinuous coefficients and concentrated factors are studied in [7-10, 12]. The rest of the paper is organized as follows: In the next Section FDS up to fourth order are described. Uniform convergence is discussed in Section 3, and numerical results in Section 4.

2 High Order Three-Point Difference Scheme

In this section we discuss a few FDS derived in [5]. All our discretizations are on an interval about the origin $x = 0$. Let first assume that there is no interface, so that p, q and f are sufficiently smooth on $[-h_-, h_+]$, $h_\mp > 0$ is the mesh step in the left and in the right of $x = 0$, respectively. Let define

$$w(r,s) = (r^2 + 2rs - 2s^2)/(24r) \text{ and } w_{-1} = w(h_-, h_+),\ w_1 = w(h_+, h_-),$$

$$w_0 = (h_- + h_+)(2h_-^2 + 7h_-h_+ + 2h_+^2)/(24h_-.h_+).$$

Then the three point scheme

$$L_h v^h \equiv \alpha_{-1}v_{-1}^h + \alpha_0 v_0^h + \alpha_1 v_1^h = \beta_{-1}f_{-1} + \beta_0 f_0 + \beta_1 f_1, \tag{8}$$

where

$$\beta_{\mp 1} = P^{\mp}\frac{h_{\mp}}{48}\left(\frac{3}{p_{\mp 1}} - \frac{1}{p_0}\right) + w_{\mp 1}, \quad P^{\mp} = 6/\left(\frac{1}{p_{\mp 1}} + \frac{4}{p_{\mp 1/2}} + \frac{1}{p_0}\right),$$

$$\beta_0 = P^{-}\frac{h_{-}}{48}\left(\frac{1}{p_{-1}} - \frac{3}{p_0}\right) + P^{+}\frac{h_{+}}{48}\left(\frac{1}{p_{+1}} - \frac{3}{p_0}\right) + w_0,$$

$$\alpha_{\mp 1} = -\varepsilon^2 P_{\mp}/h_{\mp} + \beta_{\mp 1}q_{\mp 1}, \quad \alpha_0 = \varepsilon^2\left(P^{-}/h_{-} + P^{+}/h_{+}\right) + \beta_0 q_0,$$

has local truncation error (LTE) $O(H^3)$, $H = \max(h_-, h_+)$, or $LTE = CH^2$, but $C = C(\varepsilon)$.

In the next discretization p, q and f can have jump discontinuities. The lowest order FDS, LTE = O(1) in the interface $x = 0$ is given by the formula

$$L_h v^h = 0,\ \alpha_{-1} = \frac{p(-0)}{h_-},\ \alpha_1 = \frac{p(+0)}{h+},\ \alpha_0 = \alpha_{-1} + \alpha_1.$$

The interface discretization is derived by Varga [11]

$$L_h v^h \equiv \beta_{-0} f_{-0} + \beta_{+0} f_{+0}, \tag{9}$$

where

$$\alpha_{\pm 1} = -\frac{2\varepsilon^2}{(h_- + h_+)\,h_{\pm 1}} p_{\pm 1/2},\ \alpha_0 = \alpha_{-1} + \alpha_1 + \frac{h_- q_{-0} + h_+ q_{+0}}{h_- + h_+},$$

$$\beta_{\pm 0} = \frac{h_{\pm}}{h_- + h_+}.$$

When (8) is combined with standard central differences away from the interface, it produces a symmetric discretization matrix, and in this case it gives a global $O(H^2)$ discretization error.

An improvement of formula (9) is the scheme

$$L_h v^h = \beta_{-1} f_{-1} + \beta_{-0} f_{-0} + \beta_{+0} f_{+0} + \beta_1 f_1, \tag{10}$$

where

$$\beta_{\mp 1} = P^{\mp}\frac{h_{\mp}}{24}\left(\frac{3}{p_{\mp 1}} + \frac{1}{p_{\mp 0}}\right),\ \beta_{\mp 0} = P^{\mp}\frac{h_{\mp}}{24}\left(\frac{5}{p_{\mp 1}} + \frac{3}{p_{\mp 0}}\right),$$

$$P^{\mp} = 2/\left(\frac{1}{p_{\mp 1}} + \frac{1}{p_{\mp 0}}\right),$$

$$\alpha_{\mp 1} = -\varepsilon^2 P^{\mp}/h_{\mp} + \beta_{\mp 1} q_{\mp 1}, \quad \alpha_0 = \varepsilon^2\left(P^{-}/h_{-} + P^{+}/h_{+}\right) + \beta_{-0} q_{-0} + \beta_{+0} q_{+0},$$

which is locally $O(H^2)$ and it provides a global $O(H^3)$ when is combined with higher-order scheme like (8) away from the interface. The three-point discretization

$$L_h v^h = \beta_{-1} f_{-1} + \beta_{-1/2} f_{-1/2} + \beta_{-0} f_{-0} + \beta_{+0} f_{+0} + \beta_{1/2} f_{1/2} + \beta_1 f_1, \tag{11}$$

where

$$\beta_{\mp 1} = P^{\mp} \frac{h_{\mp}}{36} \left(\frac{1}{p_{\mp 1}} - \frac{1}{p_{\mp 1/2}} \right), \; P^{\mp} = 6 / \left(\frac{1}{p_{\mp 1}} + \frac{4}{p_{\mp 1/2}} + \frac{1}{p_{\mp 0}} \right),$$

$$\beta_{\mp 1/2} = P^{\mp} \frac{h_{\mp}}{9} \frac{p_{\mp 1/2}}{p_{\mp 1/2} + \frac{1}{8} h_{\mp}^2 q_{\mp 1/2}} \left(\frac{1}{p_{\mp 1}} + \frac{2}{p_{\mp 1/2}} \right),$$

$$\beta_{\mp 0} = P^{\mp} \frac{h_{\mp}}{36} \left(\frac{1}{p_{\mp 1}} + \frac{5}{p_{\mp 1/2}} \right),$$

$$\alpha_{\mp 1} = -\varepsilon^2 P^{\mp} / h_{\mp} + \beta_{\mp 1} q_{\mp 1} + \widetilde{P}^{\mp},$$

$$\alpha_0 = \varepsilon^2 \left(P^- / h_- + P^+ / h_+ \right) + \beta_{-0} q_{-0} + \beta_{+0} q_{+0} + \widetilde{P}^- + \widetilde{P}^+$$

with

$$\widetilde{P}^{\mp} = \frac{1}{8} \frac{p_{\mp 1} + 4 p_{\mp 1/2} - p_{\mp 0}}{p_{\mp 1/2}} \beta_{\mp 1/2} q_{\mp 1/2}$$

is locally $O(H^3)$, whether or not $h_- = h_+$ and is sufficient to obtain global $O(H^4)$ discretization error when combined with the high-order scheme (8) away from the interface.

It is theoretically possible to compute three-point schemes of any desired order of accuracy if one uses enough extra evaluations of p, q and f [5].

3 Uniform Convergence

For sufficiently small ε, classical methods in uniform meshes for singularly perturbed problems only work for very large number of mesh points [8,9]. Nevertheless if these methods are defined on special fitted meshes, the convergence to the exact solution is uniform in ε [8,9]. Shishkin meshes [8,9], are simple piecewise uniform meshes of this kind, frequently used for singularly perturbed problems. For the reaction diffusion problem (7), the corresponding Shishkin mesh, is defined as follows.

The interval Ω_1 is subdivided into three subintervals

$$[-1, -1 + \sigma_1], [-1 + \sigma_1, -\sigma_1] \,\text{and}\, [-\sigma_1, 0]$$

for some σ_1 that satisfies $0 \le \sigma_1 \le 1/4$. On $[-1, -1 + \sigma_1]$ and $[-\sigma_1, 0]$ a uniform mesh with $N/8$ mesh-points is placed, while $[-1 + \sigma_1, -\sigma_1]$ has a uniform mesh with $N/4$ mesh points. The subintervals $[0, \sigma_2]$, $[\sigma_2, 1 - \sigma_2]$, $[1 - \sigma_2, 1]$ are treated analogously for some σ_2 satisfying $0 \le \sigma_2 \le 1/4$. The interior points of the mesh are denoted by

$$\begin{gathered}
\Omega^N = \{x_i : \; x_i = -1 + h_{11} i, \; 1 \le i \le N/8, \; h_{11} = 8\sigma_1/N; \\
x_i = -1 + \sigma_1 + h_{21} i, \; N/8 \le i \le 3N/8, \; h_{21} = 4(1 - 2\sigma_1)/N; \\
x_i = -\sigma_1 + h_{11} i, \; 3N/8 \le i \le N/2; \\
x_i = h_{12} i, \; N/2 \le i \le 5N/8, \; h_{12} = 8\sigma_2/N; \\
x_i = \sigma_2 + h_{22} i, \; 5N/8 \le i \le 7N/8, \; h_{22} = 4(1 - 2\sigma_2)/N; \\
x_i = 1 - \sigma_2 + h_{12} i, \; 7N/8 \le i \le N - 1\}.
\end{gathered}$$

Note that this mesh is a uniform mesh when $\sigma_1 = 1/4$ end $\sigma_2 = 1/4$. It is fitted to the singular perturbation problem (8) by choosing σ_1 and σ_2 to be the following functions of N and ε

$$\sigma_i = \min\left(\frac{1}{4}, \frac{\sigma_0 \varepsilon \ln N}{\gamma_i}\right), \ i = 1, 2.$$

Here σ_0 is a constant to be chosen which depends on the difference scheme, see Table 1. Clearly $x_{N/2} = 0$ and $\bar{\Omega}^N = \Omega^N \cup \left\{x_0 = -1, \, x_{N/2} = 0, \, x_N = 1\right\}$.

Under certain smoothness on the data the boundary value problem (8) has classical solution $u \in C\left(\bar{\Omega}\right) \cap C^4\left(\Omega\right)$ and this solution can be decomposed as

$$y\left(x\right) = \left\{ \begin{array}{l} y_1\left(x\right) = v_1\left(x\right) + w_1\left(x\right), \ x \in \Omega_1, \\ y_2\left(x\right) = v_2\left(x\right) + w_2\left(x\right), \ x \in \Omega_2, \end{array} \right. \tag{12}$$

where, for each integer $k, \ 0 \le k \le 4$, the smooth $v_i, \ i = 1, 2$ and singular $w_i, \ i = 1, 2$ components satisfy the bounds

$$\begin{aligned} \left|v_i^{(k)}\left(x\right)\right| &\le C\left(1 + \varepsilon^{2-k} e\left(x, \gamma_i\right)\right), \ x \in \Omega_i, \ i = 1, 2, \\ \left|w_i^{(k)}\left(x\right)\right| &\le C\varepsilon^{-k} e\left(x, \gamma_i\right), \ x \in \Omega_i, \ i = 1, 2, \end{aligned} \tag{13}$$

$$e\left(x, \gamma_i\right) = \exp\left(-\frac{1 - |x|}{\varepsilon}\gamma_i\right) + \exp\left(-\frac{|x|}{\varepsilon}\gamma_i\right), \ x \in \Omega_i, \ i = 1, 2,$$

where

$$\gamma_i = \beta_i - \frac{1}{2}\varepsilon_0 \delta_i, \ \beta_i = \inf_{x \in \bar{\Omega}_i} \sqrt{\frac{q\left(x\right)}{p\left(x\right)}}, \ \delta_i = \sup_{x \in \bar{\Omega}_i} \frac{\left|p'\left(x\right)\right|}{p\left(x\right)},$$

$$i = 1, 2, \ \varepsilon_0 = \min\left\{1, \frac{2\beta_1}{\delta_1}, \frac{2\beta_2}{\delta_2}\right\}$$

and the constant C is independent of ε.

We now study the local truncation error of (9). If we are away from the interface and $p, \ q$ and f are sufficiently smooth, then we directly use Taylor formula.

Let consider the solution u of (7) on $[-h_-, h_+]$, where $p, \ q$ and f are smooth to the left and right of $x = 0$ but can have jump discontinuous at the origin. We introduce left and right truncated power function s_k and t_k by [2]

$$s_k\left(x\right) = \left\{ \begin{array}{ll} x^k, & x < 0, \\ 0, & x > 0, \end{array} \right. \qquad t_k\left(x\right) = \left\{ \begin{array}{ll} 0, & x < 0, \\ x^k, & x > 0. \end{array} \right.$$

Let m be a given positive integer, and assume that

$$p \in C^m[-1, 0] \cup C^m[0, 1], \quad q, \ f \in C^{m-1}[-1, 0] \cup C^{m-1}[0, 1].$$

Then the solution u of (7) admits a local representation of the form [5]

$$u\left(x\right) = a_0 + a_1\left[p_{+0} s_1\left(x\right) + p_{-0} t_1\left(x\right)\right] + a_2 s_2\left(x\right) + ...$$

$$+a_m s_m(x) + b_2 t_2(x) + ... + b_m t_m(x) + E_{m+1}(x),$$

where

$$a_0 = u(0), \quad a_1 = u'(-0)/p_{+0} = u'(+0)/p_{-0},$$

$$a_k = u^{(k)}(-0)/k!, \quad b_k = u^{(k)}(+0)/k!, \quad k = 2, ..., m,$$

$$E_{m+1}(x) = \int_0^x \frac{(x-t)^m}{m!} u^{(m+1)}(t)\, dt.$$

Proposition 1 Assume that the functions p, q and f satisfy (2) and (8) with $m \geq 2$. Then the following estimates of the local truncation error of (9) are valid:

$$LTE = \begin{cases} C\sigma_0^2 \varepsilon^2 N^{-2} \ln^2 N, \ \ kN/8 < i < (k+1)N/8, \ k = 0, 3, 4, 7; \\ CN^{-\sigma_0}, \ \ kN/8 \leq i \leq (k+2)N/8; \ k = 1, 5; \\ CN^{-1} \ln N, \ i = N/2. \end{cases}$$

where C is a constant which doesn't depend on ε and N.

On the base of Shishkin decomposition (12), (13) and Proposition 1 the following theorem can be proved.

Theorem 3.1 *Let $u(x, \varepsilon)$ be the solution of (7) and $\{v_i^h;\ 0 \leq i \leq N\}$ the solution of the scheme (9). Then*

$$\left|u(x_i) - v_i^h\right| \leq C\left(\sigma_0^2 \varepsilon^2 N^{-2} \ln^2 N + \varepsilon N^{-1} + N^{-1} \ln N\right), \quad 0 \leq i \leq N,$$

where C is a constant independent of ε and N.

In a similar way can be studied for ε-uniform convergence the following two combined FDS (CFDS) of order higher than two on the Shishkin mesh Ω^N:

– CFDS of order three uses formula (8) away from the interface and (10) in the interface;

– CFDS of order four uses formula (8) away from the interface and (11) in the interface.

4 Numerical Experiments

Numerical experiments are discussed in this section for the example, $S = 1$, $R = 0$ and

$$-\varepsilon^2 p_i u'' + q_i u = f_i \text{ in } \Omega_i, \ i = 1, 2, \ u(-1) = 0, \ u(1) = 0,$$

check the theoretical results established in the previous section. The error of scheme (9) is measured in the discrete maximum norm. It depends on the perturbation parameter ε and the discretization parameter N:

$$e^{\varepsilon, N} = \max_{\bar{\Omega}^N} \left|u(x_i) - U^N(x_i)\right|,$$

where u is the exact solution of the differential problem and U^N is the discrete solution. In our tests the maximal error e^N (it is highlighted in each column of Table.1) and the corresponding convergence rates are estimated by

$$e^N = \max_{r=0,\ldots,8} e^{2^{-r},N} \text{ and } p^N = \frac{\ln e^N - \ln e^{2N}}{\ln 2}.$$

The first part of Table 1 illustrates the "almost" second-order estimate of Theorem 3.1.

Table 1: $p_1 = 1, p_2 = 100, q_1 = 1, q_2 = 1, f_1 = .7, f_2 = -.6, Q = 0, R = 0, \sigma_0 = 2$.

ε, N	32	64	128	256	512	1024	2048
2^{-1}	2.194E-4 1.998	5.493E-5 1.999	1.374E-5 2.000	3.436E-6 2.000	8.589E-7 2.000	2.147E-7 2.000	5.368E-8
2^{-2}	9.199E-4 1.995	2.308E-4 1.999	5.775E-5 2.000	1.444E-5 2.000	3.611E-6 2.000	9.027E-7 2.000	2.257E-7
2^{-3}	2.666E-3 1.976	6.775E-4 1.994	1.701E-4 1.998	4.256E-5 1.999	1.065E-5 2.000	2.662E-6 2.000	6.654E-7
2^{-5}	3.196E-2 1.418	1.196E-2 1.913	3.176E-3 1.943	8.262E-4 1.993	2.076E-4 1.998	5.195E-5 1.999	1.299E-5
2^{-6}	3.114E-2 1.250	1.310E-2 1.416	4.909E-3 1.588	1.633E-3 1.647	5.213E-4 1.688	1.618E-4 1.724	4.897E-5
2^{-7}	3.256E-2 1.243	1.376E-2 1.448	5.043E-3 1.602	1.661E-3 1.650	5.295E-4 1.693	1.638E-4 1.726	4.951E-5
p^N	1.198	1.416	1.588	1.647	1.688	1.724	

$\varepsilon,\ N$	32	128	512	1024	2048
$\varepsilon = 2^{-4}$, $1 \le i < N/8$	9.861E-3 1.911	6.668E-4 1.994	4.190E-5 1.999	1.048E-5 2.000	2.620E-6
$7N/8 < i < N$	1.289E-3 1.703	1.076E-4 1.946	7.102E-6 1.988	1.790E-6 1.994	4.495E-7
$3N/8 \le i \le 5N/8$	2.700E-3 1.963	1.743E-4 1.997	1.092E-5 2.000	2.730E-6 2.000	6.826E-7
$5N/8 \le i \le 7N/8$	6.184E-3 1.928	4.116E-4 1.994	2.584E-5 2.000	6.461E-6 2.000	1.615E-6
$3N/8 < i < N/2$	**1.008E-2** 1.866	**7.218E-4** 1.993	**4.538E-5** 1.999	**1.135E-5** 2.000	**2.838E-6**
$N/2 < i < 5N/8$	8.662E-3 1.850	6.255E-4 1.976	4.006E-5 1.995	1.005E-5 1.997	2.517E-6
$N/2$	9.644E-3 1.928	2.535E-3 1.995	6.424E-4 2.000	1.612E-4 1.999	4.033E-5 1.999
$\varepsilon = 2^{-8}$ $1 < i < N/8$	2.225E-2 1.198	3.751E-3 1.641	4.062E-4 1.712	1.257E-4 1.738	3.806E-5
$7N/8 < i < N$	1.746E-2 1.071	1.426E-3 1.558	9.189E-5 1.692	2.299E-5 1.724	5.748E-6
$3N/8 \le i \le 5N/8$	7.962E-4 1.686	3.654E-5 1.960	5.809E-7 1.999	4.303E-8 2.000	2.654E-9
$5N/8 \le i \le 7N/8$	7.115E-4 2.113	4.268E-5 2.716	2.659E-6 3.755	6.647E-7 4.019	1.662E-7
$N/2 < i < 5N/8$	**3.683E-2** 2.046	**5.532E-3** 2.004	**5.609E-4** 2.000	**1.712E-4** 2.000	**5.131E-5**
$N/2 < i < 5N/8$	4.261E-3 1.198	1.986E-3 1.641	3.432E-4 1.712	1.183E-4 1.738	3.835E-5
$N/2$	3.557E-3 1.198	2.645E-3 1.588	3.744E-4 1.688	1.238E-4 1.724	3.924E-5

5 Remarks on Extension to two Dimensions

Some of the results obtained in sections 2,3,4 can be extended to two dimensions. Consider the following singularly perturbed elliptic interface problem

$$-\varepsilon^2 \frac{\partial}{\partial x_1}\left(p_1(x)\frac{\partial u}{\partial x_1}\right) - \varepsilon^2 \frac{\partial}{\partial x_2}\left(p_2(x)\frac{\partial u}{\partial x_2}\right) + q(x)\,u = f(x)\,,$$

$$x \in \Omega\backslash\Gamma,\ \Omega = (0,1)^2$$

with Dirichlet boundary condition $u = 0$ on $\partial\Omega$ and jump conditions on the interface Γ

$$[u] = 0,\ \varepsilon\left[\frac{\partial u}{\partial n_p}\right] = Q(x)\,u + R(x)\ \text{ across } \Gamma.$$

Up to now, we have not found in the literature any generalization of the compact high-order FDS constructed in [5] for 2D problems. This seems possible for the case when Γ is a line parallel to one of the axis Ox_1, Ox_2. The other difficulty is the obtaining asymptotic expansion of the derivatives of u, [6], at the interface layers and corner points.

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