Gauge theory on the fuzzy torus

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Abstract

In this paper[1] a formulation of $U(1)$ gauge theory on a fuzzy torus is discussed. The theory is regulated in both the infrared and ultraviolet. It can be thought of as a non-commutative version of lattice gauge theory on a periodic lattice. The construction of Wilson loops is particularly transparent in this formulation. Following Ishibashi, Iso, Kawai and Kitazawa, we show that certain Fourier modes of open Wilson lines are gauge invariant.

We also introduce charged matter fields which can be thought of as fundamentals of the gauge group. These particles behave like charges in a strong magnetic field and are frozen into the lowest Landau levels. The resulting system is a simple matrix quantum mechanics which should reflect much of the physics of charged particles in strong magnetic fields.

1 The fuzzy torus

In this paper we will be interested in the construction of gauge invariant Wilson loops in a regularized version of non–commutative gauge theory. After this paper was written we realized that the theory we are using and much of our results have previously been discussed by Ambjorn, Makeenko, Nishimura and Szabo [2]. Our presentation is much less general than theirs but because it is also very simple we felt it was worth circulating.

The regularized theory is a non–commutative version of lattice gauge theory on the fuzzy torus. It is patterned after the the Hamiltonian form of lattice gauge theory [9].

The lattice version is an especially intuitive formulation of the non-perturbative theory. For illustrative purposes we will concentrate on the Abelian theory in $2+1$ dimensions. The generalization to higher dimensions and non-abelian gauge groups is straight forward. Our main focus will be on defining the gauge invariant quantities of the theory including closed and open Wilson lines and in formulating the theory of matter in the fundamental representation of the noncommutative algebra of functions.

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The fuzzy 2-torus is defined by non-commuting coordinates U, V satisfying

$$\begin{aligned} U^\dagger U &= V^\dagger V = 1 \\ U^N &= V^N = 1 \\ UV &= VUe^{i\theta} \end{aligned} \tag{1.1}$$

with $\theta = 2\pi/N$. These relations can be represented by $N \times N$ matrices

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & . & . \\ 0 & 0 & 1 & 0 & 0 & 0 & . & . \\ 0 & 0 & 0 & 1 & 0 & 0 & . & . \\ . & . & . & . & . & . & . & . \\ 1 & 0 & 0 & 0 & 0 & 0 & . & . \end{pmatrix} \tag{1.2}$$

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & . & . \\ 0 & e^{i\theta} & 0 & 0 & 0 & 0 & . & . \\ 0 & 0 & e^{2i\theta} & 0 & 0 & 0 & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . & e^{i(N-1)\theta} \end{pmatrix}. \tag{1.3}$$

The physical interpretation of U, V is that they are exponentials of non-commuting periodic coordinates

$$\begin{aligned} U &= \exp \frac{ix}{R} \\ V &= \exp \frac{iy}{R}. \end{aligned} \tag{1.4}$$

Formal manipulations would indicate nontrivial commutation relations for x, y of the form

$$[y, x] = i\theta R^2. \tag{1.5}$$

This relation can be satisfied by introducing a pair of operators on Hilbert space q, p with

$$p = -i\partial_q \tag{1.6}$$

and defining

$$\begin{aligned} y &= qR\theta^{\frac{1}{2}} \\ x &= pR\theta^{\frac{1}{2}}. \end{aligned} \tag{1.7}$$

This equation is sometimes a useful mnemonic but it is not strictly correct; no two finite dimensional matrices can have a commutator with a non-vanishing trace.

Functions on the fuzzy torus are defined by the non–commutative analogue of a Fourier series

$$\phi(U, V) = \sum_{n,m=0}^{N-1} c_{mn} U^n V^m. \tag{1.8}$$

It is convenient to define

$$\begin{aligned} U_{mn} &= e^{-imn\theta}U^m V^n \\ \phi_{mn} &= e^{imn\theta}c_{mn} \\ \phi(U,V) &= \sum_{n,m=0}^{N-1} \phi_{mn}U_{mn}. \end{aligned} \tag{1.9}$$

Note that the U_{mn} satisfy

$$U_{mn}U_{rs} = \exp\frac{1}{2}i\theta(ms-nr) \equiv U_{mn} * U_{rs}. \tag{1.10}$$

Equation (1.8) defines the star-product on the fuzzy torus.

The fuzzy torus is analogous to a periodic lattice with a spacing

$$a = 2\pi R/N. \tag{1.11}$$

This is because the Fourier expansion in (1.7) has only a finite number of terms. In other words there is a largest momentum in each direction

$$p_{max} = 2\pi(N-1)/R. \tag{1.12}$$

Thus the fuzzy torus has both an infrared cutoff length R and an ultraviolet cutoff length $2\pi R/N$.

The operators U, V function as shifts on the periodic lattice. Using the last of equations (1.1) one easily finds

$$\begin{aligned} U\phi(U,V)U^\dagger &= \phi(U,Ve^{i\theta}) \\ U^\dagger\phi(U,V)U &= \phi(U,Ve^{-i\theta}) \\ V\phi(U,V)V^\dagger &= \phi(Ue^{-i\theta},V) \\ V^\dagger\phi(U,V)V &= \phi(Ue^{i\theta},V). \end{aligned} \tag{1.13}$$

More generally

$$U^n V^m \phi(U,V) {V^\dagger}^m {U^\dagger}^n = \phi(Ue^{-im\theta}, Ve^{in\theta}). \tag{1.14}$$

The rule for integration on the fuzzy torus is simply

$$\int U_{mn} = 4\pi^2 R^2 \delta_{m0}\delta_{n0}. \tag{1.15}$$

Noting that

$$TrU_{mn} = N\delta_{m0}\delta_{n0} \tag{1.16}$$

we make the identification

$$\int F(U,V) = \frac{4\pi^2R^2}{N} TrF(U,V). \tag{1.17}$$

2 Gauge theory on the fuzzy torus

In what follows we will work in the temporal gauge in which the time component of the vector potential is zero.

Let us introduce gauge fields on the fuzzy torus in analogy with the link variables of lattice gauge theory [9]. We will explicitly work with the gauge group $U(1)$. The link variable in the x, y direction is called X, Y. The link variables are unitary

$$\begin{aligned} X^\dagger X &= 1 \\ Y^\dagger Y &= 1. \end{aligned} \tag{2.1}$$

The gauge invariance of the theory is patterned on that of lattice gauge theory. Let Z be a unitary, time independent function of U, V, $Z^\dagger Z = 1$. The gauge transformation induced by Z is defined to be

$$\begin{aligned} X' &= Z(U,V)X(U,V)Z^\dagger(Ue^{i\theta},V) \\ Y' &= Z(U,V)Y(U,V)Z^\dagger(U,Ve^{i\theta}) \end{aligned} \tag{2.2}$$

or

$$\begin{aligned} X' &= ZX\ V^\dagger Z^\dagger V \\ Y' &= ZYUZ^\dagger U^\dagger. \end{aligned} \tag{2.3}$$

Let us now construct Wilson loops by analogy with the conventional lattice construction. We will give some examples first. A Wilson line which winds around the x-cycle of the torus at a fixed value of y is given by

$$\begin{aligned} W_x &= TrX(U,V)X(Ue^{i\theta},V)X(Ue^{2i\theta},V)..X(Ue^{i(N-1)\theta},V) \\ &= Tr(XV^\dagger)^N. \end{aligned} \tag{2.4}$$

Similarly

$$W_y = Tr(YU)^N. \tag{2.5}$$

These expressions are gauge invariant under the transformation in (2.3).

Another example of a Wilson loop is the analogue of the plaquette in lattice gauge theory. It is given by

$$\begin{aligned} \mathcal{P} &= TrX(U,V)Y(Ue^{i\theta},V)X^\dagger(U,Ve^{i\theta})Y^\dagger(U,V) \\ &= Tr(X)(V^\dagger YV)(UX^\dagger U^\dagger)(Y^\dagger) \\ &= e^{-i\theta}Tr(XV^\dagger)(YU)(VX^\dagger)(U^\dagger Y^\dagger). \end{aligned} \tag{2.6}$$

The general rule involves drawing a closed oriented chain formed from directed links. A step in the positive (negative) x direction is described by the link operator $XV^\dagger$ ($VX^\dagger$). Similarly a step in the positive (negative) y direction gives a factor YU ($U^\dagger Y^\dagger$). The link operators are multiplied in the order specified by the chain and the trace is taken. In addition there is a factor $e^{-iA\theta}$

where A is the signed Area of the loop in units of the lattice spacing. For a simple contractable clockwise oriented loop with no crossings, A is just the number of enclosed plaquettes.

A simple Lagrangian for the gauge theory can be formed from plaquette operators and kinetic involving time derivatives. The expression

$$Tr\dot{X}^\dagger\dot{X} + \dot{Y}^\dagger\dot{Y} \tag{2.7}$$

is quadratic in time derivatives and is gauge invariant. Again, following the model of lattice gauge theory [9] we choose the action

$$\mathcal{L} = \frac{4\pi^2R^2}{g^2a^2N}Tr\left[\dot{X}^\dagger\dot{X} + \dot{Y}^\dagger\dot{Y} + \frac{e^{-i\theta}}{a^2}(XV^\dagger)(YU)(VX^\dagger)(U^\dagger Y^\dagger) + cc\right]. \tag{2.8}$$

Evidently the operators $XV^\dagger$ and YU play an important role. We therefore define

$$\begin{aligned}\mathcal{X} &= XV^\dagger \\ \mathcal{Y} &= YU.\end{aligned} \tag{2.9}$$

These operators transform simply under gauge transformations:

$$\begin{aligned}\mathcal{X} &\rightarrow Z\mathcal{X}Z^\dagger \\ \mathcal{Y} &\rightarrow Z\mathcal{Y}Z^\dagger.\end{aligned} \tag{2.10}$$

The action is now written in the form

$$\mathcal{L} = \frac{4\pi^2R^2}{g^2a^2N}Tr\left[\dot{\mathcal{X}}^\dagger\dot{\mathcal{X}} + \dot{\mathcal{Y}}^\dagger\dot{\mathcal{Y}} + \frac{e^{-i\theta}}{a^2}\mathcal{X}\mathcal{Y}\mathcal{X}^\dagger\mathcal{Y}^\dagger + cc\right] \tag{2.11}$$

or using (1.9)

$$\mathcal{L} = \frac{N}{g^2}Tr\left[\dot{\mathcal{X}}^\dagger\dot{\mathcal{X}} + \dot{\mathcal{Y}}^\dagger\dot{\mathcal{Y}} + \frac{e^{-i\theta}}{a^2}\mathcal{X}\mathcal{Y}\mathcal{X}^\dagger\mathcal{Y}^\dagger + cc\right]. \tag{2.12}$$

In this form the action is equivalent to that of a $U(N)$ lattice gauge theory formulated on a single plaquette but with periodic boundary conditions of a torus. This appears to be a form of Morita equivalence [3].

If the coupling constant is small, the ground state is determined by minimizing the plaquette term in the Hamiltonian. This is done by setting

$$e^{-i\theta}\mathcal{X}\mathcal{Y}\mathcal{X}^\dagger\mathcal{Y}^\dagger = 1. \tag{2.13}$$

Up to a gauge transformation the unique solution of this equation is

$$\begin{aligned}\mathcal{X} &= V^\dagger \\ \mathcal{Y} &= U\end{aligned} \tag{2.14}$$

or

$$X = Y = 1. \tag{2.15}$$

3 Open Wilson loops

Thus far we have constructed closed Wilson loops. Recall that the construction involves taking a trace. This is the analogue of integrating the location of the Wilson loop over all space. In other words the closed Wilson loop carries no spatial momentum. In a very interesting paper Ishibashi, Iso, Kawai and Kitazawa [7] have argued that there exist gauge invariant operators which correspond to specific Fourier modes of open Wilson lines. These objects are very closely related to the growing dipoles of non–commutative field theory whose size depends on their momentum [4, 8]. Das and Rey [5] have shown that these operators are a complete set of gauge invariant operators. Their importance has been further clarified by Gross, Hashimoto and Itzhaki [6].

Let us consider the simplest example of an open Wilson line, i.e., a single link variable, say X. From the equation (2.3) we see that X is not gauge invariant. But now consider $XV^\dagger = \mathcal{X}$. Under gauge transformations

$$\mathcal{X} \to Z\mathcal{X}Z^\dagger, \tag{3.1}$$

evidently the quantity

$$TrXV^\dagger = Tr\mathcal{X} \tag{3.2}$$

is gauge invariant. Now using (1.4) we identify this quantity as

$$TrXV^\dagger = \frac{N}{4\pi^2R^2}\int Xe^{\frac{-iy}{R}}d^2x. \tag{3.3}$$

Thus we see that a particular Fourier mode of X is gauge invariant.

Let us consider another example in which an open Wilson line consist of two adjacent links, one along the x axis and one along the y axis

$$XV^\dagger YV = \mathcal{X}\mathcal{Y}U^\dagger V. \tag{3.4}$$

Multiplying by $V^\dagger U$ and taking the trace gives

$$Tr(XV^\dagger YV)V^\dagger U = Tr\mathcal{X}\mathcal{Y} = \textit{gauge invariant}. \tag{3.5}$$

But we can also write this as

$$\frac{N}{4\pi^2R^2}\int d^2x(XV^\dagger YV)e^{\frac{-iy}{R}}e^{\frac{-ix}{R}}. \tag{3.6}$$

In other words it is again a Fourier mode of the open Wilson line. In general the particular Fourier mode is related to the separation between the endpoints of the Wilson line by the same relation as that in [4, 8] where it was shown that a particle in non–commutative field theory is a dipole oriented perpendicular to it momentum with a size proportional to the momentum.

4 Fields in the fundamental representation

In this section we will define fields in the fundamental representation of the gauge group. For simplicity we consider non-relativistic particles. Let us begin with what we do *not* mean particles in the fundamental. Define a complex valued field ϕ that takes values in the $N \times N$ dimensional matrix algebra generated by U, V. The gauge transformation properties of ϕ are given by

$$\phi \to Z\phi. \tag{4.1}$$

Note that this is left multiplication by Z and not conjugation. The field ϕ carries a single unit of abelian gauge charge. Although the field has two indices in the N-dimensional space the gauge transformations only act on the left index.

An obvious choice of gauge invariant "hopping" Hamiltonian would be

$$H \sim Tr\phi^\dagger XV^\dagger \phi V\phi^\dagger XU\phi U^\dagger + cc. \tag{4.2}$$

In a non-abelian theory a similar construction can be carried out for quark fields in the fundamental.

We shall mean something different by fields in the fundamental. Such fields have only *one* index. They are vectors rather than matrices in the Hilbert space that the represents the algebra of functions. In the present case they are N component complex vectors $|\psi\rangle$. These fields represent particles moving in a strong magnetic field which are frozen into the lowest Landau level.

Consider the case of non-relativistic particles moving on the non-commutative lattice. The conventional lattice action would be

$$L = L_0 - L_h, \tag{4.3}$$

where

$$L_0 = i\left(\langle\dot{\psi}^\dagger|\psi\rangle - cc\right) \tag{4.4}$$

and L_h is a hopping Hamiltonian. The natural non–commutative version of the hopping term is

$$L_h = \frac{1}{a}\langle\psi|XV^\dagger + YU - 2|\psi\rangle + cc. \tag{4.5}$$

The presence of the link variables X, Y is familiar from ordinary lattice field theory and the $V^\dagger, U$ are the shifts which move ψ. We may also write the hopping term as

$$L_h = \frac{1}{a}\langle\psi|\mathcal{X} + \mathcal{Y}| - 2\psi\rangle + cc. \tag{4.6}$$

Combining (2.12), (4.2) and (4.4)

$$\mathcal{L} = \frac{N}{g^2}Tr\left[\dot{\mathcal{X}}^\dagger\dot{\mathcal{X}} + \dot{\mathcal{Y}}^\dagger\dot{\mathcal{Y}} + \frac{e^{-i\theta}}{a^2}\mathcal{X}\mathcal{Y}\mathcal{X}^\dagger\mathcal{Y}^\dagger + cc\right] + i\langle\dot{\psi}^\dagger|\psi\rangle + \frac{1}{a}\langle\psi|\mathcal{X}+\mathcal{Y}-2|\psi\rangle + cc. \tag{4.7}$$

Let us consider hopping terms in (4.6). In the limit of weak coupling we may use (2.14) to give

$$L_h = \frac{1}{a}\langle\psi|V^\dagger + U - 2|\psi\rangle + cc. \tag{4.8}$$

To get some idea of the meaning of this term let us use (1.4) and expand the exponentials

$$L_h = \frac{1}{a}\langle\psi|\frac{(x^2+y^2)}{R^2}|\psi\rangle + cc \tag{4.9}$$

and using (1.7)

$$L_h = \frac{1}{a}\langle\psi|(p^2+q^2)\theta|\psi\rangle + cc. \tag{4.10}$$

Thus we recognize this term as a harmonic oscillator hamiltonian with an in spectrum of levels spaced by $\theta \sim N^{-1}$. Evidently, in this approximation the particles move in quantized circular orbits around the origin.

This phenomena is related to the fact that the fundamental particles behave like charged particles in a strong magnetic field and are frozen into their lowest Landau levels. Furthermore the LLL's are split by a force attracting the particles to $x = y = 0$. This has a natural interpretation in matrix theory in which the same system appears as a 2-brane and 0-brane with strings connecting them [1].

5 Rational Theta

Thus far we have worked with the equation (1.1) with $\theta = 2\pi/N$. Let us generalize the construction to the case $\theta = 2\pi p/N$ with p relatively prime to N. We continue to define the fuzzy torus by (1.1). Let us define two matrices u, v satisfying

$$\begin{aligned} u^\dagger u &= v^\dagger v = 1 \\ u^N &= v^N = 1 \\ uv &= vue^{\frac{2\pi i\alpha}{N}} \end{aligned} \tag{5.1}$$

such that

$$\alpha p = 1 (mod \mathcal{N}). \tag{5.2}$$

Then it follows that

$$\begin{aligned} U &= u^p \\ V &= v^p \end{aligned} \tag{5.3}$$

satisfies (1.1). Furthermore, u and $v^\dagger$ act as shifts by distance $2\pi R/N$;

$$\begin{aligned} uVu^\dagger &= V\exp\left(\frac{2\pi i}{N}\right) \\ v^\dagger Uv &= U\exp\left(\frac{2\pi i}{N}\right). \end{aligned} \tag{5.4}$$

The basic plaquette is now given by

$$\begin{aligned} \mathcal{P} &= Tr(X)(v^\dagger Y v)(uX^\dagger u^\dagger)(Y^\dagger) \\ &= e^{\frac{2\pi i\alpha}{N}} Tr(Xv^\dagger)(Yu)(vX^\dagger)(u^\dagger Y^\dagger). \end{aligned} \tag{5.5}$$

The final expression for action is essentially the same as in (2.10) except that the factor $e^{i\theta}$ is replaced by $e^{\frac{2\pi i\alpha}{N}}$.

We can now describe one approach to the continuum limit, $a \to 0$. To get to such a limit (1.9) requires that $N \to \infty$. We also want the theta parameter to approach a finite limit. This requires p/N to approach a limit. For example if we want $p/N \to 1/2$ we can choose the sequence $(p = n, N = 2n + 1)$ so that p and N remain relatively prime. In this way p/N can tend to a rational or irrational limit and the lattice spacing will approach zero.

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