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ON THE FOURTH ORDER ZERO-FINDING METHODS FOR POLYNOMIALS

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ABSTRACT. **The fourth order methods for the simultaneous approximation of simple complex zeros of a polynomial are considered. The main attention is devoted to a new method that may be regarded as a modification of the well known cubically convergent Ehrlich-Aberth method. It is proved that this method has the order of convergence equals four. Two numerical examples are given to demonstrate the convergence behavior of the studied methods.**

1. Introduction

Iterative methods for the simultaneous determination of simple or multiple zeros of a polynomial are efficient tool for solving algebraic equations, see the bibliography [6] and the corresponding updating site

`www.elsevier.com/homepage/sac/cam/ mcnamee`.

More details on simultaneous methods, their convergence properties, computational efficiency, and parallel implementation may be found in, e.g., [2], [5], [8], [9], [10] and references cited there.

In this paper we consider higher-order zero-finding methods, whose importance has grown in computer era for their efficient implementation on parallel computers. In Section 2 we give a review of the fourth order simultaneous methods constructed in the papers [4], [7], [12] and [13]. A special attention will be paid to a new fourth order method which can be regarded as an extension of the well known Ehrlich-Aberth method. The main result is concerned with the convergence analysis of the proposed method; in Section 3 we prove that its order of convergence is equal to four. Results of numerical experiments, obtained by applying the considered fourth order methods, are presented in Section 4.

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2. On some fourth order methods

Let P be a monic algebraic polynomial of degree n with simple zeros $\alpha_1, \dots, \alpha_n$,

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = \prod_{j=1}^{n}(z - \alpha_j) \quad (a_i \in \mathcal{C}).$$

Let $z_1, \dots, z_n$ be approximations to the zeros of P and let $I_n := \{1, \dots, n\}$ be the index set. In this paper we will use the abbreviations

$$N_i = \frac{1}{\delta_{1,i}} = \frac{P(z_i)}{P'(z_i)} \quad \text{(Newton's correction)},$$

$$W_i = \frac{P(z_i)}{\prod\limits_{j=1,\, j\neq i}^{n}(z_i - z_j)} \quad \text{(Weierstrass' correction)},$$

$$F_{2,i} = \sum_{j=1,\, j\neq i}^{n} \frac{N_j}{(z_i - z_j)^2}, \quad S_{1,i} = \sum_{j=1,\, j\neq i}^{n} \frac{1}{z_i - z_j},$$

$$G_{k,i} = \sum_{j=1,\, j\neq i}^{n} \frac{W_j}{(z_i - z_j)^k} \quad (k = 1, 2).$$

The corrections N_i and W_i occur in the well known iterative methods of the second order,

$$\begin{aligned} \hat{z}_i &= z_i - N_i \quad \text{(Newton's method)}, \\ \hat{z}_i &= z_i - W_i \quad \text{(Weierstrass' method)}. \end{aligned}$$

In this section we give a list of the fourth order methods developed during the last three decades.

Starting from the identity

$$\frac{1}{N_i} = \frac{P'(z_i)}{P(z_i)} = \sum_{j=1}^{n} \frac{1}{z_i - \alpha_i} = \frac{1}{z_i - \alpha_i} + \sum_{j=1, j\neq i}^{n} \frac{1}{z_i - \alpha_j},$$

we come to the following fixed point relation

$$\alpha_i = z_i - \frac{1}{\dfrac{1}{N_i} - \sum\limits_{j\neq i} \dfrac{1}{z_i - \alpha_j}} \quad (i \in I_n). \tag{1}$$

Substituting the zero α_j by its approximation z_j ($j \neq i$) in (1), we obtain the cubically convergent Ehrlich-Aberth method for the simultaneous approximation of all simple zeros of a polynomial,

$$\hat{z}_i = z_i - \frac{1}{\dfrac{1}{N_i} - \displaystyle\sum_{j \neq i} \frac{1}{z_i - z_j}} \quad (i \in I_n) \qquad (EA)$$

(see [1], [3]), where $\hat{z}_i$ is a new approximation to the zero α_i.

Instead of simple approximation z_j we can apply some better approximation to α_j. The main goal in this accelerating process is to improve convergence but without increasing the number of numerical operations. This aim can be achieved by choosing Newton's approximation $z_j - N_j$ instead of α_j in (1). In this way we obtain the Ehrlich-Aberth method with Newton's corrections (EAN) of the fourth order [7],

$$\hat{z}_i = z_i - \frac{1}{\dfrac{1}{N_i} - \displaystyle\sum_{j \neq i} \frac{1}{z_i - z_j + N_j}} \quad (i \in I_n). \qquad (EAN)$$

In 1984 Wang and Zheng [12] developed a family of simultaneous methods that generates, in a particular case, the following fourth order method

$$\hat{z}_i = z_i - \frac{1}{f(z_i) - \dfrac{P(z_i)}{2P'(z_i)}\Big(S_{1,i}^2 + S_{2,i}\Big)} \quad (i \in I_n)\,, \qquad (WZ)$$

where

$$f(z) = \frac{P'(z)}{P(z)} - \frac{P''(z)}{2P'(z)}.$$

The methods (EAN) and (WZ) use the polynomial derivatives P' and P''. A derivative free simultaneous method of the fourth order was constructed by Ellis and Watson [4],

$$\hat{z}_i = z_i - \frac{W_i}{1 + G_{1,i} + \dfrac{W_i G_{2,i}}{1 + G_{1,i}}} \quad (i \in I_n). \qquad (EW)$$

Recently, Zheng and Sun [13] simplify the method (EW) by omitting the devisor $1 + G_{1,i}$ but keeping the convergence order. This method is defined by the iterative formula

$$\begin{aligned} \hat{z}_i &= z_i - \frac{W_i}{1 + \displaystyle\sum_{j \neq i} \frac{W_j}{z_i - z_j} + W_i \sum_{j \neq i} \frac{W_j}{(z_i - z_j)^2}} \\ &= z_i - \frac{W_i}{1 + G_{1,i} + W_i G_{2,i}} \quad (i \in I_n). \qquad (ZS) \end{aligned}$$

Now we derive a new fourth order method for the simultaneous determination of simple zeros of a polynomial. Let $z_1, \dots, z_n$ be reasonably good approximations to the zeros $\alpha_1, \dots, \alpha_n$ of the polynomial P, which means that $|\varepsilon| = \max_{1 \le i \le n} |z_i - \alpha_i|$ is sufficiently small quantity. Let us return to the fixed point relation (1) and substitute the exact zero α_j in the right hand side of the relation (1) by its Newton's approximation $z_j - N_j$. One obtains

$$\frac{1}{z_i - \alpha_j} \cong \frac{1}{z_i - z_j + N_j} = \frac{1}{(z_i - z_j)\Big(1 + \dfrac{N_j}{z_i - z_j}\Big)}.$$

Assuming that $|\varepsilon|$ is small enough to provide $|N_j/(z_i - z_j)| < 1$, we use the development into geometric series and get

$$\frac{1}{z_i - \alpha_j} \cong \frac{1}{z_i - z_j}\left(1 - \frac{N_j}{z_i - z_j} + \Big(\frac{N_j}{z_i - z_j}\Big)^2 + O(|\varepsilon|^3)\right). \tag{2}$$

Having in mind (2) we return to the fixed point relation (1) and obtain

$$\alpha_i = z_i - \frac{N_i}{1 - N_i \sum_{j \ne i} \dfrac{1}{z_i - z_j}\Big(1 + \dfrac{N_j}{z_i - z_j} + \Big(\dfrac{N_j}{z_i - z_j}\Big)^2 + O(|\varepsilon|^3)\Big)}$$

for all $i \in I_n$. Neglecting terms of higher order in the last formula, we find

$$\begin{aligned} \hat{z}_i &= z_i - \frac{1}{\dfrac{1}{N_i} - \sum_{j \ne i} \dfrac{1}{z_i - z_j} + \sum_{j \ne i} \dfrac{N_j}{(z_i - z_j)^2}} \\ &= z_i - \frac{N_i}{1 - N_i S_{1,i} + N_i F_{2,i}} \quad (i \in I_n). \end{aligned} \tag{NM}$$

As far as we know, the method defined by (NM) is the new one.

Remark 1. The method (NM) can be regarded as a modification of the cubically convergent Ehrlich-Aberth's method (EA). The additional term $N_i \sum_{j \ne i} N_j (z_i - z_j)^{-2}$ in (NM) increases the convergence order from three to four.

3. Convergence analysis

In this section we prove that the iterative method (NM) has the order of convergence equals four. First, let us introduce the notations

$$d = \min_{\substack{i,j \\ i \ne j}} |\alpha_i - \alpha_j|, \quad q = \frac{2n-1}{d},$$

and suppose that the conditions

$$|\varepsilon_i| < \frac{d}{2n+1} = \frac{1}{q} \quad (i = 1, \dots, n) \tag{3}$$

are satisfied. In the sequel, we will always assume in our analysis and estimate procedures that $n \geq 3$.

Lemma 1. *Let $z_1 \dots, z_n$ be distinct approximations to the zeros $\alpha_1, \dots, \alpha_n$, and let $\varepsilon_i = z_i - \alpha_i$, $\hat{\varepsilon}_i = \hat{z}_i - \alpha_i$, where $\hat{z}_1, \dots, \hat{z}_n$ are new approximations produced by the iterative methods* (NM). *If* (3) *holds, then*

(*i*) $\displaystyle |\hat{\varepsilon}_i| \leq \frac{q^3}{2(n-1)} |\varepsilon_i|^2 \sum_{j \neq i} |\varepsilon_j|^2 \quad (i \in I_n);$

(*ii*) $\displaystyle |\hat{\varepsilon}_i| < \frac{d}{2n-1} = \frac{1}{q} \quad (i = 1, \dots, n).$

Proof. Taking into account (3) we find

$$|z_i - \alpha_j| \geq |\alpha_i - \alpha_j| - |z_i - \alpha_i| > d - \frac{d}{2n-1} = \frac{2n-2}{2n+1} d = \frac{2n-2}{q}, \tag{4}$$

and, by (3) and (4),

$$|z_i - z_j| \geq |z_i - \alpha_j| - |z_j - \alpha_j| > \frac{2n-2}{q} - \frac{1}{q} = \frac{2n-3}{q}. \tag{5}$$

According to (4) we estimate

$$|\Sigma_{1,i}| \leq \sum_{j \neq i} \frac{1}{|z_i - \alpha_j|} < \frac{(n-1)q}{2n-2} = \frac{q}{2}. \tag{6}$$

Since

$$\frac{1}{N_i} = \delta_{1,i} = \sum_{j=1}^{n} \frac{1}{z_i - \alpha_j} = \frac{1}{\varepsilon_i} + \sum_{j \neq i} \frac{1}{z_i - \alpha_j} = \frac{1}{\varepsilon_i} + \Sigma_{1,i},$$

we get

$$N_i = \frac{\varepsilon_i}{1 + \varepsilon_i \Sigma_{1,i}}. \tag{7}$$

Now, using (4), (6) and (7), we estimate

$$|N_j| \leq \left| \frac{\varepsilon_j}{1 + \varepsilon_j \Sigma_{1,j}} \right| < \frac{|\varepsilon_j|}{1 - |\varepsilon_j||\Sigma_{1,j}|} < \frac{1/q}{1 - \dfrac{1}{q} \cdot \dfrac{q}{2}} = \frac{2}{q}. \tag{8}$$

Starting from the iterative formula (NM) we obtain

$$\hat{\varepsilon}_i = \hat{z}_i - \alpha_i = z_i - \alpha_i - \frac{1}{\dfrac{1}{\varepsilon_i} + \displaystyle\sum_{j \neq i} \frac{1}{z_i - \alpha_j} - \sum_{j \neq i} \frac{1}{z_i - z_j} + \sum_{j \neq i} \frac{N_j}{(z_i - z_j)^2}},$$

wherefrom

$$\hat{\varepsilon}_i = \varepsilon_i - \frac{1}{\dfrac{1}{\varepsilon_i} - \displaystyle\sum_{j\neq i} \frac{\varepsilon_j}{(z_i-\alpha_j)(z_i-z_j)} + \sum_{j\neq i} \frac{N_j}{(z_i-z_j)^2}},$$

which reduces to

$$\hat{\varepsilon}_i = \varepsilon_i - \frac{\varepsilon_i}{1+\varepsilon_i \displaystyle\sum_{j\neq i} \frac{N_j(z_i-\alpha_j)-\varepsilon_j(z_i-z_j)}{(z_i-z_j)^2(z_i-\alpha_j)}}. \tag{9}$$

Substituting (7) in (9) we obtain

$$\hat{\varepsilon}_i = \varepsilon_i - \frac{\varepsilon_i}{1+\varepsilon_i \displaystyle\sum_{j\neq i} \frac{\varepsilon_j\big((z_i-\alpha_j)-(z_i-z_j)(1+\varepsilon_j\Sigma_{1,j})\big)}{(z_i-z_j)^2(z_i-\alpha_j)(1+\varepsilon_j\Sigma_{1,j})}},$$

or, after short rearrangement,

$$\hat{\varepsilon}_i = \varepsilon_i - \frac{\varepsilon_i}{1+\varepsilon_i \displaystyle\sum_{j\neq i} \frac{\varepsilon_j^2(1-(z_i-z_j)\Sigma_{1,j})}{(z_i-z_j)^2(z_i-\alpha_j)(1+\varepsilon_j\Sigma_{1,j})}}. \tag{10}$$

Let us put

$$A_{ij} = \frac{1-(z_i-z_j)\Sigma_{1,j}}{(z_i-z_j)^2(z_i-\alpha_j)(1+\varepsilon_j\Sigma_{1,j})}. \tag{11}$$

Then (10) can be written in the shorter form

$$\hat{\varepsilon}_i = \varepsilon_i - \frac{\varepsilon_i}{1+\varepsilon_i \displaystyle\sum_{j\neq i}\varepsilon_j^2 A_{ij}} = \frac{\varepsilon_i^2 \displaystyle\sum_{j\neq i}\varepsilon_j^2 A_{ij}}{1+\varepsilon_i \displaystyle\sum_{j\neq i}\varepsilon_j^2 A_{ij}}. \tag{12}$$

Using the inequalities (4), (5) and (6), and the fact that the function $f(x) = (1+ax)/(bx^2)$ $(a, b > 0)$ is monotonically decreasing for $x > 0$, let us find the upper bound of the absolute values of A_{ij} given by (11). We obtain

$$|A_{ij}| \le \frac{1+|z_i-z_j||\Sigma_{1,i}|}{|z_i-z_j|^2|z_i-\alpha_j||1-\varepsilon_j\Sigma_{1,i}|} < \frac{1+\dfrac{2n-3}{q}\cdot\dfrac{q}{2}}{\Big(\dfrac{2n-3}{q}\Big)^2\cdot\dfrac{2n-2}{q}\cdot\dfrac{1}{2}}$$

$$= \frac{q^3}{n-1}\cdot\frac{2n-1}{2(2n-3)^2}.$$

Hence

$$|A_{ij}| < \frac{1}{3}\cdot\frac{q^3}{n-1}. \tag{13}$$

Using (3) and (13) we bound

$$\Big|\varepsilon_i \sum_{j\neq i} \varepsilon_j^2 A_{ij}\Big| \le |\varepsilon_i| \sum_{j\neq i} |\varepsilon_j|^2 |A_{ij}| < \frac{1}{q}\cdot (n-1)\frac{1}{q^2}\cdot \frac{1}{3}\frac{q^3}{n-1} = \frac{1}{3}.$$

By the last estimate we get

$$\Big|1+\varepsilon_i \sum_{j\neq i} \varepsilon_j^2 A_{ij}\Big| \ge 1 - \Big|\varepsilon_i \sum_{j\neq i} \varepsilon_j^2 A_{ij}\Big| > 1-\frac{1}{3} = \frac{2}{3}. \tag{14}$$

Now from (12) it follows

$$|\hat{\varepsilon}_i| \le \frac{|\varepsilon_i|^2 \sum_{j\neq i} |\varepsilon_j|^2 |A_{ij}|}{1-|\varepsilon_i| \sum_{j\neq i} |\varepsilon_j|^2 A_{ij}|} < \frac{q^3}{2(n-1)} |\varepsilon_i|^2 \sum_{j\neq i} |\varepsilon_j|^2 \quad (i \in I_n), \tag{15}$$

which proves (i) of Lemma 1.

From (15) we obtain

$$|\hat{\varepsilon}_i| \le \frac{q^3}{2(n-1)} |\varepsilon_i|^2 \sum_{j\neq i} |\varepsilon_j|^2 < \frac{1}{q^4}\cdot \frac{(n-1)q^3}{2(n-1)} < \frac{1}{q}.$$

Therefore, we proved the implication

$$|\varepsilon_i| < \frac{d}{2n-1} = \frac{1}{q} \quad \Rightarrow \quad |\hat{\varepsilon}_i| < \frac{d}{2n-1} = \frac{1}{q},$$

which means that the assertion (ii) of Lemma 1 is valid. □

Let $z_1^{(0)}, \dots, z_n^{(0)}$ be reasonably good initial approximations to the zeros $\alpha_1, \dots, \alpha_n$ of the polynomial P, and let $\varepsilon_i^{(m)} = z_i^{(m)} - \alpha_i$, where $z_1^{(m)}, \dots, z_n^{(m)}$ are approximations obtained in the mth iterative step by the simultaneous method (NM). Using the assertions of Lemma 1, we are now able to state the main convergence theorem concerned with the iterative methods (NM).

Theorem 1. *Under the conditions*

$$|\varepsilon_i^{(0)}| = |z_i^{(0)} - \alpha_i| < \frac{d}{2n-1} = \frac{1}{q} \quad (i = 1, \dots, n), \tag{16}$$

the iterative method (NM) *is convergent with the convergence order equal to four.*

Proof. In the proof of Lemma 1 we have derived the relation (15) under the condition (3). Using the same argumentation, under the condition (16) we derive

$$|\varepsilon_i^{(1)}| \le \frac{q^3}{2(n-1)} |\varepsilon_i^{(0)}|^2 \sum_{j\neq i} |\varepsilon_j^{(0)}|^2 < \frac{1}{q} \quad (i \in I_n),$$

which means that the implications

$$|\varepsilon_i^{(0)}| < \frac{d}{2n-1} = \frac{1}{q} \quad \Rightarrow \quad |\varepsilon_i^{(1)}| < \frac{d}{2n-1} = \frac{1}{q} \quad (i \in I_n)$$

hold. Using the mathematical induction we can prove that the condition (16) implies

$$|\varepsilon_i^{(m+1)}| < \frac{q^3}{2(n-1)} |\varepsilon_i^{(m)}|^2 \sum_{j \neq i} |\varepsilon_j^{(m)}|^2 < \frac{1}{q} \quad (i \in I_n) \tag{17}$$

for each $m = 0, 1, \ldots$ and $i = 1, \ldots, n$. Putting $|\varepsilon_i^{(m)}| = t_i^{(m)}/q$, the inequality (17) becomes

$$t_i^{(m+1)} < \frac{\big(t_i^{(m)}\big)^2}{2(n-1)} \sum_{j \neq i} \big(t_j^{(m)}\big)^2 \quad (i \in I_n). \tag{18}$$

Let $t^{(m)} = \max\limits_{1 \leq i \leq n} t_i^{(m)}$. By virtue of (16) it follows

$$q|\varepsilon_i^{(0)}| = t_i^{(0)} \leq t^{(0)} < 1 \ (i \in I_n)$$

and from (18) one obtains $t_i^{(m)} < 1$ for all $i = 1, \ldots, n$ and $m = 1, 2, \ldots$. According to this we obtain from (18)

$$t_i^{(m+1)} < \big(t_i^{(m)}\big)^2 \big(t^{(m)}\big)^2 \leq \big(t^{(m)}\big)^4 \quad (i \in I_n) \tag{19}$$

and conclude that the sequences $\{t_i^{(m)}\}$ $(i \in I_n)$ converge to 0. Consequently, the sequences $\{|\varepsilon_i^{(m)}|\}$ also converge to 0, which means that $z_i^{(m)} \to \alpha_i$ $(i \in I_n)$. Finally, from (19) we may infer that the iterative method (NM) has the convergence order four. □

4. Numerical results

To demonstrate the convergence rate of the presented fourth order methods, we tested a lot of algebraic polynomials with complex zeros. To save all significant digits of the produced approximations, we implemented these methods on PC PENTIUM IV using the programming package *Mathematica* 5 with multiple precision arithmetic.

For comparison purpose, beside the method (NM), we also tested the Zheng-Sun method (ZS), the Ellis-Watson method (EW), the Ehrlich-Aberth method with Newton's corrections (EAN) and the Wang-Zheng method (WZ).

The performed numerical experiments demonstrated very fast convergence and good behavior of the method (NM). For illustration, among a lot of numerical experiments, we present two numerical examples.

Example 1. Iterative methods (EAN), (WZ), (EW), (ZS) and (NM) were applied for the simultaneous approximation of the eigenvalues of Hessenberg's matrix (see Stoer and Bulirsch [11])

$$H = \begin{bmatrix} 1{+}2\mathrm{i} & 1 & 0 & 0 \\ 0 & 2{+}3\mathrm{i} & 1 & 0 \\ 0 & 0 & 3{+}4\mathrm{i} & 1 \\ 1 & 0 & 0 & 4{+}5\mathrm{i} \end{bmatrix},$$

whose characteristic polynomial is

$$f(\lambda) = \lambda^4 - (10 + 14\,i)\lambda^3 - (36 - 100\,i)z^2 + (270 - 66\,i)\lambda - 180 - 180\,i.$$

It is known that so-called Gerschgorin's disks, associated to a matrix $X = [x_{ij}]_{n\times n}$, are of the form $\{x_{ii}; R_i\}$ $(i = 1, \dots, n)$, where x_{ii} are the diagonal elements of the matrix X and $R_i = \sum_{j\neq i} |x_{ij}|$. The union of Gerschgorin's disks contain all eigenvalues of X, which is very convenient for the application of iterative methods (especially inclusion methods) if these disks are mutually disjoint. Namely, in that case each of them contains one and only one eigenvalue. If we apply some method in ordinary complex arithmetic, then we choose initial approximations to be $z_i^{(0)} = x_{ii}$ $(i = 1, \dots, n)$. For this reason, finding eigenvalues of the given matrix $H = [h_{ij}]$, we have taken the diagonal elements h_{ii} of H as starting approximations, that is,

$$z_1^{(0)} = 1 + 2i, \quad z_2^{(0)} = 2 + 3i, \quad z_3^{(0)} = 3 + 4i, \quad z_4^{(0)} = 4 + 5i.$$

As a measure of closeness of approximations with regard to the exact zeros, we have calculated Euclid's norm

$$e^{(m)} := \left(\sum_{i=1}^{n} |z_i^{(m)} - \alpha_i|^2\right)^{1/2} \qquad (m = 0, 1, 2, \dots).$$

The results of the first three iterations are displayed in Table 1, where $A(-q)$ means $A \times 10^{-q}$.

	New method (NM)	(EAN)	(ZS)	(WZ)	(EW)
$e^{(1)}$	1.66(−3)	2.21(−4)	1.66(−3)	1.66(−3)	5.28(−4)
$e^{(2)}$	5.87(−15)	6.78(−19)	1.56(−14)	1.02(−14)	1.12(−16)
$e^{(3)}$	1.04(−60)	1.99(−77)	1.01(−58)	9.42(−60)	1.22(−67)

Table 1 Euclid's norm of errors; $A(-q)$ means $A \times 10^{-q}$.

From Table 1 we observe that all applied methods converge very fast. The explanation for this extremely rapid convergence lies in the fact that the eigenvalues of Hessenberg's matrix are very close to the diagonal elements. The dimension of H is greater, the approximation h_{ii} $(\cong \alpha_i)$ is better.

Example 2. We have applied the same iterative methods as in Example 1 for finding zeros of the monic polynomial P of degree $n = 11$ given by

$$\begin{aligned} P(z) \;\; = \;\; & z^{11} + (0.223 - 0.439\,i)z^{10} + (-0.757 + 0.109\,i)z^9 \\ & +(0.773 + 0.965\,i)z^8 + (0.123 - 0.088\,i)z^7 \\ & +(-0.655 - 0.618\,i)z^6 + (0.738 + 0.645\,i)z^5 \\ & +(-0.018 - 0.438\,i)z^4 + (0.174 - 0.956\,i)z^3 \\ & +(-0.115 + 0.577\,)z^2 + (0.177 - 0.946\,i)z \\ & +(-0.369 - 0.682\,i). \end{aligned}$$

The coefficients $a_k \in \mathcal{C}$ of P (except the leading coefficient) were chosen by the random generator as $Re(a_k) = \texttt{random(x)}$, $Im(a_k) = \texttt{random(x)}$, where $\texttt{random(x)} \in (-1, 1)$.

All tested methods have started with Aberth's initial approximations

$$z_\nu^{(0)} = -\frac{a_1}{n} + r_0 \exp(i\theta_\nu), \quad i = \sqrt{-1}, \quad \theta_\nu = \frac{\pi}{n}\Big(2\nu - \frac{3}{2}\Big) \quad (\nu = 1, \dots, n)$$

(see [1]), taking in this concrete case $n = 11$, $a_1 = 0.223 - 0.439\,i$. In this example the stopping criterion was given by

$$E^{(m)} = \max_{1 \le i \le 11} |P(z_i^{(m)})| < \tau = 10^{-12}.$$

We have performed several experiments taking $r_0 = 0.2,\ 0.5,\ 1,\ 2,\ 4,\ 6,\ 8$ and 100. The aim of such a choice was to investigate the behavior of the tested methods for initial approximations of various magnitudes. Table 2 gives the number of iterative steps for the considered iterative procedures.

	New method (NM)	(EAN)	(ZS)	(WZ)	(EW)
$r_0 = 0.2$	$*)$	17	$*)$	$*)$	9
$r_0 = 0.5$	$*)$	10	$*)$	$*)$	6
$r_0 = 1$	5	5	5	5	4
$r_0 = 2$	7	6	7	7	6
$r_0 = 4$	10	10	10	10	9
$r_0 = 6$	12	11	12	12	10
$r_0 = 8$	13	13	13	13	11
$r_0 = 100$	25	24	25	25	21

Table 2 The number of iterations for various initial approximations and $\tau = 10^{-12}$

From Table 2 we notice that all methods, except (EW), have fulfilled the required stopping criterion after almost same number of iterations. This example, as well as a lot of other numerical experiments, show that the methods (NM), (EAN), (ZS) and (WZ) possess approximately the same ability if initial approximations are chosen using Aberth's approach. The Ellis-Watson method (EW) requires slightly less iterative steps, but its computational cost is somewhat greater.

The symbol $*$ in Table 2 indicates that the stopping criterion was not satisfied in less than 100 iterations. Such behavior of some methods appeared for initial approximations of small magnitudes. Fortunately, the tested methods have been the most efficient in the region containing all zeros of a given polynomial. For a given polynomial $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$ (a_0, $a_n \neq 0$) this region has the the form of the annulus $\{z \in \mathcal{C} \ : \ r < |z| < R\}$, where the radii r and R are calculated by the formulas

$$r = 2 \max_{1 \le k \le n} \left| \frac{a_{n-k}}{a_n} \right|^{1/k}, \quad R = 2 \max_{1 \le k \le n} \left| \frac{a_k}{a_0} \right|^{1/k}$$

(see, e.g., [8, Ch. 1]).

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