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ON SOME GENERALIZED DIFFERENCE SEQUENCE SPACES DEFINED BY A MODULUS FUNCTION

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Abstract. **The idea of difference sequence spaces was introduced by Kızmaz [9] and generalized by Et and Çolak [6]. In this paper we introduce the sequence spaces $[V,\lambda,f,p]_0(\Delta^r,E)$, $[V,\lambda,f,p]_1(\Delta^r,E)$, $[V,\lambda,f,p]_\infty(\Delta^r,E)$, $S_\lambda(\Delta^r,E)$ and $S_{\lambda_0}(\Delta^r,E)$, where E is any Banach space, examine them and give various properties and inclusion relations on these spaces. We also show that the space $S_\lambda(\Delta^r,E)$ may be represented as a $[V,\lambda,f,p]_1(\Delta^r,E)$ space.**

1. Introduction

Let w be the set of all sequences real or complex numbers and ℓ_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x=(x_k)$ with the usual norm $\|x\|=\sup|x_k|$, where $k\in\mathbb{N}=\{1,2,\dots\}$, the set of positive integers.

Let $\lambda=(\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1}\le\lambda_n+1$, $\lambda_1=1$.

The generalized de la Vallée-Poussin mean is defined by

$$t_n(x)=\frac{1}{\lambda_n}\sum_{k\in I_n}x_k,$$

where $I_n=[n-\lambda_n+1,n]$ for $n=1,2,\dots$.

A sequence $x=(x_k)$ is said to be (V,λ) −summable to a number L [11] if $t_n(x)\to L$ as $n\to\infty$.

If $\lambda_n=n$, then (V,λ) −summability and strongly (V,λ) −summability are reduced to $(C,1)$ −summability and $[C,1]$ −summability, respectively.

The idea of difference sequence spaces was introduced by Kızmaz [9]. In 1981, Kızmaz [9] defined the sequence spaces

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for $X = \ell_\infty$, c and c_0, where $\Delta x = (x_k - x_{k+1})$.
Then Et and Çolak [6] generalized the above sequence spaces to the sequence spaces

$$X(\Delta^r) = \{x = (x_k) : \Delta^r x \in X\}$$

for $X = \ell_\infty$, c and c_0, where $r \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^r x = (\Delta^r x_k - \Delta^r x_{k+1})$, and so $\Delta^r x_k = \sum_{v=0}^{r} (-1)^v \binom{r}{v} x_{k+v}$.

Later on difference sequence spaces were studied by Malkowsky and Parashar [15], Et and Başarır [4], Et and Bektas [5].

We recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

i) $f(x) = 0$ if and only if $x = 0$,

ii) $f(x+y) \le f(x) + f(y)$ for $x, y \ge 0$,

iii) f is increasing,

iv) f is continuous from the right at 0.

It follows that f must be continuous everwhere on $[0,\infty)$. A modulus may be unbounded or bounded. Ruckle [17] and Maddox [14], used a modulus f to construct some sequence spaces.

Subsequently modulus function has been discussed in [1], [16], [19] and many others.

Let $X, Y \subset w$. Then we shall write

$$M(X,Y) = \bigcap_{x \in X} x^{-1} * Y = \{a \in w : ax \in Y \quad \text{for all } x \in X\} \text{ [20].}$$

The set $X^\alpha = M(X, \ell_1)$ is called Köthe-Toeplitz dual space or $\alpha-$dual of X.

Let X be a sequence space. Then X is called

i) *Solid* (or *normal*), if $(\alpha_k x_k) \in X$ for all sequences (α_k) of scalars with $|\alpha_k| \le 1$ for all $k \in \mathbb{N}$, whenever $(x_k) \in X$.

ii) *Symmetric*, if $(x_k) \in X$ implies $(x_{\pi(k)}) \in X$, where $\pi(k)$ is a permutation of $\mathbb{N}$.

iii) *Perfect* if $X = X^{\alpha\alpha}$.

iv) *Sequence algebra* if $x.y \in X$, whenever $x, y \in X$.

It is well known that if X is perfect then X is normal [8].

The following inequality will be used throughout this paper.

$$|a_k + b_k|^{p_k} \le C\{|a_k|^{p_k} + |b_k|^{p_k}\}, \tag{1}$$

where $a_k, b_k \in \mathbb{C}$, $0 < p_k \leq \sup_k p_k = H$, $C = \max\left(1, 2^{H-1}\right)$ [13].

2. Main Results

In this section we prove some results involving the sequence spaces

$$[V,\lambda,f,p]_0(\Delta^r,E),\ [V,\lambda,f,p]_1(\Delta^r,E) \text{ and } [V,\lambda,f,p]_\infty(\Delta^r,E).$$

Definition 2.1. Let E be a Banach space. We define $w(E)$ to be the vector space of all $E-$valued sequences that is $w(E) = \{x = (x_k) : x_k \in E\}$. Let f be a modulus function and $p = (p_k)$ be any sequence of strictly positive real numbers. We define the following sequence sets

$$[V,\lambda,f,p]_1(\Delta^r,E) = \left\{x \in w(E) : \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} [f(\|\Delta^r x_k - L\|)]^{p_k} = 0, \text{ for some } L\right\},$$

$$[V,\lambda,f,p]_0(\Delta^r,E) = \left\{x \in w(E) : \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} [f(\|\Delta^r x_k\|)]^{p_k} = 0\right\},$$

$$[V,\lambda,f,p]_\infty(\Delta^r,E) = \left\{x \in w(E) : \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} [f(\|\Delta^r x_k\|)]^{p_k} < \infty\right\}.$$

If $x \in [V,\lambda,f,p]_1(\Delta^r,E)$ then we will write $x_k \to L\,[V,\lambda,f,p]_1(\Delta^r,E)$ and L will be called λ_E- difference limit of x with respect to the modulus f.

Throughout the paper Z will denote any one of the notation 0, 1, or ∞.

In the case $f(x) = x$, $p_k = 1$ for all $k \in \mathbb{N}$ and $p_k = 1$ for all $k \in \mathbb{N}$, we shall write $[V,\lambda]_Z(\Delta^r,E)$ and $[V,\lambda,f]_Z(\Delta^r,E)$ instead of $[V,\lambda,f,p]_Z(\Delta^r,E)$, respectively.

Theorem 2.2. *Let the sequence (p_k) be bounded. Then the sequence spaces $[V,\lambda,f,p]_Z(\Delta^r,E)$ are linear spaces.*

Proof. We shall prove it for $[V,\lambda,f,p]_0(\Delta^r,E)$. The others can be proved by the same way. Let $x, y \in [V,\lambda,f,p]_0(\Delta^r,E)$ and $\beta, \mu \in \mathbb{C}$. Then there exist positive numbers M_β and N_μ such that $|\beta| \leq M_\beta$ and $|\mu| \leq N_\mu$. Since f is subadditive and Δ^r is linear

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\|\Delta^r(\beta x_k+\mu y_k)\right\|\right)\right]^{p_k}$$
$$\leq \frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(|\beta|\left\|\Delta^r x_k\right\|\right)+f\left(|\mu|\left\|\Delta^r y_k\right\|\right)\right]^{p_k}$$
$$\leq C\left(M_\beta\right)^H\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\|\Delta^r x_k\right\|\right)\right]^{p_k}+C\left(N_\mu\right)^H\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left\|\Delta^r y_k\right\|\right]^{p_k}\to 0$$

as $n\to\infty$. This proves that $[V,\lambda,f,p]_0(\Delta^r,E)$ is a linear space. □

Theorem 2.3. *Let f be a modulus function, then*

$$[V,\lambda,f,p]_0(\Delta^r,E)\subset[V,\lambda,f,p]_1(\Delta^r,E)\subset[V,\lambda,f,p]_\infty(\Delta^r,E).$$

Proof. The first inclusion is obvious. We establish the second inclusion. Let $x\in[V,\lambda,f,p]_1(\Delta^r,E)$. By definition of f we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\|\Delta^r x_k\right\|\right)\right]^{p_k}=\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\|\Delta^r x_k-L+L\right\|\right)\right]^{p_k}$$
$$\leq C\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\|\Delta^r x_k-L\right\|\right)\right]^{p_k}+C\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\|L\|\right)\right]^{p_k}.$$

There exists a positive integer K_L such that $\|L\|\leq K_L$. Hence we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\|\Delta^r x_k\right\|\right)\right]^{p_k}\leq\frac{C}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\|\Delta^r x_k-L\right\|\right)\right]^{p_k}+\frac{C}{\lambda_n}\left[K_L f(1)\right]^H\lambda_n.$$

Since $x\in[V,\lambda,f,p]_1(\Delta^r,E)$ we have $x\in[V,\lambda,f,p]_\infty(\Delta^r,E)$ and this completes the proof. □

Theorem 2.4. *$[V,\lambda,f,p]_0(\Delta^r,E)$ is a paranormed (need not total paranorm) space with*

$$g_\Delta(x)=\sup_n\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\|\Delta^r x_k\right\|\right)\right]^{p_k}\right)^{\frac{1}{M}}$$

where $M=\max(1,\sup p_k)$.

Proof. From Theorem 2.3, for each $x\in[V,\lambda,f,p]_0(\Delta^r,E)$, $g_\Delta(x)$ exists. Clearly $g_\Delta(x)=g_\Delta(-x)$. It is trivial that $\Delta^r x_k=0$ for $x=0$. Since $f(0)=0$, we get $g_\Delta(x)=0$ for $x=0$. Since $p_k/M\leq 1$ and $M\geq 1$, using the Minkowski's inequality and definition of f, for each n, we have

$$\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\|\Delta^r x_k+\Delta^r y_k\|\right)\right]^{p_k}\right)^{\frac{1}{M}}$$
$$\leq\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\|\Delta^r x_k\|\right)+f\left(\|\Delta^r y_k\|\right)\right]^{p_k}\right)^{\frac{1}{M}}$$
$$\leq\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f(\|\Delta^r x_k\|)\right]^{p_k}\right)^{\frac{1}{M}}+\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\|\Delta^r y_k\|\right)\right]^{p_k}\right)^{\frac{1}{M}}$$

Hence $g_{\Delta}(x)$ is subadditive. Finally, to check the continuity of multiplication, let us take any complex number β. By definition of f we have

$$g_{\Delta}(\beta x)=\sup_n\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\|\Delta^r(\beta x_k\|)\right)\right]^{p_k}\right)^{\frac{1}{M}}\leq K_\beta^{\frac{H}{M}}g_\Delta(x)$$

where K_β is a positive integer such that $|\beta|<K_\beta$. Now, let $\beta\to 0$ for any fixed x with $g_{\Delta}(x)\neq 0$. By definition of f for $|\beta|<1$, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\|\beta\Delta^r x_k\|\right)\right]^{p_k}<\varepsilon\quad\text{for } n>n_0(\varepsilon). \tag{2}$$

Also, for $1\leq n\leq n_0$, taking β small enough, since f is continuous we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\|\beta\Delta^r x_k\|\right)\right]^{p_k}<\varepsilon. \tag{3}$$

(2) and (3) together imply that $g_{\Delta}(\beta x)\to 0$ as $\beta\to 0$. □

Theorem 2.5. *If* $r\geq 1$, *then the inclusion*

$$[V,\lambda,f]_Z\left(\Delta^{r-1},E\right)\subset[V,\lambda,f]_Z\left(\Delta^r,E\right)$$

is strict. In general $[V,\lambda,f]_Z(\Delta^i,E)\subset[V,\lambda,f]_Z(\Delta^r,E)$ *for all* $i=1,2,\dots,$ $r-1$ *and the inclusion is strict.*

Proof. We give the proof for $Z=\infty$ only. It can be proved in a similar way for $Z=0$ and $Z=1$. Let $x\in[V,\lambda,f]_\infty(\Delta^{r-1},E)$. Then we have

$$\sup_n\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\|\Delta^{r-1}x_k\right\|\right)\right]<\infty$$

By definition of f, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\|\Delta^r x_k\right\|\right)\right] \leq$$

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\|\Delta^{r-1} x_k\right\|\right)\right] + \frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\|\Delta^{r-1} x_{k+1}\right\|\right)\right] < \infty$$

Thus $[V,\lambda,f]_\infty\left(\Delta^{r-1},E\right) \subset [\Delta^r,\lambda,f]_\infty\left(\Delta^r,E\right)$. Proceeding in this way one will have $[V,\lambda,f]_\infty\left(\Delta^i,E\right) \subset [V,\lambda,f]_\infty\left(\Delta^r,E\right)$ for $i = 1,2,\ldots,r-1$. Let $E = \mathbb{C}$, and $\lambda_n = n$ for each $n\in\mathbb{N}$. Then the sequence $x = (k^r)$, for example, belongs to $[V,\lambda,f]_\infty\left(\Delta^r,E\right)$, but does not belong to $[V,\lambda,f]_\infty\left(\Delta^{r-1},E\right)$ for $f(x) = x$. (If $x = (k^r)$, then $\Delta^r x_k = (-1)^r r!$ and $\Delta^{r-1}x_k = (-1)^{r+1}r!(k+\frac{(r-1)}{2})$ for all $k\in\mathbb{N}$).

□

The proof of the following result is a routine work.

Proposition 2.6. $[V,\lambda,f,p]_1\left(\Delta^{r-1},E\right) \subset [V,\lambda,f,p]_0\left(\Delta^r,E\right)$.

Theorem 2.7. *Let f, f_1, f_2 be modulus functions. Then we have*

i) $[V,\lambda,f_1,p]_Z\left(\Delta^r,E\right) \subset [V,\lambda,f\circ f_1,p]_Z\left(\Delta^r,E\right)$,

ii) $[V,\lambda,f_1,p]_Z\left(\Delta^r,E\right) \cap [V,\lambda,f_2,p]_Z\left(\Delta^r,E\right) \subset [V,\lambda,f_1+f_2,p]_Z\left(\Delta^r,E\right)$.

Proof. i) We shall only prove (i). Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \leq t \leq \delta$. Write $y_k = f_1\left(\left\|\Delta^r x_k\right\|\right)$ and consider

$$\sum_{k\in I_n}[f(y_k)]^{p_k} = \sum_1 [f(y_k)]^{p_k} + \sum_2 [f(y_k)]^{p_k}$$

where the first summation is over $y_k \leq \delta$ and second summation is over $y_k > \delta$. Since f is continuous, we have

$$\sum_1 [f(y_k)]^{p_k} < \lambda_n \varepsilon^H \tag{4}$$

and for $y_k > \delta$, we use the fact that

$$y_k < \frac{y_k}{\delta} \leq 1 + \frac{y_k}{\delta}.$$

By the definition of f we have for $y_k > \delta$,

$$f(y_k) < 2f(1)\frac{y_k}{\delta}.$$

Hence

$$\frac{1}{\lambda_n}\sum_2 [f(y_k)]^{p_k} \leq \max\left(1, \left(2f(1)\delta^{-1}\right)^H\right)\frac{1}{\lambda_n}\sum_{k\in I_n} y_k. \tag{5}$$

From (4) and (5), we obtain $[V,\lambda,f,p]_0\left(\Delta^r\right) \subset [V,\lambda,f\circ f_1,p]_0\left(\Delta^r\right)$.

The proof of (ii) follows from the following inequality

$$[(f_1+f_2)(\|\Delta^r x_k\|)]^{p_k} \le C\left[f_1(\|\Delta^r x_k\|)\right]^{p_k} + C\left[f_2(\|\Delta^r x_k\|)\right]^{p_k}.$$

□

The following result is a consequence of Theorem 2.7 (i).

Proposition 2.8. *Let f be a modulus function. Then $[V,\lambda,p]_Z(\Delta^r,E) \subset [V,\lambda,f,p]_Z(\Delta^r,E)$.*

3. Statistical Convergence

The notion of statistical convergence was introduced by Fast [3] and studied by various authors ([2],[7],[10],[12],[16],[18]).

In this section we give some inclusion relations between $S_\lambda(\Delta^r,E)$ and $[V,\lambda,f,p]_1(\Delta^r,E)$.

Definition 3.1. A sequence $x=(x_k)$ is said to be λ_E^r- statistically convergent to the number L if for every $\varepsilon>0$,

$$\lim_n \frac{1}{\lambda_n}\left|\{k\in I_n : \|\Delta^r x_k - L\| \ge \varepsilon\}\right| = 0.$$

In this case we write $S_\lambda(\Delta^r,E)-\lim x = L$ or $x_k \to LS_\lambda(\Delta^r,E)$.

In the case $\lambda_n=n$ and $L=0$ we shall write $S(\Delta^r,E)$ and $S_{\lambda_0}(\Delta^r,E)$ instead of $S_\lambda(\Delta^r,E)$.

Theorem 3.2. *Let $\lambda=(\lambda_n)$ be the same as in Section 1, then*
i) If $x_k \to L\,[V,\lambda]_1(\Delta^r,E)$ then $x_k \to LS_\lambda(\Delta^r,E)$,
ii) If $x\in\ell_\infty(\Delta^r,E)$ and $x_k\to LS_\lambda(\Delta^r,E)$, then $x_k \to L\,[V,\lambda]_1(\Delta^r,E)$,
iii) $S_\lambda(\Delta^r,E)\cap\ell_\infty(\Delta^r,E) = [V,\lambda]_1(\Delta^r,E)\cap\ell_\infty(\Delta^r,E)$.
where $\ell_\infty(\Delta^r,E)=\{x\in w(E) : \sup_k\|\Delta^r x_k\|<\infty\}$.

Proof. i) Let $\varepsilon>0$ and $x_k\to L\,[V,\lambda]_1(\Delta^r,E)$. Then we have

$$\sum_{k\in I_n}\|\Delta^r x_k - L\| \ge \varepsilon\left|\{k\in I_n : \|\Delta^r x_k - L\|\ge\varepsilon\}\right|.$$

Hence $x_k\to LS_\lambda(\Delta^r,E)$.

In fact the set $[V,\lambda]_1(\Delta^r,E)$ is a proper subset of $S_\lambda(\Delta^r,E)$. To show this, let $E=\mathbb{C}$ and define $x=(x_k)$ such that

$$\Delta^r x_k = \begin{cases} k, & \text{for } n-[\sqrt{n}]+1\le k\le n \\ 0, & \text{otherwise.}\end{cases}$$

Then $x\notin\ell_\infty(\Delta^r,E)$, $x_k\to 0S_\lambda(\Delta^r,E)$, and $x\notin[V,\lambda]_1(\Delta^r,E)$.

ii) Suppose that $x_k\to LS_\lambda(\Delta^r,E)$ and $x\in\ell_\infty(\Delta^r,E)$, say $\|\Delta^r x_k - L\|\le M$. Given $\varepsilon>0$, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\|\Delta^r x_k - L\| =$$
$$\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ \|\Delta^r x_k - L\|\geq\varepsilon}}\|\Delta^r x_k - L\| + \frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ \|\Delta^r x_k - L\|<\varepsilon}}\|\Delta^r x_k - L\|$$
$$\leq \frac{M}{\lambda_n}\{k\in I_n : \|\Delta^r x_k - L\|\geq\varepsilon\} + \varepsilon$$

Hence x is λ_E^r- statistically convergent to the number L.

iii) This immediately follows from (i) and (ii).

□

Theorem 3.3. *If* $\liminf \frac{\lambda_n}{n} > 0$,*then* $S(\Delta^r, E) \subseteq S_\lambda(\Delta^r, E)$.

Proof. For given $\varepsilon > 0$, we get

$$\{k \leq n : \|\Delta^r x_k - L\| \geq \varepsilon\} \supset \{k \in I_n : \|\Delta^r x_k - L\| \geq \varepsilon\}.$$

Hence

$$\begin{aligned}\frac{1}{n}|\{k \leq n : \|\Delta^r x_k - L\| \geq \varepsilon\}| &\geq \frac{1}{n}|\{k \leq n : \|\Delta^r x_k - L\| \geq \varepsilon\}| \\ &\geq \frac{\lambda_n}{n}.\frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^r x_k - L\| \geq \varepsilon\}|.\end{aligned}$$

Therefore $x \in S_\lambda(\Delta^r, E)$.

□

Theorem 3.4. *Let* f *be a modulus function and* $\sup_k p_k = H$. *Then* $[V,\lambda,f,p]_1(\Delta^r, E) \subset S_\lambda(\Delta^r, E)$.

Proof. Let $x \in [V,\lambda,f,p]_1(\Delta^r, E)$ and $\varepsilon > 0$ be given. Let Σ_1 denote the sum over $k \leq n$ such that $\|\Delta^r x_k - L\| \geq \varepsilon$ and Σ_2 denote the sum over $k \leq n$ such that $\|\Delta^r x_k - L\| < \varepsilon$. Then

$$\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^r x_k - L\|)]^{p_k} =$$
$$\frac{1}{\lambda_n}\sum_1[f(\|\Delta^r x_k - L\|)]^{p_k} + \frac{1}{\lambda_n}\sum_2[f(\|\Delta^r x_k - L\|)]^{p_k}$$
$$\geq \frac{1}{\lambda_n}\sum_1[f(\|\Delta^r x_k - L\|)]^{p_k} \geq \frac{1}{\lambda_n}\sum_1[f(\varepsilon)]^{p_k}$$
$$\geq \frac{1}{\lambda_n}\sum_1\min\left([f(\varepsilon)]^{\inf p_k}, [f(\varepsilon)]^H\right)$$
$$\geq \frac{1}{\lambda_n}|\{k \in I_n : \|\Delta^r x_k - L\| \geq \varepsilon\}| \min\left([f(\varepsilon)]^{\inf p_k}, [f(\varepsilon)]^H\right).$$

Hence $x \in S_\lambda(\Delta^r, E)$. □

Theorem 3.5. *Let f be bounded and $0 < h = \inf_k p_k \le p_k \le \sup_k p_k = H < \infty$. Then $S_\lambda(\Delta^r, E) \subset [V, \lambda, f, p]_1 (\Delta^r, E)$.*

Proof. Suppose that f is bounded. Let $\varepsilon > 0$ be given and Σ_1 and Σ_2 be in previous theorem. Since f is bounded there exists an integer K such that $f(x) < K$, for all $x \ge 0$. Then

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(|\Delta^r x_k - L|\right)\right]^{p_k} =$$
$$\frac{1}{\lambda_n}\sum_1\left[f\left(\|\Delta^r x_k - L\|\right)\right]^{p_k} + \frac{1}{\lambda_n}\sum_2\left[f\left(\|\Delta^r x_k - L\|\right)\right]^{p_k}$$
$$\le \frac{1}{\lambda_n}\sum_1 \max\left(K^h, K^H\right) + \frac{1}{\lambda_n}\sum_2\left[f(\varepsilon)\right]^{p_k}$$
$$\le \max\left(K^h, K^H\right)\frac{1}{\lambda_n}\left|\{k \in I_n : \|\Delta^r x_k - L\| \ge \varepsilon\}\right|$$
$$+ \max\left(f(\varepsilon)^h,\ f(\varepsilon)^H\right).$$

Hence $x \in [V, \lambda, f, p]_1 (\Delta^r, E)$. □

Theorem 3.6. $S_\lambda(\Delta^r, E) = [V, \lambda, f, p]_1 (\Delta^r, E)$ *if and only if f is bounded.*

Proof. Let f be bounded. By Theorems 3.4 and 3.5 we have $S_\lambda(\Delta^r, E) = [V, \lambda, f, p]_1 (\Delta^r, E)$.

Conversely suppose that f is unbounded. Then there exists a sequence (t_k) of positive numbers with $f(t_k) = k^2$, for $k = 1, 2, \ldots$. If we choose

$$\Delta^r x_i = \begin{cases} t_k, & i = k^2,\ i = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

then we have

$$\frac{1}{\lambda_n}\left|\{k \in I_n : |\Delta^r x_k| \ge \varepsilon\}\right| \le \frac{\sqrt{\lambda_{n-1}}}{\lambda_n}$$

for all n and so $x \in S_\lambda(\Delta^r, E)$, but $x \notin [V, \lambda, f, p]_1 (\Delta^r, E)$ for $E = \mathbb{C}$. This contradicts to $S_\lambda(\Delta^r, E) = [V, \lambda, f, p] (\Delta^r, E)$.

□

Theorem 3.7. *The sequence spaces $[V, \lambda, f, p]_0 (\Delta^r, E)$, $[V, \lambda, f, p]_1 (\Delta^r, E)$, $[V, \lambda, f, p]_\infty (\Delta^r, E)$, $S_\lambda(\Delta^r, E)$ and $S_{\lambda_0}(\Delta^r, E)$ are not solid for $r \ge 1$.*

Proof. Let $E = \mathbb{C}$, $p_k = 1$ for all k, $f(x) = x$ and $\lambda_n = n$ for all $n \in \mathbb{N}$. Then $(x_k) = (k^r) \in [V, \lambda, f, p]_\infty (\Delta^r, E)$ but $(\alpha_k x_k) \notin [V, \lambda, f, p]_\infty (\Delta^r, E)$ when $\alpha_k = (-1)^k$ for all $k \in \mathbb{N}$. Hence $[V, \lambda, f, p]_\infty (\Delta^r, E)$ is not solid. The other cases can be proved on considering similar examples.

□

From the above theorem we may give the following corollary.

Corollary 3.8. *The sequence spaces* $[V,\lambda,f,p]_0(\Delta^r,E)$, $[V,\lambda,f,p](\Delta^r,E)$ *and* $[V,\lambda,f,p]_\infty(\Delta^r,E)$ *are not perfect for* $r \geq 1$.

Theorem 3.9. *The sequence spaces* $[V,\lambda,f,p]_1(\Delta^r,E)$, $[V,\lambda,f,p]_\infty(\Delta^r,E)$, $S_\lambda(\Delta^r,E)$ *and* $S_{\lambda_0}(\Delta^r,E)$ *are not symmetric for* $r \geq 1$.

Proof. Let $E = \mathbb{C}$, $p_k = 1$ for all k, $f(x) = x$ and $\lambda_n = n$ for all $n \in \mathbb{N}$. Then $(x_k) = (k^r) \in [V,\lambda,f,p]_\infty(\Delta^r,E)$. Let (y_k) be a rearrangement of (x_k), which is defined as follows

$$(y_k) = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10,...}\}.$$

Then $(y_k) \notin [V,\lambda,f,p]_\infty(\Delta^r,E)$.
For the space $S_{\lambda_0}(\Delta^r,E)$, consider the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 1, & \text{if } (2i-1)^2 \leq k < (2i)^2, \quad i = 1,2,... \\ 4, & \text{otherwise.} \end{cases}$$

Then $(x_k) \in S_0(\Delta)$. Let (y_k) be the same as above, then $(y_k) \notin S_0(\Delta)$. □

Remark 3.10. *The space* $[V,\lambda,f,p]_0(\Delta^r,E)$ *is not symmetric for* $r \geq 2$.

Theorem 3.11. *The sequence spaces* $[V,\lambda,f,p]_Z(\Delta^r,E)$, $S_\lambda(\Delta^r,E)$ *and* $S_{\lambda_0}(\Delta^r,E)$ *are not sequence algebras.*

Proof. Let $E = \mathbb{C}$, $p_k = 1$ for all $k \in \mathbb{N}$, $f(x) = x$ and $\lambda_n = n$ for all $n \in \mathbb{N}$. Then $x = (k^{r-2})$, $y = (k^{r-2}) \in [V,\lambda,f,p]_Z(\Delta^r,E)$, but $x.y \notin [V,\lambda,f,p]_Z(\Delta^r,E)$. The other cases can be proved on considering similar examples. □

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