

**A THEOREM ON A REPRESENTATION OF
*–REGULARLY VARYING SEQUENCES**

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Abstract. In this paper we shall prove a theorem on a representation of *–regularly varying sequences in the sense of Karamata [1].

1. Introduction and results

Consider the sequences (c_n) , $c_n > 0$ ($n \in N$) which are nondecreasing and satisfy the following asymptotic condition

$$(1) \quad \lim_{\lambda \rightarrow 1+} \overline{\lim}_{n \rightarrow +\infty} \frac{c_{[n\lambda]}}{c_n} = 1.$$

Such sequences are called *–regularly varying, and they have an important role in the analysis of divergent sequential processes (see e.g. [6]).

Condition (1) is weaker than the Karamata condition of regular variability and stronger than the condition of \mathcal{O} –regular variability (see e.g. [2], [5], [8] and [9]).

Relation (1) obviously means that for any such sequence, the function

$$k_0(\lambda) = \overline{\lim}_{n \rightarrow +\infty} \frac{c_{[n\lambda]}}{c_n}$$

is right continuous at $\lambda = 1$.

We shall first define an important class of functions.

CRV is the class of all measurable functions $F : [a, +\infty) \mapsto (0, +\infty)$ ($a > 0$) such that $F(x(t)) \sim F(y(t))$ as $t \rightarrow +\infty$, for any two functions x, y with the properties

$$\lim_{t \rightarrow +\infty} x(t) = \lim_{t \rightarrow +\infty} y(t) = +\infty,$$

and $x(t) \sim y(t)$ as $t \rightarrow +\infty$.

This class is investigated in detail in the papers [3] and [4].

Proposition 1. *If (c_n) is an arbitrary nondecreasing sequence of positive numbers, then the following assertions are equivalent:*

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- (a) $c_{[x]}$ ($x \geq 1$) belongs to the class CRV ;
- (b) (c_n) is a $*$ -regularly varying sequence.

Corollary 1. *The function $k_0(\lambda)$ is defined for every $\lambda > 0$.*

Corollary 2. *For any $*$ -regularly varying sequence (c_n) we have*

$$\lim_{\lambda \rightarrow 1} k_0(\lambda) = 1.$$

Corollary 3. *For any $*$ -regularly varying sequence (c_n) and arbitrary positive s, t we have $k_0(st) \leq k_0(s)k_0(t)$.*

Corollary 4. *For an arbitrary $*$ -regularly varying sequence (c_n) , the function $k_0(\lambda)$ is continuous in $\lambda > 0$.*

Corollary 5. *If (c_n) is an arbitrary nondecreasing sequence of positive numbers, then the following assertions are equivalent:*

- (a) (c_n) is a $*$ -regularly varying sequence;
- (b) For arbitrary mappings $K_1, K_2: \mathbb{N} \mapsto \mathbb{N}$ with the properties

$$\lim_{n \rightarrow +\infty} K_1(n) = \lim_{n \rightarrow +\infty} K_2(n) = +\infty$$

and $K_1(n) \sim K_2(n)$ as $n \rightarrow +\infty$, one has $c_{K_1(n)} \sim c_{K_2(n)}$ as $n \rightarrow +\infty$;

- (c) $\lim_{n \rightarrow +\infty} \lim_{\lambda \rightarrow 1} \frac{c_{[\lambda n]}}{c_n} = 1$.

The asymptotic condition (c) is in fact the Schmidt convergence condition (see e.g. [7]).

The next theorem is a representation theorem for $*$ -regularly varying sequences.

Theorem 1. *Let (c_n) be an arbitrary nondecreasing sequence of positive numbers. Then the following assertions are equivalent:*

- (a) (c_n) is a $*$ -regularly varying sequence;
- (b) There is an $n_0 \in \mathbb{N}$ such that

$$c_n = \exp \left\{ \tilde{\mu}_n + r_1(\log n) + \sum_{k=n_0}^n \frac{\delta_k}{k} \right\}$$

for every $n \geq n_0$, where the sequence $\tilde{\mu}_n \rightarrow 0$ as $n \rightarrow +\infty$, r_1 is a bounded and uniformly continuous function on the interval $[\log n_0, +\infty)$, and (δ_n) is a bounded sequence.

Remark. It can be proved that all previous results remain true for sequences (c_n) which are not necessarily nondecreasing if the condition $\lambda \rightarrow 1+$ in (1) is replaced by $\lambda \rightarrow 1$.

2. Proofs of results

Proof of Proposition 1. By some results from [4,p.454] the implication (a) \implies (b) is trivial.

(b) \implies (a). Let (c_n) be a $*$ -regularly varying sequence. From (1) we find that

$$\overline{\lim}_{n \rightarrow +\infty} \frac{c_{[\lambda n]}}{c_n} < +\infty$$

for some $a > 1$ and all $\lambda \in [1, a)$. Hence, the function $k_0(\lambda)$ is finite for all $\lambda \in [1, a)$. For an arbitrary fixed $\lambda \in [1, a)$ define

$$k(\lambda) = \overline{\lim}_{x \rightarrow +\infty} \frac{c_{[\lambda x]}}{c_{[x]}}.$$

Then

$$\begin{aligned} k(\lambda) &\leq \overline{\lim}_{x \rightarrow +\infty} \frac{c_{[\lambda[x]]}}{c_{[x]}} \cdot \overline{\lim}_{x \rightarrow +\infty} \frac{c_{[\lambda x]}}{c_{[\lambda[x]]}} \leq \\ &\leq k_0(\lambda) \cdot \overline{\lim}_{x \rightarrow +\infty} \frac{c_{[\lambda x]}}{c_{[\lambda[x]]}}. \end{aligned}$$

Since $(\lambda x)/[\lambda[x]] \rightarrow 1+$ as $x \rightarrow +\infty$, we find that for a fixed λ and all $x \geq x_0$, $(\lambda x)/[\lambda[x]] \in [1, 1 + \delta]$. Hence,

$$\overline{\lim}_{x \rightarrow +\infty} \frac{c_{[\lambda x]}}{c_{[\lambda[x]]}} = \overline{\lim}_{x \rightarrow +\infty} \frac{c_{\left[\frac{\lambda x}{[\lambda[x]]} [\lambda[x]]\right]}}{c_{[\lambda[x]]}} \leq k_0(1 + \delta)$$

for all $\delta > 0$. Thus for a given λ and all $\delta \in (0, a - 1)$ we have $k(\lambda) \leq k_0(\lambda) \cdot k_0(1 + \delta)$. Since the function k_0 is right continuous at $\lambda = 1$, we have that $k(\lambda) \leq k_0(\lambda)$ for a fixed $\lambda \in [1, a)$. Therefore $\lim_{\lambda \rightarrow 1+} k(\lambda) \leq \lim_{\lambda \rightarrow 1+} k_0(\lambda) \leq 1$. Since $c_{[x]}$ is an nondecreasing function, we have that $\lim_{\lambda \rightarrow 1+} k(\lambda) = 1$ and $\lim_{\lambda \rightarrow 1-} k(\lambda) \leq 1$. Finally, since the function $k(\lambda)$ is defined on an interval $[1 - a', 1 + a'']$ ($a', a'' > 0$), we get that $c_{[x]}$ belongs to the class of \mathcal{O} -regularly varying functions ([1]), that is we have

$$k(\lambda) = \overline{\lim}_{x \rightarrow +\infty} \frac{c_{[\lambda x]}}{c_{[x]}} < +\infty$$

for all $\lambda > 0$.

Hence we get $1 = k(1) \leq k(\lambda) \cdot k(1/\lambda)$, and consequently $k(1/\lambda) \rightarrow c \geq 1$ as $\lambda \rightarrow 1+$. This gives that $\lim_{\lambda \rightarrow 1-} k(\lambda) = 1$, that is $\lim_{\lambda \rightarrow 1} k(\lambda) = 1$, which by some results from [4,p.454] yields that $c_{[x]}$ belongs to the class *CRV*. \square

Remark. Since $k_0(\lambda) \leq k(\lambda)$ for $\lambda > 0$, and $k(\lambda) \leq k_0(\lambda) \cdot k_0(1 + \delta)$ for all $\delta > 0$, we find that $k_0(\lambda) = k(\lambda)$ for all $\lambda > 0$. Therefore, using the properties of the index function of *CRV* functions, Corollaries 2, 3 and 4 follow immediately.

Proof of Corollary 5. (a) \implies (b). If (c_n) is a $*$ -regularly varying sequence, then the function $F(x) = c_{[x]}$ ($x \geq 1$) belongs to the class *CRV*. Next, let K_1 and K_2 be arbitrary functions with the properties in (b). Then, it is easily seen that the functions k_1, k_2 defined by

$$k_i(x) = K_i(n) \quad (x \in [n, n+1); n \in \mathbb{N}; i = 1, 2)$$

have the properties $\lim_{x \rightarrow +\infty} k_1(x) = \lim_{x \rightarrow +\infty} k_2(x) = +\infty$, and $k_1(x) \sim k_2(x)$ as $x \rightarrow +\infty$. Hence $\lim_{x \rightarrow +\infty} (F(k_1(x))/F(k_2(x))) = 1$, and consequently $c_{K_1(n)} \sim c_{K_2(n)}$ as $n \rightarrow +\infty$.

(b) \implies (c). Assume (b), and let (λ_k) be an arbitrary sequence such that $\lim_{k \rightarrow +\infty} \lambda_k = 1$. Then

$$S_k = \frac{[\lambda_k k] + 1}{k} \geq \lambda_k \geq \bar{S}_k = \frac{[\lambda_k k]}{k}$$

for every $k \in \mathbb{N}$, and $S_k, \bar{S}_k \rightarrow 1$ as $k \rightarrow +\infty$. Putting $n_1(k) = [\lambda_k k]$, $n_2(k) = [\lambda_k k] + 1$, $n_3(k) = k$, we have that $n_1, n_2, n_3 : \mathbb{N} \mapsto \mathbb{N}$, $n_1(k), n_2(k), n_3(k) \rightarrow +\infty$ as $k \rightarrow +\infty$ and

$$\lim_{k \rightarrow +\infty} \frac{n_1(k)}{n_3(k)} = \lim_{k \rightarrow +\infty} \frac{n_2(k)}{n_3(k)} = 1.$$

Since

$$\begin{aligned} 1 &= \lim_{k \rightarrow +\infty} \frac{c_{\frac{n_1(k)}{n_3(k)} n_3(k)}}{c_{n_3(k)}} = \lim_{k \rightarrow +\infty} \frac{c_{[\lambda_k k]}}{c_k} \leq \\ &\leq \lim_{k \rightarrow +\infty} \frac{c_{\frac{n_2(k)}{n_3(k)} n_3(k)}}{c_{n_3(k)}} = 1, \end{aligned}$$

we get condition (c).

(c) \implies (a). Assuming (c), and taking an arbitrary $\epsilon > 0$, we find some $k_0 \in \mathbb{N}$ and a $\delta > 0$ such that $1 - \epsilon \leq c_{[\lambda k]}/c_k \leq 1 + \epsilon$ whenever $k \geq k_0$ and $|1 - \lambda| < \delta$. Hence

$$1 - \epsilon \leq \overline{\lim}_{k \rightarrow +\infty} \frac{c_{[\lambda k]}}{c_k} \leq 1 + \epsilon,$$

thus $|k_0(\lambda) - 1| \leq \epsilon$ if $|1 - \lambda| < \delta$. This means that

$$\lim_{\lambda \rightarrow 1} \overline{\lim}_{k \rightarrow +\infty} \frac{c_{[\lambda k]}}{c_k} = 1,$$

so that we have (a). \square

Proof of Theorem 1. (a) \implies (b). Let (c_n) be a *-regularly varying sequence. Then, by Proposition 1, the function $F(x) = c_{[x]}$ ($x \geq 1$) belongs to the class CRV . By [4] there is a $B > 0$ such that for any $n \geq B$ one has

$$c_n = F(n) = \exp \left\{ \tilde{\mu}(n) + r(\log n) + \int_B^n \frac{\epsilon(t)}{t} dt \right\},$$

where the functions $\epsilon(x)$ and $\tilde{\mu}(x)$ are bounded measurable functions in $[B, +\infty)$, $r(x)$ is a uniformly continuous bounded function in $[\log B, +\infty)$, and we have $\lim_{x \rightarrow +\infty} \tilde{\mu}(x) = 0$. Putting $n_0 = [B] + 1$ and $s = \int_B^{n_0} \frac{\epsilon(t)}{t} dt \in \mathbb{R}$, we get that the function $r_1(t) = r(t) + s$ is bounded and uniformly continuous in $t \in [\log n_0, +\infty)$. Hence, for $n \geq n_0$ we have

$$c_n = \exp \left\{ \tilde{\mu}_n + r_1(\log n) + \sum_{k=n_0}^n \frac{\delta_k}{k} \right\},$$

where $\lim_{n \rightarrow +\infty} \tilde{\mu}_n = 0$, r_1 is a bounded and uniformly continuous function on the interval $[\log n_0, +\infty)$, $\delta_k = k \int_{k-1}^k \frac{\epsilon(t)}{t} dt$ ($k \geq n_0 + 1$) and $\delta_{n_0} = 0$. Finally, we find that

$$\begin{aligned} |\delta_k| &= k \cdot \left| \int_{k-1}^k \frac{\epsilon(t)}{t} dt \right| \leq k \cdot \sup_{t \geq k-1} |\epsilon(t)| \cdot \log \left(1 + \frac{1}{k-1} \right) \leq \\ &\leq 2 \sup_{t \geq k-1} |\epsilon(t)| < M, \end{aligned}$$

for any $k \geq n_0 + 1$, since the function $\epsilon(t)$ is bounded on $[B, +\infty)$.

(b) \implies (a). Assuming (b), let $\lambda > 1$ and $n \geq n_0$. Then

$$\frac{c_{[\lambda n]}}{c_n} = \exp \left\{ \tilde{\mu}_{[\lambda n]} - \tilde{\mu}_n + r_1(\log [\lambda n]) - r_1(\log n) + \sum_{k=n+1}^{[\lambda n]} \frac{\delta_k}{k} \right\}.$$

Since

$$\lim_{\lambda \rightarrow 1+} \overline{\lim}_{n \rightarrow +\infty} \left(\tilde{\mu}_{[\lambda n]} - \tilde{\mu}_n \right) = 0,$$

$$\left| \sum_{k=n+1}^{[\lambda n]} \frac{\delta_k}{k} \right| \leq \sup_{k \geq n+1} |\delta_k| \cdot \int_{n+1}^{[\lambda n]+1} \frac{dt}{t-1} = \sup_{k \geq n+1} |\delta_k| \cdot \log \frac{[\lambda n]}{n},$$

that is $\lim_{\lambda \rightarrow 1+} \overline{\lim}_{n \rightarrow +\infty} \left| \sum_{k=n+1}^{[\lambda n]} \frac{\delta_k}{k} \right| = 0$, and r_1 is a uniformly continuous function on the interval $[\log n_0, +\infty)$, we get that

$$\lim_{\lambda \rightarrow 1+} \overline{\lim}_{n \rightarrow +\infty} \frac{c_{[\lambda n]}}{c_n} = 1.$$

In other words, (c_n) is a $*$ -regularly varying sequence. \square

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