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ON SOME SECOND ORDER CESÀRO DIFFERENCE SPACES OF NON–ABSOLUTE TYPE

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ABSTRACT. The second order Cesàro sequence spaces of non–absolute type $X_p(\Delta^2)$ for $1 \leq p \leq \infty$ were defined and studied in [1]. It seems, however, that the characterizations of their β–duals given there do not hold for $1 < p \leq \infty$. In this paper, we determine the β–duals $(X_p(\Delta^2))^\beta$ for $1 \leq p \leq \infty$.

1. INTRODUCTION

Let ω denote the set of all complex sequences $x = (x_k)_{k=1}^\infty$. We write ℓ_∞, c, c_0 and cs and ℓ_1 for the sets of all bounded, convergent, null sequences and for the sets of all convergent and absolutely convergent series, respectively, and $\ell_p = \{x \in \omega : \sum_{k=1}^\infty |x_k|^p < \infty\}$ for $1 < p < \infty$. As usual, e and $e^{(n)}$ $(n = 1, 2, \dots)$ are the sequences with $e_k = 1$ $(k = 1, 2, \dots)$, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \neq n)$. We write $\mathbf{n}^\lambda = (n^\lambda)_{n=1}^\infty$, $\mathbf{k}^\lambda = (k^\lambda)_{k=1}^\infty$ for $\lambda \in \mathbb{R}$ and $\mathbf{1/n} = (1/n)_{n=1}^\infty$.

Let $x, y \in \omega$ and $X \subset \omega$. We write $xy = (x_k y_k)_{k=1}^\infty$, $x^{-1} * Y = \{a \in \omega : ax \in Y\}$, $x^\beta = x^{-1} * cs$ and

$$X^\beta = \bigcap_{x \in X} x^\beta = \{a \in \omega : \textstyle\sum_{k=1}^\infty a_k x_k \text{ converges for all } x \in X\}$$

for the *β–dual of* X.

Given any infinite matrix $A = (a_{nk})_{n,k=1}^\infty$ of complex numbers and any sequence x, we write $A_n = (a_{nk})_{k=1}^\infty$ for the sequence in the n–th row of A, $A_n(x) = \sum_{k=1}^\infty a_{nk} x_k$ $(n = 1, 2, \dots)$ and $A(x) = (A_n(x))_{n=1}^\infty$, provided $A_n \in x^\beta$ for all n. Furthermore, $X_A = \{x \in \omega : A(x) \in X\}$ denotes the *matrix domain of* A *in* X. We define the matrices Σ, Δ and E by $\Sigma_{nk} = 1$ $(1 \leq k \leq n)$, $\Sigma_{nk} = 0$ $(k > n)$, $\Delta_{nn} = 1$, $\Delta_{n,n+1} = -1$, $\Delta_{nk} = 0$ (otherwise), $e_{nk} = 1$ $(k \geq n+1)$ and $e_{nk} = 0$ $(1 \leq k \leq n)$ for all n, and write $\Delta^2 = \Delta\Delta$. Then the sets $X_p(\Delta^2) = ((\mathbf{1/n})^{-1} * \ell_p)_{\Sigma\Delta^2}$ are the *second order Cesàro*

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difference sequence spaces of non–absolute type (cf. [1]). Throughout, let $q = \infty$ for $p = 1$, $q = p/(p-1)$ for $1 < p < \infty$ and $q = 1$ for $p = \infty$. We write $R = E(a)$ for $a \in cs$, that is $R_n = \sum_{k=n+1}^{\infty} a_k$ $(n = 1, 2 \dots)$. For any subset X of ω, we write $S(X) = \{x \in X : x_1 = x_2 = 0\}$. Since $(\Delta\Sigma)(x) = (-x_{n+1})_{n=1}^{\infty}$ for all $x \in S(X)$, we have $S(X_p(\Delta^2)) = S((\mathbf{1}/\mathbf{n})^{-1} * \ell_p)_\Delta)$. Furthermore, obviously $(X_p(\Delta^2))^\beta = (S(X_p(\Delta^2)))^\beta$. In [1], it was stated that

$$(X_p(\Delta^2))^\beta = (\mathbf{n}^{-1} * \ell_q)_E \text{ for } 1 \le p \le \infty. \tag{1.1}$$

First we observe that (1.1) does not hold for $1 < p \le \infty$. To see this, we put,

$$R_k = \begin{cases} \dfrac{1}{n^{2/q}} \dfrac{1}{2^n} & (k = 2^n) \\ 0 & (k \ne 2^n) \end{cases} \text{ and } x_k = \begin{cases} 0 & (k = 1, 2) \\ k^{1+\frac{1}{2q}} & (k = 3, 4, \dots). \end{cases}$$

This yields

$$\sum_{k=1}^{\infty} |kR_k|^q = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty, \text{ that is } R \in \mathbf{n}^{-1} * \ell_q,$$

$$|\Delta_k(x)| = (k+1)^{1+\frac{1}{2q}} - k^{1+\frac{1}{2q}} \le \left(1 + \frac{1}{2q}\right)(k+1)^{\frac{1}{2q}} \quad (k = 3, 4, \dots)$$

by the mean value theorem, hence, for $1 < p < \infty$ with $M = (1 + 1/(2q))^p$,

$$\sum_{k=3}^{\infty} \left(\frac{|\Delta_k(x)|}{k}\right)^p \le 2^p M \cdot \sum_{k=1}^{\infty} \frac{1}{(k+1)^{(1-1/q)p + \frac{p}{2q}}} = 2^p M \cdot \sum_{k=1}^{\infty} \frac{1}{(k+1)^{1+\frac{p}{2q}}} < \infty,$$

that is $x \in S(X_p(\Delta^2))$, and we have for $p = \infty$, that is $q = 1$

$$\frac{|\Delta_k(x)|}{k} \le \frac{3}{2} \frac{\sqrt{k+1}}{k} \ (k = 3, 4, \dots), \text{ that is } x \in S(X_\infty(\Delta^2)),$$

but

$$|a_{2^n} x_{2^n}| = |(R_{2^n - 1} - R_{2^n})\, x_{2^n}| = \frac{1}{n^{2/q}} \frac{1}{2^n} 2^{n(1+\frac{1}{2q})} = \frac{2^{\frac{n}{2q}}}{n^{2/q}} \to \infty \ (n \to \infty),$$

that is $ax \notin cs$.

In this paper, we determine the β-duals of $X_p(\Delta^2)$ for $1 \le p \le \infty$.

2. The β–duals of the sets $X_p(\Delta^2)$

Now we determine $(S(X_p(\Delta^2)))^\beta$ for $1 \le p \le \infty$. We start with

Remark 2.1. *We put*

$$D_p^{(1)} = \left(\mathbf{n}^{-1} * \ell_q\right)_E \text{ and } D_p^{(2)} = \begin{cases} \left(\left(\mathbf{n}^{2-1/p}\right)^{-1} * \ell_\infty\right)_E & (1 \le p < \infty) \\ \left((\mathbf{n}^2)^{-1} * c_0\right)_E & (p = \infty). \end{cases}$$

Then $D_1^{(1)} = D_1^{(2)}$, *but if* $1 < p \le \infty$ *then neither* $D_p^{(1)} \subset D_p^{(2)}$ *nor* $D_p^{(2)} \subset D_p^{(1)}$.

Proof. The identity $D_1^{(1)} = D_1^{(2)}$ is obvious from the definition of the sets $D_p^{(1)}$ and $D_p^{(2)}$. Let $1 < p \le \infty$. We define the sequence a by $a_1 = a_2 = 0$ and

$$a_n = \frac{1}{(\log(n-1))^{1/q}} \frac{1}{(n-1)^{1+1/q}} - \frac{1}{(\log n)^{1/q}} \frac{1}{n^{1+1/q}} \text{ for } n \ge 3.$$

Since $1/p + 1/q = 1$ for $1 < p < \infty$ and $1/p = 0$ for $p = \infty$, we obtain

$$R_n = \frac{1}{(\log n)^{1/q}} \frac{1}{n^{1+1/q}} \text{ for } n \ge 2,$$

$$\left|n^{2-1/p} R_n\right| = \left|n^{1+1/q} R_n\right| = \frac{1}{(\log n)^{1/q}} \to 0 \ (n \to \infty) \text{ and}$$

$$\sum_{n=2}^\infty |nR_n|^q = \sum_{n=2}^\infty \frac{1}{n \log n} = \infty, \text{ hence } a \in D_p^{(2)} \setminus D_p^{(1)}.$$

We define the sequence a by $a_{2^n} = -a_{2^n+1} = -n^{-2}\, 2^{-n}$ $(n = 1, 2, \dots)$ and $a_k = 0$ otherwise. This yields $R_{2^n} = n^{-2}\, 2^{-n}$ $(n = 1, 2, \dots)$ and $R_k = 0$ otherwise, $\sum_{k=1}^\infty |kR_k|^q = \sum_{n=1}^\infty n^{-2q} < \infty$ and

$$\left|(2^n)^{2-1/p} R_{2^n}\right| = (2^n)^{1+1/q} \frac{1}{n^2\, 2^n} = \frac{2^{n/q}}{n^2} \to \infty \ (n \to \infty),$$

hence $a \in D_p^{(1)} \setminus D_p^{(2)}$. □

Theorem 2.1. *We put* $M_1 = D_1^{(1)}$ *and* $M_p = D_p^{(1)} \cap D_p^{(2)}$ *for* $1 <\le \infty$. *Then we have* $(S(X_p(\Delta^2)))^\beta = M_p$. *Moreover, if* $a \in (S(X_p(\Delta^2)))^\beta$ *then*

$$\sum_{k=1}^\infty a_k x_k = \sum_{k=1}^\infty R_k \Delta(x_k) \text{ for all } x \in S(X_p(\Delta^2)). \tag{2.1}$$

Proof. We write $Y = S(X_p(\Delta^2))$.
First we assume $a \in M_p$. Let $y \in Y$. Then $x = (\mathbf{n})^{-1}\Delta(y) \in \ell_p$, and $a \in D_p^{(1)}$, that is $\mathbf{n}R \in \ell_q$, implies $\|R\Delta(y)\|_1 \le \|\mathbf{n}R\|_q \|x\|_p < \infty$, hence

$$R\Delta(y) \in cs. \tag{2.2}$$

We define the matrix A and the sequence z by

$$a_{mk} = \begin{cases} -kR_m & (1 \le k \le m) \\ 0 & (k > m) \end{cases} \quad \text{and } z_m = A_m(x) \ (m = 1, 2, \dots). \tag{2.3}$$

If $p = 1$, then $\mathbf{n}R \in \ell_\infty$ implies that there is a constant C such that

$$|a_{mk}| = k|R_m| \le \frac{k}{m} C \text{ for all } m, \tag{2.4}$$

and since obviously $D_p^{(1)} \subset D_1^{(1)}$ for all $p \ge 1$, this inequality holds for all p. Thus

$$\lim_{m\to\infty} a_{mk} = 0 \text{ for each fixed } k. \tag{2.5}$$

Now (2.4) and (2.5) together imply $A \in (l_1, c_0)$ by [2, Example 8.4.1A, p. 126].

If $1 < p < \infty$, then $a \in D_p^{(2)}$ yields

$$\sup_m \sum_{k=1}^\infty |a_{mk}|^q = \sup_m \left(|R_m|^q \sum_{k=1}^m k^q \right) \le \sup_m \left| m^{1+1/q} R_m \right|^q < \infty.$$

This and (2.5) together imply $A \in (\ell_p, c_0)$ by [2, Example 8.4.5D, p. 129].

If $p = \infty$, $a \in D_\infty^{(2)}$ yields

$$\lim_{m\to\infty} \sum_{k=1}^\infty |a_{mk}| = \lim_{m\to\infty} R_m \frac{m(m+1)}{2} = 0.$$

This and (2.5) together imply $A \in (\ell_\infty, c_0)$ by [2, Theorem 1.7.19, p. 17].

Finally, (2.2), $A \in (\ell_p, c_0)$ and Abel's summation by parts

$$\sum_{k=1}^m a_k y_k = -\sum_{k=1}^m R_k \Delta_k(y) - R_m y_{m+1} = -\sum_{k=1}^m R_k \Delta_k(y) - z_m \text{ for all } m \tag{2.6}$$

together imply $ay \in cs$, that is $a \in Y^\beta$. Thus we have shown $M_p \subset Y^\beta$.

Conversely we assume $a \in Y^\beta$. Then $ax \in cs$ for all $x \in Y$. First $b = (0, 0, 1, \dots) \in Y$ implies $ab \in cs$, and so the sequence R is defined. Let $y \in Y$ be given. Then $x = \Delta(y) \in (\mathbf{1}/\mathbf{n})^{-1} * \ell_p$, $y_k = -\sum_{j=1}^{k-1} x_j$ for $k = 1, 2, \dots$, and we have

$$\sum_{k=1}^n a_k y_k = -\sum_{k=1}^{n-1} \left(\sum_{j=k+1}^n a_j \right) x_k \text{ for } n = 1, 2, \dots. \tag{2.7}$$

Defining the matrix $B = (b_{nk})_{n,k=1}^{\infty}$ by

$$b_{nk} = \begin{cases} -\sum\limits_{j=k+1}^{n} a_j & (1 \le k \le n-1) \\ 0 & (k > n) \end{cases} \qquad (n = 1, 2, \dots),$$

we conclude $B \in ((\mathbf{1}/\mathbf{n})^{-1} * \ell_p, c) \subset ((\mathbf{1}/\mathbf{n})^{-1} * \ell_p, \ell_\infty)$. Now $B \in ((\mathbf{1}/\mathbf{n})^{-1} * \ell_p, \ell_\infty)$ if and only if $\tilde{B} \in (\ell_p, \ell_\infty)$ where $\tilde{b}_{nk} = kb_{nk}$ for all n and k, and so

$$S_q = \sup_n \|\tilde{B}_n\|_q < \infty \tag{2.8}$$

by [2, Example 8.4.1A, p. 126 ($p = 1$), 8.4.5D, p. 129 ($1 < p < \infty$) and 8.4.5A, p. 129 ($p = \infty$)].

If $p = 1$, then (2.8) yields $|kR_k| = \lim_{n\to\infty} |k \sum_{j=k+1}^{n} a_j| \le S_1$ for all k, that is $R \in \mathbf{k}^{-1} * \ell_\infty$, hence $a \in M_1$. Thus we have shown $Y^\beta \subset D_1^{(1)} = M_1$.

Now let $p > 1$ and $m \in \mathbb{N}$ be given. Then (2.8) yields

$$\sum_{k=1}^{m-1} k^q \left| \sum_{j=k+1}^{n} a_j \right|^q \le \|\tilde{B}_n\|_q^q \le S_q^q \text{ for all } n \ge m,$$

and so $\sum_{k=1}^{m-1} |kR_k|^q = \lim_{n\to\infty} \sum_{k=1}^{m-1} k^q |\sum_{j=k+1}^{n} a_j|^q \le S_q^q$. Since $m \in \mathbb{N}$ was arbitrary, we have $\mathbf{k}R \in \ell_q$. that is $a \in D_p^{(1)}$. We define the matrix A and the sequence z as in (2.3). Then we have $z \in c$ by (2.3), that is $A \in (\ell_p, c)$.

For $1 < p < \infty$, there is a constant C such that

$$C \left(|R_m| m^{1+1/q} \right)^q \le |R_m|^q \sum_{k=1}^{m} k^q = \sup_n \|A_n\|_q^q \text{ for all } m,$$

and $\sup_n \|A_n\|_q^q < \infty$ by [2, Example 8.4.5D, p. 129], that is $R \in (\mathbf{n}^{1+1/q})^{-1} * \ell_\infty$, hence $a \in D_p^{(2)}$. Thus we have shown $Y^\beta \subset D_p^{(1)} \cap D_p^{(2)} = M_p$.

If $p = \infty$, then $\sum_{k=1}^{\infty} |a_{mk}| = 0$ by [2, Theorem 1.7.18 (ii), p. 15], since $\lim_{m\to\infty} a_{mk} = k \lim_{m\to\infty} R_m = 0$ for each fixed k. But we have $\sum_{k=1}^{\infty} |a_{mk}| = R_m \sum_{k=1}^{m} k = R_m m(m+1)/2$ and so $R \in (\mathbf{n}^2)^{-1} * c_0$, that is $a \in D_\infty^{(2)}$. Thus we have shown $Y^\beta \subset D_\infty^{(1)} \cap D_\infty^{(2)} = M_\infty$.

If $a \in (S(X_p(\Delta^2)))^\beta$, then (2.1) is obvious from (2.6). □

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