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ARENS-ROYDEN AND THE SPECTRAL LANDSCAPE

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Abstract. In this note we attempt to deconstruct the Arens-Royden Theorem, and to offer an abstraction of the spectral picture of an operator. We also show dramatically why the invertible group of operators on certain product Banach spaces is not connected.

Suppose A is a Banach algebra (by default complex, with identity 1): we shall write

0.1 $$A^{-1} = \{a \in A : 1 \in Aa \cap aA\}$$

for the open subgroup of *invertible* elements, and A_0^{-1} for the connected component of the identity in A^{-1}: it turns out ([10], [13] Theorem 7.11.4) that

0.2 $$A_0^{-1} = \mathrm{Exp}(A) = \{e^{c_1}e^{c_2}\ldots e^{c_k} : k \in \mathbf{N}, c \in A^k\}$$

coincides with the *generalized exponentials*, the subgroup generated by the exponentials. $\mathrm{Exp}(A)$ is open, relatively closed in A^{-1}, connected and a normal subgroup: thus we can form the quotient group,

0.3 $$\kappa(A) = A^{-1}/\mathrm{Exp}(A),$$

the *abstract index group* [7] of A. Now we can state [2];[7];[8];[22];[24];[25];[26]

1. Theorem (Arens-Royden Mark I) *If A is commutative then*

1.1 $$\kappa(A) \cong H^1(\sigma(A), \mathbf{Z}),$$

the first Cech cohomology group of the "maximal ideal space" of A.

Specifically we shall interpret elements of the maximal ideal space $\sigma(A) \subseteq A^*$ as bounded multiplicative linear functionals on A; this includes sending $1 \in A$ to $1 \in \mathbf{C}$. We offer no formal definition of Cech cohomology: but if we believe the Arens-Royden theorem Mark I then it must apply to the algebra $C(\sigma(A))$ of continuous functions on $\sigma(A)$, which has of course the same maximal ideal space

1.2 $$\sigma\ C(\sigma(A)) \cong \sigma(A)\ ,$$

and whose abstract index group therefore offers an interpretation of the Cech cohomology. We arrive at

2. Theorem (Arens-Royden Mark II) *If A is commutative then*

2.1 $$\kappa(A) \cong \kappa\ C(\sigma(A)).$$

We can sharpen the statement a little more: the isomorphism is not any old isomorphism (remember the James space !), but a specific isomorphism derived from the Gelfand mapping. Stepping back a little, suppose $T : A \to B$ is a bounded multiplicative linear mapping of Banach algebras: in particular, for arbitrary $a, a' \in A$,

2.2 $$T(a'a) - T(a')T(a) = 0 = T(1) - 1.$$

For example if $B = \mathbf{C}$ then $T \in \sigma(A)$. It is clear - whether or not T is bounded - that

2.3 $$T(A^{-1}) \subseteq B^{-1};$$

if T is also bounded (or not [21]!) then $T(A_0^{-1}) \subseteq B_0^{-1}$ and also - of course the same thing -

2.4 $$T\ \mathrm{Exp}(A) \subseteq \mathrm{Exp}(B).$$

Thus $T : A \to B$ induces a mapping of abstract index groups,

2.5 $$\kappa(T) : \kappa(A) \to \kappa(B) :$$

κ god bless it is a functor.

All this applies in particular to the *Gelfand mapping*: we define

2.6 $$\Gamma_A : A \to C(\sigma(A))$$

by setting - whether or not A is commutative -

2.7 $$\Gamma_A(a)(\varphi) = \varphi(a)\ (\varphi \in \sigma(A), a \in A).$$

It is this sort of thing that gives abstract linear analysis a bad name!

If $\sigma(A)$ is empty we will not trouble ourselves about the interpretation of (2.7). Our sharpened version of the Arens-Royden theorem says that the isomorphism is induced by the Gelfand mapping:

3. Theorem (Arens-Royden Mark III) *If A is commutative then*

3.1 $$\kappa(\Gamma_A) : \kappa(A) \to \kappa\, C(\sigma(A)) \text{ is one-one onto.}$$

Stepping back again, suppose $T : A \to B$ is a homomorphism of Banach algebras. From (2.3) it follows that there is inclusion

3.2 $$A^{-1} \subseteq T^{-1}(B^{-1}) \subseteq A.$$

It is natural - think of the Calkin homomorphism and Atkinson's theorem - to describe ([11];[12];[13] Definition 7.6.1) $T^{-1}(B^{-1}) \subseteq A$ as the *T-Fredholm* elements of A. We are tempted to make a definition: we shall say that a homomorphism $T : A \to B$ *has the Gelfand property* ([13] (9.6.0.1)) iff

3.3 $$T^{-1}(B^{-1}) \subseteq A^{-1}.$$

Thus *Gelfand's theorem* can be succinctly stated:

4. Theorem (Gelfand) *If A is commutative then $\Gamma_A : A \to C(\sigma(A))$ has the Gelfand property.*

It is now tempting to try and deconstruct the Arens-Royden theorem, and to divide the statement into an "Arens theorem" and a "Royden theorem". Let us - tentatively - suggest that a homomorphism $T : A \to B$ have the *Arens property* if the index mapping $\kappa(T)$ is one-one, and the *Royden property* if $\kappa(T)$ is onto. Thus we say that $T : A \to B$ has the Arens property provided there is inclusion

4.1 $$A^{-1} \cap T^{-1}\mathrm{Exp}(B) \subseteq \mathrm{Exp}(A),$$

and that $T : A \to B$ has the Royden property provided

4.2 $$B^{-1} \subseteq T(A^{-1}) \cdot \mathrm{Exp}(B).$$

The Arens-Royden theorem therefore says that if A is commutative then the Gelfand mapping has both the Arens and the Royden properties.

5. Example $A = \mathrm{Holo}(\mathbf{S}) \subseteq C(\mathbf{S}) = B$ *the algebra of functions holomorphic in a neighbourhood of the circle* $\mathbf{S} = \partial \mathbf{D}$, *embedded* $T : A \to B$ *in the continuous functions.*

It is familiar ([17];[13] Theorem 7.10.7) that the abstract index group $\kappa(B) \cong \mathbf{Z}$ is essentially the integers. Now the "Arens condition" (4.1) says that if a function $b \in B$ invertible on $\mathbf{S}$ is holomorphic near $\mathbf{S}$ and has a continuous logarithm on $\mathbf{S}$ then that logarithm is holomorphic there.

In contrast the "Royden condition" (4.2) says that every continuous function $b \in B^{-1}$ invertible on the circle has holomorphic functions in its coset $b\mathrm{Exp}(B)$. Indeed if $b \in B^{-1}$ we can take $a = z^n$ with $n \in \mathbf{Z}$ given by the *topological degree* or "winding number" of $b/|b| : \mathbf{S} \to \mathbf{S}$.

6. Example *The Calkin homomorphism* $T : A \to B$, *where* $A = B(X)$ *is the bounded operators on a Banach space and* $B = B(X)/K(X)$ *is its quotient by the ideal of compact operators.*

Generally if $T : A \to B$ is onto there is ([10];[25] §4.8;[13] Theorem 7.11.5) equality

6.1 $$T\ \mathrm{Exp}(A) = \mathrm{Exp}(B);$$

for such T the "Arens condition" (4.1) takes the form

6.2 $$A^{-1} \cap (\mathrm{Exp}(A) + T^{-1}(0)) \subseteq \mathrm{Exp}(A),$$

while the "Royden condition" reduces to

6.3 $$B^{-1} \subseteq T(A^{-1}).$$

For example if $A = B(X)$ for a Hilbert space X then Kuiper's theorem ([4] Theorem I.6.1) says that the invertible group of $A = B(X)$ is connected: $A^{-1} = \mathrm{Exp}(A)$. This makes the "Arens property" (4.1) a triviality. The "Royden property" in this case reduces to the connectedness of B^{-1}, which never happens. If instead $B^{-1} = \mathrm{Exp}(B)$ is connected then the "Royden property" (4.2) becomes a triviality, and the "Arens property" only happens when A^{-1} is also connected.

By a *spectrum* K we shall understand, in the first instance, a nonempty compact subset $K \subseteq \mathbf{C}$ of the complex plane: this works because every compact set is the spectrum of something. If $K \subseteq \mathbf{C}$ is a spectrum then so is its topological boundary ∂K and so is its *connected hull*

6.4 $$\eta K = K \cup \bigcup \{H : H \in \mathrm{Hole}(K)\},$$

where [10],[15] we write $\mathrm{Hole}(K)$ for the (possibly empty) set of bounded components of the complement of K in $\mathbf{C}$: thus $\mathbf{C}\setminus\eta K$ is the unique unbounded component of $\mathbf{C} \setminus K$.

7. Definition *By a "spectral picture" we shall understand an ordered pair (K, ν) in which K is a spectrum and ν is a mapping from* $\mathrm{Hole}(K)$ *to the integers* $\mathbf{Z}$.

If $K \subseteq \mathbf{C}$ is a spectrum and if $f : U \to \mathbf{C}$ is a continuous mapping whose domain $U \subseteq \mathbf{C}$ includes K then it is clear that $f(K)$ is again a spectrum, where of course

7.1 $$f(K) = \{f(\lambda) : \lambda \in K\}.$$

We shall pay particular attention to functions

7.2 $$f \in \mathrm{Holo}(\eta K),$$

for which $U \supseteq \eta K$ is open in $\mathbf{C}$ and on which f is *holomorphic*. Recall ([15] Proposition 2.2)

7.3 $$\partial f(K) \subseteq f(\partial K) \text{ and } f(\eta K) \subseteq \eta f(K);$$

also if $L \in \mathrm{Hole}\ f(K)$ and $H \in \mathrm{Hole}\ K$ then ([5] Proposition 3.1; Lemma 3.5)

7.4 $$\text{either } L \cap f(H) = \emptyset \text{ or } L \subseteq f(H).$$

8. Definition *If (K, ν) is a spectral picture and if $f \in \mathrm{Holo}(\eta K)$ then*

8.1 $$f(K, \nu) = (f(K), \nu_f),$$

where for each hole $L \in \mathrm{Hole}\ f(K)$ we set

8.2 $$\nu_f(L) = \sum\{N_f(L, H)\nu(H) : L \subseteq f(H)\},$$

where if $\mu \in L \subseteq f(H)$ the equation $f(\lambda) = \mu$ has exactly $N_f(L, H)$ solutions $\lambda \in H$:

8.3 $$N_f(L, H) = \#\{\lambda \in H : f(\lambda) = \mu\} = \#\ f^{-1}(\mu) \cap H.$$

Of course, via Rouché's theorem [17] from complex analysis, the number $N_f(L,H)$ is independent of the choice of $\mu \in L$. If $L \in \mathrm{Hole}\ f(K)$ is not a subset of $f(H)$ for any hole $H \in \mathrm{Hole}(K)$ then we interpret the right hand side of (2.2) as the integer 0 .

The fundamental example of a spectral picture comes from Fredholm theory:

9. Example *If X is a Banach space and $A = B(X)/K(X)$ the Calkin algebra on X and $T \in B(X)$ is a bounded linear operator on X then take*

9.1 $$a = [T]_{K(X)}\ ;\ K = \sigma_A(a) = \sigma_{ess}(T)$$

and for each $H \in \mathrm{Hole}(K)$ define

9.2 $$\nu(H) = \mathrm{index}(T - \lambda I) \text{ with } \lambda \in H.$$

Fredholm theory guarantees that $\nu(H)$ is well-defined (independent of the choice of $\lambda \in H$); then according to [5](3.7) or [18](8.8) the spectral picture of $f(T)$ is the image, in the sense of (8.1), of the spectral picture of T. Indeed writing $f(z) - \mu = g(z)\prod_j (z - \lambda_j)$ argue

9.3 $$\nu_f(L) = \mathrm{index}(f(T) - \mu I) = \mathrm{index}\ g(T) + \sum_j \mathrm{index}(T - \lambda_j I).$$

A "spectral landscape" in a Banach algebra A will be an ordered pair (K,ν) in which K is a spectrum and ν is a mapping from $\mathrm{Hole}(K)$ to the abstract index group $\kappa(A)$; as a favour we ask that the cosets $\nu(H)$ mutually commute. Then the "spectral landscape" of an element $a \in A$ is what we would expect:

10. Definition *The spectral landscape of an element $a \in A$ of a Banach algebra A is the ordered pair $(K,\nu) = (\sigma_A(a), \iota_\sigma(a))$ where $\iota_\sigma(a) : \mathrm{Hole}(\sigma(a)) \to \kappa(A)$ is defined by setting*

10.1 $$\iota_\sigma(a)(\lambda) = \mathrm{Exp}(A)(a - \lambda) \text{ if } \lambda \in H \in \mathrm{Hole}(A).$$

The image of a spectral landscape (K,ν) by a polynomial $f : \mathbf{C} \to \mathbf{C}$, or more generally a holomorphic function $f \in \mathrm{Holo}(\eta K)$, will be the ordered pair $(f(K), \nu_f)$, where

10.2 $$\nu_f(L) = \prod\{\nu(H)^{N_f(L,H)} : L \subseteq f(H)\}.$$

The spectral mapping theorem follows by the argument of (9.3):

11. Theorem *If $a \in A$ and $f \in \mathrm{Holo}(\eta\sigma(a))$ then the spectral landscape of $f(a)$ is the image of the spectral landscape of a:*

11.1 $$(\sigma f(a), \iota_\sigma f(a)) = (f\sigma(a), (\iota_\sigma a)_f).$$

Proof. If $f : U \to \mathbf{C}$ is holomorphic on an open set $U \supseteq \eta\sigma(a))$ containing the spectrum and all its holes and if $\mu \in L \in \mathrm{Hole}\ \sigma f(a)$ then

11.2 $$f(z) - \mu = g(z) \prod_j (z - \lambda_j),$$

where $g : U \to \mathbf{C}$ is holomorphic and nonvanishing on $\eta\sigma(a)$, giving

11.3 $$(f(a) - \mu)\mathrm{Exp}(A) = \prod_j \mathrm{Exp}(A)(a - \lambda_j) \bullet$$

The spectral landscape also co-operates with passage to a subalgebra: if $T : A \to B$ is a homomorphism we recall (2.5) $\kappa(T) : \kappa(A) \to \kappa(B)$. If T is bounded, and also bounded below, so that effectively A is a closed subalgebra of B, then for arbitrary $a \in A$

11.4 $$\partial\sigma_A(a) \subseteq \sigma_B(Ta) \subseteq \sigma_A(a) \subseteq \eta\sigma_B(Ta)$$

with

11.5 $$\mathrm{Hole}\ \sigma_A(a) \subseteq \mathrm{Hole}\ \sigma_B(Ta).$$

Now the behaviour of the spectral landscape under passage to a closed subalgebra $A \subseteq B$ is that for arbitrary $a \in A$

11.6 $$\iota_{B\sigma}(Ta)J = \kappa(T)\iota_{A\sigma}(a),$$

where J is the restriction mapping.

If for example $A = B(X)$ for a Banach space X then the spectral picture of an operator $a \in A$ is what we might call the *essential spectral landscape* of the element a, or rather the combination (K, ν) where $K = \sigma_{ess}(a)$ and $\nu = \mathrm{Index} \circ \iota_\sigma^{ess}$: there is a well defined mapping $\mathrm{Index} : \kappa(B(X)/K(X)) \to \mathbf{Z}$ from the abstract index group of a Calkin algebra to the integers. The actual spectral landscape of $a \in A$ maps holes in the actual spectrum of A into the quotient group $\kappa(A)$: when $A = B(X)$ for a Hilbert space X then $A^{-1} = \mathrm{Exp}(A)$ is connected, so that this becomes trivial.

In real life the "spectral picture" [23] needs to be augmented by the addition of "pseudo-holes", and the group of integers extended to include ∞ and $-\infty$. Thus the spectral landscape might be seen as the superposition of a "left" and a "right" landscape, while the abstract index group becomes the intersection of a left and a right semigroup:

12. Definition *If A is a Banach algebra then the abstract left index semi-group of A is the quotient*

12.1 $$\kappa_{left}(A) = A^{-1}_{left}/\mathrm{Exp}(A) = \{\mathrm{Exp}(A)a : a \in A^{-1}_{left}\},$$

while the abstract right index semi-group of A is the quotient

12.2 $$\kappa_{right}(A) = A^{-1}_{left}/\mathrm{Exp}(A) = \{a\mathrm{Exp}(A) : a \in A^{-1}_{right}\}.$$

The left spectral landscape of $a \in A$ is the ordered pair (K, ν) where $K = \sigma^{left}_A(a)$ is the left spectrum of a in A and $\nu = \iota^{left}_\sigma : \mathrm{Hole}(K) \to \kappa_{left}(A)$ takes right cosets:

12.3 $$\nu(H) = \mathrm{Exp}(A)(a - \lambda) \textit{ if } \lambda \in H \in \mathrm{Hole}(K).$$

Continuity and the discrete topology ensure that ν is well defined. We should remark that if $1 \in G \subseteq H$, G a subgroup of the semigroup H, then the set of left cosets $\{xG : x \in H\}$ forms a partition of H, in the sense that any two are either disjoint or coincide; the same is true of right cosets. When we specialise to the semigroup $H = A^{-1}_{left} \subseteq A$ of left invertibles then the left cosets are subsets of the right:

13. Theorem *If A is a Banach algebra then $\kappa_{left}(A)$ and $\kappa_{right}(A)$ are semigroups, with the discrete topology.*

Proof. Suppose $x'x = 1$: then if $0 \neq \lambda \in \mathbf{C}$

13.1 $$xA^{-1}x' \subseteq A^{-1} + \lambda(1 - xx') \subseteq x'A^{-1}x :$$

the inverse of $xax' - \lambda(1 - xx')$ is $xa^{-1}x' - \lambda^{-1}(1 - xx')$. Also (cf [13] Theorem 7.11.2)

13.2 $$\textit{the sets } A^{-1} \textit{ and } A^{-1}_{left} \setminus A^{-1} \textit{ are open in } A :$$

if $a \in A^{-1}$ then $\{a(1-x) : \|x\| < 1\} \subseteq A^{-1}$ and if $a \in A^{-1}_{left} \setminus A^{-1}$ then $\{a(1-x) : \|x\| < 1\} \subseteq A^{-1}_{left} \setminus A^{-1}$.

From the first part of (13.1) it follows

13.3 $$x'x = 1 \Longrightarrow xA^{-1} \subseteq A^{-1}x.$$

From (13.3) we are able to successfully multiply right cosets to form the semigroup A^{-1}_{left}/A, which by (13.2) acquires the discrete topology.

All this holds equally well with the generalized exponentials $\mathrm{Exp}(A)$ in place of A^{-1}: for example (cf [13] (7.11.3.4)) $xe^c x' + 1 - xx' = e^{xcx'}$. In addition

13.4 $\mathrm{Exp}(A)$ *is the connected component of* 1 *in* A^{-1}_{left} $\bullet$

When we specialise to the Calkin algebra $A = B(X)/K(X)$ then there are well-defined mappings $\mathrm{Index} : \kappa_{left}(A) \to \mathbf{Z} \cup \{-\infty\}$ and $\mathrm{Index} : \kappa_{right}(A) \to \mathbf{Z} \cup \{\infty\}$. The argument of Theorem 11 extends to the left spectral landscape:

13.5 $$(\sigma^{left} f(a), \iota^{left}_\sigma f(a)) = (f\sigma^{left}(a), (\iota^{left}_\sigma a)_f).$$

The original Arens-Royden theorem has an extension to operator matrices [3], [25]: if A is commutative and $T = \Gamma_A$ is the Gelfand homomorphism then

13.6 $$\kappa(T^{n\times n}) : \kappa(A^{n\times n}) \to \kappa(C\sigma(A)^{n\times n})$$

is an isomorphism.

Gonzalez and Aiena [1],[9] have used operator matrices to throw light on one way in which the invertible group of Banach space operators can fail to be connected:

14. Theorem *If* $G = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ *is a Banach algebra with blocks then*

14.1 $$1 - MN \subseteq A^{-1} \text{ and } 1 - NM \subseteq B^{-1}$$

if and only if there is equality

14.2 $$G^{-1} = \begin{pmatrix} A^{-1} & M \\ N & B^{-1} \end{pmatrix},$$

in which case

14.3 $$\mathrm{Exp}(G) \subseteq \begin{pmatrix} \mathrm{Exp}(A) & M \\ N & \mathrm{Exp}(B) \end{pmatrix}.$$

Proof. Recall [14] that for G to be a Banach algebra the diagonal blocks A and B must also be Banach algebras while the off diagonals M and N must be A B bimodules; products MN and NM lie in A and B respectively. Now if $1 - MN \subseteq A^{-1}$ and $1 - NM \subseteq B^{-1}$ then

$$\begin{pmatrix} 1 & m \\ n & 1 \end{pmatrix}\begin{pmatrix} 1 & -m \\ -n & 1 \end{pmatrix} = \begin{pmatrix} 1-mn & 0 \\ 0 & 1-nm \end{pmatrix} = \begin{pmatrix} 1 & -m \\ -n & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ n & 1 \end{pmatrix}$$

and then

$$\begin{pmatrix} a & m \\ n & b \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} 1 & mb^{-1} \\ na^{-1} & 1 \end{pmatrix};$$
$$\begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix}\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} 1 & a^{-1}m \\ b^{-1}n & 1 \end{pmatrix};$$

also

$$\begin{pmatrix} a & m \\ n & b \end{pmatrix} \in G^{-1} \Longrightarrow \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & m \\ n & b \end{pmatrix} + \begin{pmatrix} 0 & -m \\ -n & 0 \end{pmatrix}$$
$$\in \begin{pmatrix} a & m \\ n & b \end{pmatrix}(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} A & M \\ N & B \end{pmatrix}\begin{pmatrix} 0 & -m \\ -n & 0 \end{pmatrix})$$
$$= \begin{pmatrix} a & m \\ n & b \end{pmatrix}\begin{pmatrix} 1-Mn & -Am \\ -Bn & 1-Nm \end{pmatrix} \subseteq \begin{pmatrix} A & M \\ N & B \end{pmatrix}^{-1}.$$

This shows that (14.1) implies (14.2); conversely

$$\begin{pmatrix} 1-mn & 0 \\ 0 & 1-nm \end{pmatrix} = \begin{pmatrix} 1 & m \\ n & 1 \end{pmatrix}\begin{pmatrix} 1 & -m \\ -n & 1 \end{pmatrix} \in \begin{pmatrix} A & M \\ N & B \end{pmatrix}^{-1}$$
$$\Longrightarrow 1 - mn \in A^{-1}\ ,\ 1 - nm \in B^{-1}.$$

Now if $\begin{pmatrix} a & m \\ n & b \end{pmatrix}$ is in $\mathrm{Exp}(G)$ then there is $\begin{pmatrix} a_t & m_t \\ n_t & b_t \end{pmatrix}_{(0 \le t \le 1)}$ connecting $\begin{pmatrix} a & m \\ n & b \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that (a_t) and (b_t) connect $a \in A^{-1}$ and $b \in B^{-1}$ to $1 \in A$ and $1 \in B$, giving (14.3) •

In fact each of the two conditions in (14.1) implies the other, and one of the inclusions in (14.2) implies the other. It is also clear from (14.1) that

14.4 $$1 - MN \subseteq \mathrm{Exp}(A) \textit{ and } 1 - NM \subseteq \mathrm{Exp}(B);$$

thus also (cf [22]!) each of the two conditions in (14.4) implies the other.

These arguments can be used to show (cf [20]) that the invertible group on certain Banach spaces is not connected:

15. Example *If* $X = Y \times Z$ *with* $Y = \ell_p$ *and* $Z = \ell_q$ *with* $q \neq p$ *then*

15.1 $$T = \begin{pmatrix} u & r \\ 0 & v \end{pmatrix} \in BL^{-1}(X,X) \setminus \mathrm{Exp}\ BL(X,X),$$

where u *and* v *are the forward and backward shifts on* Y *and* Z *respectively and* $r : Z \to Y$ *is the rank one projection on the first co-ordinate.*

Proof. If u' and v' are the forward and backward shifts on Z and Y respectively and $r' : Y \to Z$ the same projection then

15.2 $$v'u = 1 \neq uv' = 1 - rr' \text{ and } vu' = 1 \neq u'v = 1 - r'r,$$

so that T is invertible with

15.3 $$T^{-1} = \begin{pmatrix} v' & 0 \\ r' & u' \end{pmatrix}.$$

At the same time [1],[9] the whole of $BL(Y,Z)$ and of $BL(Z,Y)$ consist of inessential operators. By Theorem 14 therefore, for the Calkin quotient of T to be in the connected component of the identity it would be necessary for the Calkin quotients of u and v to be generalized exponentials, and hence in particular for

15.4 $$\mathrm{index}(u) = \mathrm{index}(v) = 0.$$

Since this is not the case T cannot be a generalized exponential •

Alternatively the Calkin mapping

$$\Phi : BL(X,X) = \begin{pmatrix} A' & M' \\ N' & B' \end{pmatrix} \to \begin{pmatrix} A & M \\ N & B \end{pmatrix}$$

has the property that for arbitrary $a' \in A' = BL(Y,Y)$, $b' \in B' = BL(Z,Z)$ there is ([12];[16];[13] Theorem 7.6.2) implication

15.5 $$\Phi(a') \in \mathrm{Exp}(A) \Longrightarrow a' \in a'(A')^{-1}a' \ , \ \Phi(b') \in \mathrm{Exp}(B) \Longrightarrow b' \in b'(B')^{-1}b' \ ,$$

and now ([12];[16];[13] (9.3.4.3)) a left invertible element with an invertible generalized inverse must also be right invertible.

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