On Q-analogies of generalized Hermite's polynomials

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Abstract

In this paper[1], we discuss the properties of q - polynomials which are hold in any way, especially difference-differential equation and similar relations. Also, the distribution of zeros is studied. At last, we illustrate all by a few examples and make some conjectures.

1 Introduction

In the theory of q-analogies and q-extensions of classical formulas and functions, for $0 < q < 1$, we investigate some new classes of polynomials. These polynomials are *bibasic*, i.e. in their definition participate both, ordinary and q-numbers.

We discuss the properties of q - polynomials which are hold on in any way with respect to standard ones, especially difference-differential relations. This research deals with a problem which makes connection between combinatorial algebra and polynomial theory.

Let

$$[n]_\alpha = \frac{1-q^{\alpha n}}{1-q^\alpha}, \quad [n]_1 = [n],$$

$$[n]_\alpha! = [n]_\alpha [n-1]_\alpha \cdots [1]_\alpha, \qquad [0]_\alpha! = 1$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_\alpha = \frac{[n]_\alpha!}{[k]_\alpha! [n-k]_\alpha!}.$$

The q-exponential function $e_\alpha(x)$ is defined by

$$e_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_\alpha!}, \quad e_1(x) = e(x). \tag{1.1}$$

Lemma 1.1 *The q-exponential function $e_\alpha(x)$ has the properties:*

$$(a)\ \frac{1}{e_\alpha(x)} = e_{-\alpha}(-x) \quad (b)\ e_\alpha(q^\alpha x) = \Big(1 + (q^\alpha - 1)x\Big) e_\alpha(x)$$

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We define q-derivative with index α by

$$D_\alpha = \frac{f(q^\alpha x) - f(x)}{x(q^\alpha - 1)}, \quad D_1 = D.$$

Lemma 1.2 *For q-derivative of the pair of functions is valid:*

$$\begin{aligned}
&(1) \quad D\big(\lambda a(x) + \mu b(x)\big) = \lambda Da(x) + \mu Db(x)\\
&(2) \quad D\Big(a(x)\cdot b(x)\Big) = a(qx)Db(x) + Da(x)b(x)\\
&(3) \quad D\left(\frac{a(x)}{b(x)}\right) = \frac{Da(x)b(x) - a(x)Db(x)}{b(x)b(qx)}.
\end{aligned}$$

Remark. Unfortunately, in q-analysis does not exist theorem about q-derivative of composition, corresponding to known rule in standard analysis. For example, if we want to evaluate q-derivative of $e(x^2)$ we must find it using expansion by series. So, we find

$$De(x^2) \;=\; x(qe(qx^2) + e(x^2)),$$

which is not an "expected" result.

The main problem. Let $\{P_n(x)\}$ be a sequence of the polynomials determined by the generating function $G(x,t)$, i.e.

$$\sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!} = G(x,t).$$

Find a similar generating function $\mathcal{G}(x,t,q)$ $(0 < q < 1)$ which generates a sequence $\{P_n(x;q)\}$ such that

$$\sum_{n=0}^{\infty} P_n(x;q)\frac{t^n}{[n]_\alpha!} = \mathcal{G}(x,t,q),$$

holding on the majority of properties of the polynomial sequence and reducing to known ones when $q \uparrow 1$.

Evidently the q-analogies need not to be unique (for example, see J. Cigler [3]). All papers, we could find, deals with the sequences of polynomials which are orthogonal and satisfy three-term recurrence relation. Our purpose is to make q-extensions of the polynomials satisfying m-th order recurrence relation. That is why we start from the papers of G.V. Milovanović and G.D. Djordjević about generalized Hermite's polynomials [4].

2 On q-analogies of generalized Hermite's polynomials

The generalized Hermite's polynomials are defined by the generating function (see G. Djordjević, G.V. Milovanović [4])

$$\sum_{n=0}^{\infty} h_{n,m}(x)\frac{t^n}{n!} = e^{xt-t^m/m} = G(x,t,m).$$

This function has the property

$$G_t'(x,t,m) = (x-t^{m-1})G(x,t,m). \tag{2.1}$$

In our purpose to determine $\mathcal{G}(x,t,m,q)$ like we mentioned defining our main problem, the troubles come from the fact that we do not have q-derivative rule for composition of functions.

We start with a function which satisfies

$$f(t) = \sum_{k=0}^{\infty} a_k t^k : \quad Df(t) = Ct^{m-1}f(t),\ f(0)=1,\ C = \text{const}.$$

From the condition $f(0) = 1$, we get $a_0 = 1$. Then,

$$\sum_{k=1}^{\infty} a_k[k]t^{k-1} = Ct^{m-1}\sum_{k=0}^{\infty} a_k t^k,$$

where from $a_k = 0,\ k = 1,2,\dots,m-1$ and $[k+1]a_k = Ca_{k-(m-1)}$. Using $[km] = [m][k]_m$, we have

$$a_{km} = \frac{C}{[km]}\ a_{(k-1)m} \Rightarrow a_{km} = \frac{C^k}{[m]^k[k]_m!},$$

i.e.

$$f(t) = \sum_{k=0}^{\infty}\frac{1}{[k]_m!}\Big(\frac{Ct^m}{[m]}\Big)^k = e_m(Ct^m/[m]).$$

We can get a class of generalizations choosing different values for C. We will use $C = q$.

Let the polynomials $\{h_{n,m}(x;q)\}_{n=0}^{\infty}$ $(m \in N,\ q \in (0,1))$ be defined by the next generating function

$$\sum_{n=0}^{\infty} h_{n,m}(x;q)\frac{t^n}{[n]!} = \frac{e(xt)}{e_m(qt^m/[m])} = \mathcal{G}(x,t,m,q). \tag{2.2}$$

Using lemma 1.2 about quotient, we have

$$D\mathcal{G}(x,t,m,q) = \frac{xe(xt)e_m(qt^m/[m]) - e(xt)qt^{m-1}e_m(qt^m/[m])}{e_m(qt^m/[m])e_m(q(qt)^m/[m])}.$$

At last, by lemma 1.1, we note that this function has the property

$$D\mathcal{G}(x,t,m,q) = \frac{x - qt^{m-1}}{1 + q(q-1)t^m}\mathcal{G}(x,t,m,q). \tag{2.3}$$

In the next theorems, we will use $\lfloor n/m \lfloor$ to denote the nearest integer smaller or equal to n/m.

Theorem 2.1 *Every polynomial $h_{n,m}(x;q)$ can be expressed by the next sum relation*

$$h_{n,m}(x;q) = [n]! \sum_{s=0}^{\lfloor n/m \lfloor} (-1)^s \frac{q^s}{[m]^s\ [s]_{-m}!}\ \frac{x^{n-ms}}{[n-ms]!}.$$

Proof. Expanding the generating function in series, we yield

$$\begin{aligned}
\frac{e(xt)}{e_m(qt^m/[m])} &= e(xt)e_{-m}(-qt^m/[m]) \\
&= \sum_{i=0}^{\infty} \frac{(xt)^i}{[i]!} \sum_{j=0}^{\infty} \frac{(-qt^m/[m])^j}{[j]_{-m}!} \\
&= \sum_{k=0}^{m-1} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(xt)^{mj+k}}{[mj+k]!} \frac{(-qt^m/[m])^{n-j}}{[n-j]_{-m}!}.
\end{aligned}$$

Comparing with (2.2), we find

$$h_{mn+k,m}(x;q) = [mn+k]! \sum_{j=0}^{n} \frac{x^{mj+k}}{[mj+k]!} \frac{(-q/[m])^{n-j}}{[n-j]_{-m}!}.$$

Putting $N = mn + k$ and $s = n - j$, we get wanted relation. ∎

Theorem 2.2 *For the polynomial sequence $\{h_{n,m}(x;q)\}_{n=0}^{\infty}$ is valid the next q-differential relation*

$$Dh_{n,m}(x;q) = [n]h_{n-1,m}(x;q).$$

Proof. Using q-deriving of the both sides of (2.2) by x and using the property of the function $e(z)$: $De(az) = a\ e(az)$, where a is a constant. ∎

Theorem 2.3 *The polynomial sequence $\{h_{n,m}(x;q)\}_{n=0}^{\infty}$ satisfies the next m-th order recurrence relation*

$$h_{n+1,m}(x;q) = xh_{n,m}(x;q) - q^{n-m+2}\frac{[n]!}{[n-m+1]!}h_{n-m+1,m}(x;q),$$

$$h_{k,m}(x;q) = x^k, \quad k = 0, 1, \ldots, m-1.$$

Proof. Using q-deriving of the both sides of (2.2) by t and formula (2.3), we yield

$$\sum_{n=1}^{\infty} h_{n,m}(x;q)\frac{t^{n-1}}{[n-1]!} = \frac{x-qt^{m-1}}{1+q(q-1)t^m}\mathcal{G}(x,t,m,q),$$

where from

$$(1+q(q-1)t^m)\sum_{n=0}^{\infty} h_{n+1,m}(x)\frac{t^n}{[n]!} = (x-qt^{m-1})\sum_{n=0}^{\infty} h_{n,m}(x;q)\frac{t^n}{[n]!}.$$

After equalizing the coefficients by t^n in all series, the recurrence relation follows. ∎

Theorem 2.4 *The polynomial* $y = h_{n,m}(x;q)$ *satisfy the* m*-th order* q*-differential equation*

$$q^{n-m+1}D^m y - xDy + [n]y = 0.$$

Proof. By successive applying of theorem 2.2, we get

$$D^m h_{n,m}(x;q) = [n][n-1]\cdots[n-m+1]h_{n-m,m}(x;q).$$

Substituting $n+1$ with n in recurrence relation in theorem 2.3 and applying the previous expression, we yield differential equation for $h_{n,m}(x;q)$. ∎

Theorem 2.5 *For polynomial* $h_{n,m}(x;q)$ *holds*

$$x^n = [n]!\sum_{k=0}^{\lfloor n/m \rfloor} \frac{q^k}{[m]^k\,[k]_m![n-mk]!}\ h_{n-mk,m}(x;q).$$

Proof. Using generating function of polynomials $h_{n,m}(x;q)$ and definition expressions for functions $e(xt)$ and $e_m(qt^m/[m])$ we have

$$e_m(qt^m/[m])\sum_{n=0}^{\infty} h_{n,m}(x;q)\frac{t^n}{[n]!} = e(xt)\ ,$$

i.e.

$$\left(\sum_{n=0}^{\infty}\frac{q^n}{[m]^n[n]_m!}t^{mn}\right)\left(\sum_{n=0}^{\infty} h_{n,m}(x;q)\frac{t^n}{[n]!}\right) = \sum_{n=0}^{\infty}\frac{(xt)^n}{[n]!}\ .$$

Multiplying sums on the left-hand side of the identity we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{x^n}{[n]!}\,t^n &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{q^k}{[m]^k[k]_m![j]!}h_{j,m}(x;q)t^{mk+j}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/m \rfloor}\frac{q^k}{[m]^k\,[k]_m![n-mk]!}\ h_{n-mk,m}(x;q)\,t^n\end{aligned}$$

where from the statement of theorem follows. ∎

Theorem 2.6 *For polynomial* $h_{n,m}(x;q)$ *and a real value* u*, holds*

$$u^n h_{n,m}\big(x/u;q\big) = [n]! \sum_{k=0}^{\lfloor n/m \rfloor} \frac{q^k}{[m]^k\ [k]_m![n-mk]!}\ A_k\ h_{n-mk,m}(x;q),$$

where

$$A_k = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_m u^{mj}.$$

Proof. Using generating function of polynomials

$$\begin{aligned} \mathcal{G}\big(x/u, tu, m, q\big) &= \frac{e_m(qt^m/[m])}{e_m(qt^m u^m/[m])} \mathcal{G}(x,t,m,q) \\ &= e_m(qt^m/[m]) e_{-m}(-qt^m u^m/[m]) \mathcal{G}(x,t,m,q). \end{aligned}$$

The product on the right–hand side is

$$\begin{aligned} &\left(\sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^j q^k u^{mj}}{[m]^k [j]_m! [k-j]_m!} t^{mk} \right) \left(\sum_{i=0}^{\infty} h_{i,m}(x;q) \frac{t^i}{[i]!} \right) \\ &\qquad = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/m \rfloor} \sum_{j=0}^{k} \frac{(-1)^j q^k u^{mj}}{[m]^k [j]_m! [k-j]_m! [n-mk]!} h_{n-mk,m}(x;q) t^n. \end{aligned}$$

Comparing with

$$\mathcal{G}\big(x/u, tu, m, q\big) = \sum_{n=0}^{\infty} \frac{u^n}{[n]!} h_{n,m}\big(x/u;q\big) t^n$$

and assigning

$$A_k = [k]_m! \sum_{j=0}^{k} (-1)^j \frac{u^{mj}}{[j]_m! [k-j]_m!} = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_m u^{mj}$$

state holds. ∎

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