New ordering rule and Lie bracket of quantum mechanics

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Abstract

The algebraic multiplication of quantum mechanics is defined as two step operation.[1] The first step is ordinary multiplication while the second is application of the superoperator - symmetrizer. This allows introduction of Poisson bracket as the Lie bracket of quantum mechanics which results in obstruction free quantization.

1 Introduction

The Hermitian operators $\hat{q}$ and $\hat{p}$, acting in the Hilbert space of states, represent coordinate and momentum of quantum mechanical system and correspond to the coordinate q and momentum p of classical mechanical system. For these operators it holds $[\hat{q}, \hat{p}] = i\hbar\hat{I}$. This is one part of the quantization procedure other parts of which are the following. One has to say which operator should correspond to, let say, $q^n p^m$ of classical mechanics. That is where the problem of symmetrized product (ordering rule) of quantum mechanics firstly enters. In Cohen (1966) and Kerner and Sutcliffe (1970) one can find a short review of different propositions for symmetrized product while in Shewell (1959) there is a critical discussion of many ordering rules introduced in there cited references.

On the other hand, the classical mechanical variables have structure of both the algebra and Lie algebra. The Lie product, or bracket, of classical mechanics is the Poisson bracket. It is desired to equip the set of quantum mechanical observables not just with the algebraic (symmetrized) product, but with the Lie algebraic product as well. After Dirac (1958), the canonical quantization prescription says that the Poisson bracket of classical variables has to be translated into the $\frac{1}{i\hbar}$ times the commutator of corresponding quantum observables. Discussions on the algebraic and Lie algebraic structures in quantum mechanics one can find in Emch (1972) and Joseph (1970). In Arens and Babit (1965), Gotay (1980) and (1996) and Chernoff (1995), it was found that these two are interrelated in such a way that there is an obstruction to quantization which

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is manifested through the existence of some contradiction. One can conclude, see Chernoff (1995), that the problem of quantization, which we have sketched above, is impossible.

2 The symmetrized product

In Kerner and Sutcliffe (1970) it was shown that there given propositions for symmetrized product start to differ for quartic monomials in $\hat{q}$ and $\hat{p}$ while they all agree in the most trivial case: $\frac{1}{2}(\hat{q}\hat{p}+\hat{p}\hat{q})$. The way in which this expression was constructed from $\hat{q}$ and $\hat{p}$, that are basic elements of the algebra, could be: $\hat{q}$ and $\hat{p}$ were multiplied as they were ordinary numbers, which has resulted in $\hat{q}\hat{p}$, and then some ordering procedure was applied. Taken in this way, the symmetrized product becomes a two step operation. The second step is the ordering procedure which we shall treat as application of some superoperator $\mathbf{S}$ on the operatorial expressions consisting of $\hat{q}$'s and $\hat{p}$'s.

The action of $\mathbf{S}$ on operatorial sequences is defined by:

$$\mathbf{S}(\hat{q}^{a_1}\hat{p}^{b_1}\cdots\hat{q}^{a_n}\hat{p}^{b_n}) = \frac{(\sum_i a_i)!(\sum_i b_i)!}{(\sum_i a_i+\sum_i b_i)!} \sideset{}{'}\sum_{\substack{c_1,\cdots,c_m \\ d_1,\cdots,d_m \\ \sum_j c_j=\sum_i a_i \\ \sum_j d_j=\sum_i b_i}} \hat{q}^{c_1}\hat{p}^{d_1}\cdots\hat{q}^{c_m}\hat{p}^{d_m}, \tag{1}$$

where $a_i, b_i, c_j, d_j \in \mathbf{N}_o,\ i\in\{1,\cdots,n\}, j\in\{1,\cdots,m\},\ n\in\mathbf{N},\ m=\sum_i a_i+\sum_i b_i+1$ and where the prime over the sum indicates the absence of repeated combinations.

The second defining property of $\mathbf{S}$ is linearity:

$$\mathbf{S}(k\hat{q}^{a_1}\hat{p}^{b_1}\cdots\hat{q}^{a_n}\hat{p}^{b_n}+l\hat{q}^{c_1}\hat{p}^{d_1}\cdots\hat{q}^{c_m}\hat{p}^{d_m}) =$$

$$= k\mathbf{S}(\hat{q}^{a_1}\hat{p}^{b_1}\cdots\hat{q}^{a_n}\hat{p}^{b_n})+l\mathbf{S}(\hat{q}^{c_1}\hat{p}^{d_1}\cdots\hat{q}^{c_m}\hat{p}^{d_m}), \tag{2}$$

where $a_i, b_i, c_j, d_j \in \mathbf{N}_o,\ i\in\{1,\cdots,n\}, j\in\{1,\cdots,m\},\ n,m\in\mathbf{N},\ k,l\in\mathbf{C}$. The third defining property of $\mathbf{S}$ is:

$$\mathbf{S}(k\hbar^m\hat{q}^{a_1}\hat{p}^{b_1}\cdots\hat{q}^{a_n}\hat{p}^{b_n}) = 0, \tag{3}$$

where $a_i, b_i \in \mathbf{N}_o,\ i\in\{1,\cdots,n\},\ n\in\mathbf{N}, k\in\mathbf{C},\ m\in\mathbf{N}$.

The symmetrized product of observables and the partial derivatives of state $\hat{\rho}$ is defined in way which is a slight modification of the above. Precisely, (1) becomes:

$$\mathbf{S}(\hat{q}^{a_1}\hat{p}^{b_1}(\frac{\partial\hat{\rho}}{\partial\hat{r}})^{e_1}\cdots\hat{q}^{a_n}\hat{p}^{b_n}(\frac{\partial\hat{\rho}}{\partial\hat{r}})^{e_n}) =$$

$$= \frac{(\sum_i a_i)!(\sum_i b_i)!}{(\sum_i a_i + \sum_i b_i + \sum_i e_i)!} \sum'_{\substack{c_1,\cdots,c_m \\ d_1,\cdots,d_m \\ f_1,\cdots,f_m \\ \sum_j c_j = \sum_i a_i \\ \sum_j d_j = \sum_i b_i \\ \sum_j f_j = \sum_i e_i = 1}} \hat{q}^{c_1}\hat{p}^{d_1}(\frac{\partial\hat{\rho}}{\partial\hat{r}})^{f_1}\cdots\hat{q}^{c_m}\hat{p}^{d_m}(\frac{\partial\hat{\rho}}{\partial\hat{r}})^{f_m}, \quad (4)$$

where $a_i, b_i, c_j, d_j \in \mathbf{N}_o,\ i \in \{1,\cdots,n\}, j \in \{1,\cdots,m\},\ n \in \mathbf{N},\ m = \sum_i a_i + \sum_j b_j + 1,\ e_i, f_j \in \{0,1\}$ and $\hat{r} \in \{\hat{q},\hat{p}\}$. Modifications of the other properties follow straightforwardly.

One can easily answer what shall be the symmetrized product of two monomials, let say, $\hat{q}^a \circ \hat{p}^b$ and $\hat{q}^c \circ \hat{p}^d$ (both monomials are symmetrized in the above way). Namely, the result should be the expression gained after the expressions similar to the RHS of (1) were multiplied as it would be done in the c-number case and after $\mathbf{S}$ was applied on that. The final result would be:

$$\frac{(a+c)!(b+d)!}{(a+b+c+d)!}(\hat{q}^{a+c}\hat{p}^{b+d} + \cdots + \hat{p}^{b+d}\hat{q}^{a+c}),$$

where, in the parenthesis, there should stand only different combinations of $a+b+c+d$ operators, where $a+c$ are of the one kind and $b+d$ are of the other. In more compact notation, this reads:

$$(\hat{q}^a \circ \hat{p}^b) \circ (\hat{q}^c \circ \hat{p}^d) = \mathbf{S}(\mathbf{S}(\hat{q}^a \cdot \hat{p}^b) \cdot \mathbf{S}(\hat{q}^c \cdot \hat{p}^d)) = \hat{q}^{a+c} \circ \hat{p}^{b+d}.$$

Similar result would appear in the case of general quantum mechanical observables $\sum_i c_i \hat{q}^{n_i} \circ \hat{p}^{m_i}$ and $\sum_j d_j \hat{q}^{r_j} \circ \hat{p}^{s_j}$.

3 The Lie bracket of quantum mechanics

We propose the substitution of the commutator divided by $i\hbar$ with the operatorial form of Poisson bracket:

$$\{f(\hat{q},\hat{p}), g(\hat{q},\hat{p})\}_{\mathbf{S}} = \frac{\partial f(\hat{q},\hat{p})}{\partial\hat{q}} \circ \frac{\partial g(\hat{q},\hat{p})}{\partial\hat{p}} - \frac{\partial g(\hat{q},\hat{p})}{\partial\hat{q}} \circ \frac{\partial f(\hat{q},\hat{p})}{\partial\hat{p}}. \quad (5)$$

One can convince oneself that the just defined "symmetrized" Poisson bracket has all properties of the Lie product. On the other hand, one can easily show that:

$$\{F(\hat{q},\hat{p}), |\psi\rangle\langle\psi|\}_{\mathbf{S}} = \frac{1}{i\hbar}[F(\hat{q},\hat{p}), |\psi\rangle\langle\psi|], \quad (6)$$

where $F(\hat{q},\hat{p})$ is the general element of the quantum mechanical algebra. From this expression it follows that the dynamical equation of quantum mechanics can be reexpressed. Obviously, resulting equation is operatorial version of the Liouville equation:

$$\frac{\partial\rho(\hat{q},\hat{p},t)}{\partial t} = \{H(\hat{q},\hat{p}), \rho(\hat{q},\hat{p},t)\}_{\mathbf{S}}. \quad (7)$$

It is understood that the Hamiltonian is $H(\hat{q},\hat{p}) = \sum_i c_i \hat{q}^{n_i} \circ \hat{p}^{m_i}$.

Regarding the problem of possibility for obstruction to quantization if it is based on the symmetrized Poisson bracket as the Lie product, let us just mention that if there was some equation for classical variables, then the same equation will hold for their quantum counterparts since the symmetrized product and the symmetrized Poisson bracket imitate the adequate operations in c-number case. Said in more descriptive way, the equation in quantum mechanics will hold since it differs from the corresponding equation of classical mechanics only in that there are hats above coordinate and momentum and there is $\circ$ instead of $\cdot$. Consequently, this quantization is, we believe, unambiguous, *i.e.*, obstruction free *in toto*.

4 Concluding remarks

For the introduction of the symmetrized product that we have proposed it was necessary to look on the operators of coordinate and momentum as on the basic elements. The only way to make the approach consistent and, we believe, to avoid ambiguities is to apply the same ordering procedure in all situations. The unavoidable second step of all multiplications, *i.e.*, the application of symmetrizer **S**, due to properties that it produces all distinct combinations of coordinate and momentum and annihilates $\hbar$, makes the approach obstruction free. When the operators introduced at the beginning of quantization are looked from the point of view of either algebra or Lie algebra, there will be no contradictory statements because **S** within $\circ$ and $\{\ ,\ \}_{\mathbf{S}}$ forms these two operations to be the complete imitation of the corresponding ones of classical mechanics. The generalization of the symmetrized product and Poisson bracket for more than one degree of freedom is influenced by requirements coming from physics, see Prvanović and Marić (2000) and references therein. This topic we shall consider in the forthcoming article.

References

[1] R. Arens and D. Babit, Journal of Mathematical Physics **6** (1965), 1071.

[2] P.R. Chernoff, Journal of Functional Analysis **130** (1995), 255.

[3] L. Cohen, Journal of Mathematical Physics **7** (1966), 781.

[4] P.A.M. Dirac, *Principles of Quantum Mechanics*, Oxford University Press, New York, 1958.

[5] G.G. Emch, *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*, Wiley, New York, 1972.

[6] M.J. Gotay, International Journal of Theoretical Physics **19** (1980), 139.

[7] M.J. Gotay, H.B. Grundling and G.M. Tuynman, Journal of Nonlinear Sciences **6** (1996), 469.

[8] A. Joseph, Communications in Mathematical Physics **17** (1970), 210.

[9] E.H. Kerner and W.G. Sutcliffe, Journal of Mathematical Physics **11** (1970), 391.

[10] S. Prvanović and Z. Marić, International Journal of Theoretical Physics **39** (2000), 1469.

[11] J.R. Shewell, American Journal of Physics **27** (1959), 16.

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